Approximation Algorithms for Quantum Max-d-Cut

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Arxiv

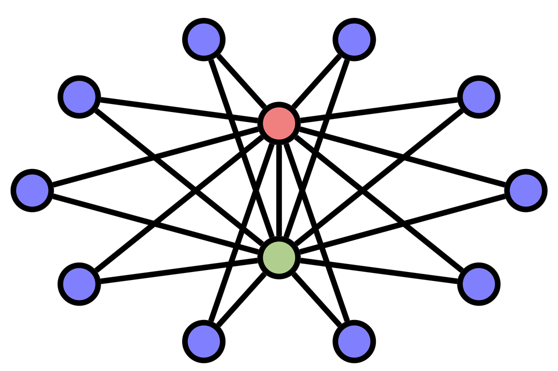
2-Local Hamiltonian Problems

Given a graph $G = (V, E, w)$ and a set of edge-interactions acting on two qudits, $\{h_{uv} \mid (u, v) \in E\}$, we seek to maximize the total energy:

$$\max_{\rho \in \mathcal{D}((\mathbb{C}^d)^{\otimes n})} \sum_{(u,v)\in E} w_{uv} \operatorname{tr}(\rho h_{uv})$$

Max-d-Cut

Max-d-Cut asks to partition the vertices into d sets that maximizes the number of cut edges.



As a 2LHP, Max-d-Cut has an edge-interaction, which is the projector $h = \sum_{a \neq b} |ab\rangle\langle ab|$.

Quantum Max-d-Cut

The Quantum Max-d-Cut problem has an edge-interaction being the projector onto the anti-symmetric subspace of $(\mathbb{C}^d)^{\otimes 2}$:

$$h := \sum_{1 \le a < b \le d} \left(\frac{1}{\sqrt{2}} |ab\rangle - \frac{1}{\sqrt{2}} |ba\rangle \right) \left(\frac{1}{\sqrt{2}} \langle ab| - \frac{1}{\sqrt{2}} \langle ba| \right)$$

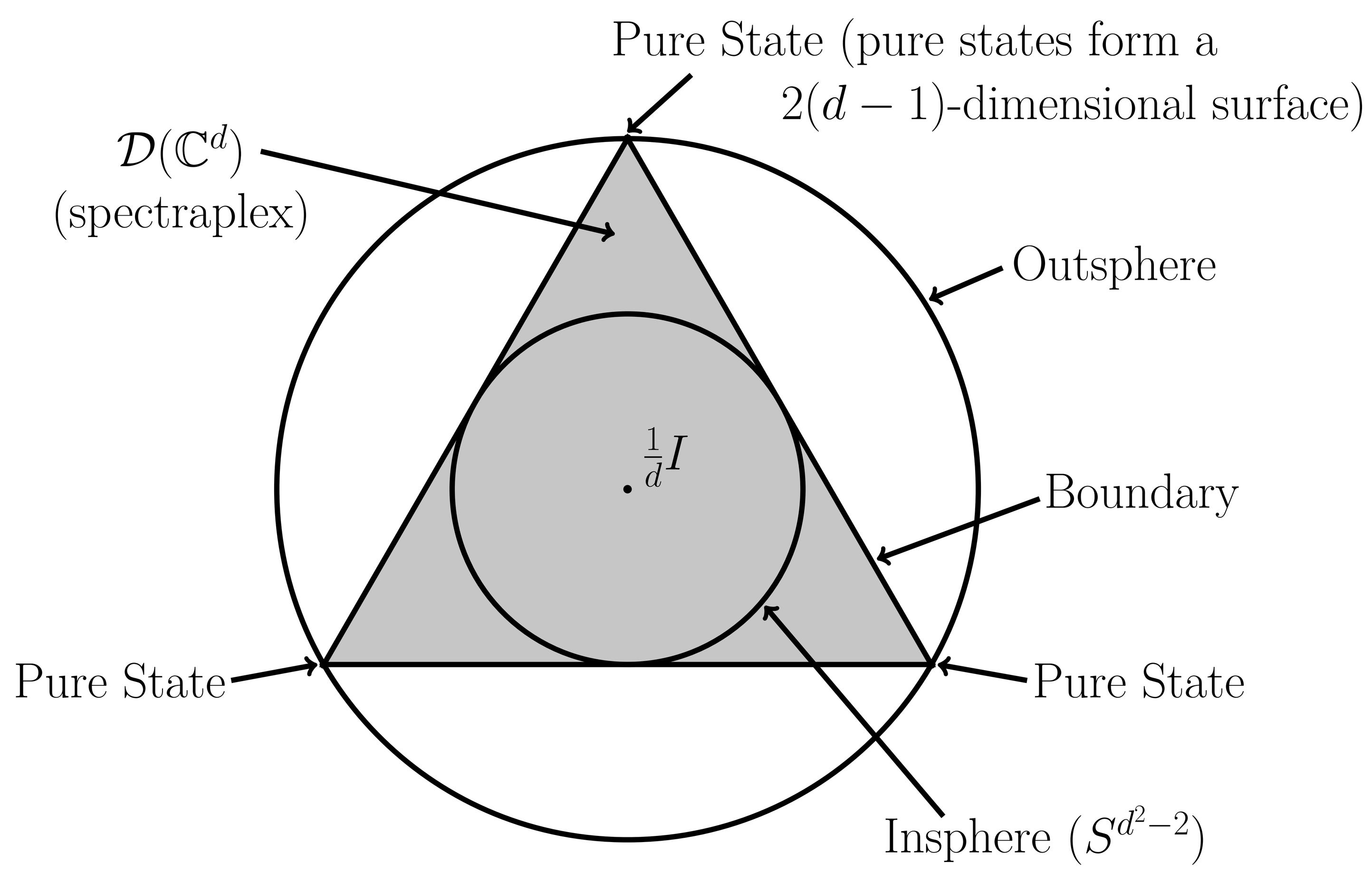
Equivalently, $h = \frac{1}{2}\left(\frac{d-1}{d}\right) I - \frac{1}{4} \sum_{a=1}^{d^2-1} \Lambda^a \otimes \Lambda^a$, where the summation term is known as the $SU(d)$-Heisenberg model.

This problem is known to be QMA-hard [PM21].

When $d = 2$, this problem is well studied as Quantum Max-Cut with the best known approximation ratio of 0.562 [Lee22] (and 0.582 for triangle free graphs [Kin22]).
For product state solution, an algorithm with optimal approximation ratio of $1/2$ is known [PT22].

Geometry of Quantum States

We can understand the geometry of $\mathcal{D}(\mathbb{C}^d)$ pictorially by looking at the $d = 3$ case and considering a planar slice through $\frac{1}{d}I$.



Key Idea

For each qudit, we round to a Bloch vector on the insphere, which is guaranteed to correspond to a density matrix.

The SDP

For a 2-local Hamiltonian problem defined over $G = (V, E, w)$, the SDP relaxation is (some constraints omitted):

$$\begin{aligned} \underset{\substack{\rho_{uv} \in \mathcal{D}((\mathbb{C}^d)^{\otimes 2}) \\ |\Lambda_u^a\rangle \in \mathbb{R}^{(d^2-1)n}}}{\text{maximize}} \quad & \sum_{(u,v)\in E} w_{uv} \operatorname{tr}(\rho_{uv} h_{uv}) \\ \text{subject to} \quad & \langle \Lambda_u^a | \Lambda_u^b \rangle = \frac{2}{d} \delta_{ab} \qquad \forall u \in V,\ \forall a, b \in [d^2-1], \\ & \langle \Lambda_u^a | \Lambda_v^b \rangle = \operatorname{tr}(\rho_{uv} \Lambda^a \otimes \Lambda^b) \quad \forall u, v \in V,\ \forall a, b \in [d^2-1] \end{aligned}$$

We create the "SDP vectors" by concatenating all of the above SDP vectors for a given vertex. $|u\rangle := \frac{1}{\sqrt{\frac{2}{d}(d^2-1)}} \oplus_{a=1}^{d^2-1} |\Lambda_u^a\rangle$

Rounding

We extent the results of [GP19], which considers the $d = 2$ case.
Our Algorithm:
Input $(\forall u \in V)$: SDP vector $|u\rangle \in S^{\ell-1}$ (where $\ell = (d^2-1)^2 n$).
(1) Pick a random matrix, $\mathbf{Z} \sim \mathcal{N}(0,1)^{(d^2-1)\times \ell}$.
(2) *Output:* Bloch Vector $\vec{b}_u := \frac{1}{\sqrt{2d(d-1)}} \mathbf{Z}|u\rangle / \|\mathbf{Z}|u\rangle\|$

Main Theorem

For all $d \geq 2$, there exists an efficient approximation algorithm for Quantum Max-d-Cut that admits an α_d-approximation, where the constants α_d satisfy,

1. $\alpha_d > \frac{1}{2}(1 - 1/d)$ (i.e., non-trivial performance guarantee)
2. $\alpha_d - \frac{1}{2}(1 - 1/d) \sim \frac{1}{2d^3}$
3. $\alpha_2 \geq 0.4987$ [GP19], $\alpha_3 \geq 0.3729$, $\alpha_4 \geq 0.3884$, $\alpha_{100} \geq 0.4950005$
4. for $d \geq 3$ these constants match the algorithmic gap

We note that the Frieze-Jerrum algorithm, adapted to Quantum Max-d-Cut, additively improves on the trivial bound, $\frac{1}{2}(1 - 1/d)$, by $\Theta\left(\frac{\ln d}{2d^2}\right)$ [FJ97], beating our algorithm.

References

[FJ97] A. Frieze and M. Jerrum. Improved approximation algorithms for MAXk-CUT and MAX BISECTION. May 1997.

[GP19] Sevag Gharibian and Ojas Parekh. Almost Optimal Classical Approximation Algorithms for a Quantum Generalization of Max-Cut. 2019.

[Kin22] Robbie King. An Improved Approximation Algorithm for Quantum Max-Cut, September 2022.

[Lee22] Eunou Lee. Optimizing Quantum Circuit Parameters via SDP. 2022.

[PM21] Stephen Piddock and Ashley Montanaro. Universal Qudit Hamiltonians. 2021.

[PT22] Ojas Parekh and Kevin Thompson. An Optimal Product-State Approximation for 2-Local Quantum Hamiltonians with Positive Terms, June 2022.