Problem 1: Recall that for $s \in \mathbb{R}$ we have
$$H^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) : (1 + |\xi|^2)^{s/2} \hat{u} \in L^2(\mathbb{R}^n) \right\}.$$  

(a) Let $u \in \mathcal{E}'(\mathbb{R}^n)$, i.e. $u$ is a compactly supported distribution. Show that there exists $s \in \mathbb{R}$ such that $u \in H^s(\mathbb{R}^n)$.

(b) Let $u \in H^s(\mathbb{R}^n)$ and $\chi \in C_\infty^\infty(\mathbb{R}^n)$. Show that $\chi u \in H^s(\mathbb{R}^n)$.

(c) Show that $\mathcal{S}(\mathbb{R}^n) \subset \cap_{s \in \mathbb{R}} H^s(\mathbb{R}^n) \subset C_\infty^\infty(\mathbb{R}^n)$.

Problem 2: For $u \in \mathcal{D}'(\mathbb{R}^n)$, we say that $u \in H^s_{\text{loc}}(\mathbb{R}^n)$ (for some $s \in \mathbb{R}$) if $\chi u \in H^s(\mathbb{R}^n)$ for all $\chi \in C_\infty^\infty(\mathbb{R}^n)$.

(a) Suppose $v \in \mathcal{E}'(\mathbb{R}^{n+1}_{t,x})$ satisfies the property that
$$(\partial_t - \Delta)v = v^0 + \sum_{i=1}^n \partial_{x_i} v^i$$
for some distributions $v^0, v^1, \ldots, v^n \in \mathcal{D}'(\mathbb{R}^{n+1}_{t,x})$, such that for some $s$ we have
$$v^0 \in H^{s-1}(\mathbb{R}^{n+1}), \quad v^i \in H^{s-1/2}(\mathbb{R}^{n+1}) \text{ for } 1 \leq i \leq n.$$
Show that $v \in H^s(\mathbb{R}^{n+1})$. (Hint: because $v$ is compactly supported, it suffices to make an estimate on $\hat{v}(\tau, \xi)$ when $|\tau, \xi|$ is large. You may find it useful to note that $|||\xi|^2 + i\tau| \geq \max(|\xi|^2, |\tau|)$.)

(b) Suppose $u \in \mathcal{D}'(\mathbb{R}^{n+1}_{t,x})$ satisfies the property that
$$(\partial_t - \Delta)u \in H^{s_0}_{\text{loc}}(\mathbb{R}^{n+1}).$$
Show that $u \in H^{s_0+1}_{\text{loc}}(\mathbb{R}^{n+1})$. (Hint: For a fixed bounded set $U \subset \mathbb{R}^{n+1}$, first show, without knowing anything about $u \in \mathcal{D}'(\mathbb{R}^{n+1})$, that there exists some fixed $s'$ such that $\chi u \in H^{s'}(\mathbb{R}^{n+1})$ for all $\chi \in C_\infty^\infty(U)$. Then, use the assumption on $(\partial_t - \Delta)u$ to show by induction that if
$$\chi u \in H^s(\mathbb{R}^{n+1}) \text{ for all } \chi \in C_\infty^\infty(U),$$
then
$$\chi u \in H^{s+1/2}(\mathbb{R}^{n+1}) \text{ for all } \chi \in C_\infty^\infty(U).$$
Do so by rewriting $(\partial_t - \Delta)(\chi u)$ for $\chi \in C_\infty^\infty(U)$ in terms of sums of derivatives of other functions in $C_\infty^\infty(U)$ times derivatives of $u$.)

(c) Suppose $u$ is a distributional solution to the heat equation $(\partial_t - \Delta)u = 0$ on all of $\mathbb{R}^{n+1}_{t,x}$. Show that $u$ must be smooth on $\mathbb{R}^{n+1}_{t,x}$.
Problem 3: Recall that given a constant coefficient operator $P$ on $\mathbb{R}^{n+1}$, we say that $E \in \mathcal{D}'(\mathbb{R}^n)$ is a backward fundamental solution if $PE = \delta_{(0,0)}$ and

$$\text{supp } E \subseteq \{(t, x) \in \mathbb{R}^{n+1} : t \leq 0\}.$$ 

Show that there does not exist a backward fundamental solution for the heat operator $P = \partial_t - \Delta$. (Hint: Let $E_+$ be the forward fundamental solution derived in class. Show that any backward fundamental solution $E$ must satisfy $E - E_+ \in C^\infty(\mathbb{R}^{n+1})$. Show that this is incompatible with the support condition on $E$.)

Problem 4: Verify directly that the fundamental solution $E$ derived in class

$$E(t, x) = \begin{cases} (4\pi t)^{-n/2}e^{-|x|^2/(4t)} & t > 0 \\ 0 & t \leq 0 \end{cases}$$ 

satisfies $(\partial_t - \Delta)E = \delta_{(0,0)}$. That is, verify that for any $\phi \in C^\infty_c(\mathbb{R}^{n+1})$ we have

$$\int_{\mathbb{R}^{n+1}} E(t, x)(-\partial_t \phi - \Delta \phi)(t, x) \, dx \, dt = \phi(0, 0).$$

Problem 5: Let $(a^{ij})_{i,j=1}^n$ be a positive-definite symmetric real-valued matrix, and consider the constant-coefficient differential operator on $\mathbb{R}^{n+1}$

$$P = \partial_t - \sum_{i,j=1}^n a^{ij} \partial_{x_i} \partial_{x_j}.$$ 

Find a formula for (a) forward fundamental solution of $P$.

Problem 6: Let $g \in C^\infty(\mathbb{R})$, and consider the power series

$$u(t, x) = \sum_{n=0}^\infty g^n(t) \frac{x^{2n}}{(2n)!} \quad (x \in \mathbb{R}).$$

(a) Suppose for each $t$ there exist constants $C_t, C'_t > 0$ such that

$$|g^n(t)| \leq C_t (C'_t)^k k!.$$ 

Show then that the power series defines a smooth function $u(t, x)$ which solves the heat equation.

(b) For $a > 0$, let $g(t) = \begin{cases} e^{-1/ta} & t > 0 \\ 0 & t \leq 0 \end{cases}$. Show that $g$ satisfies the assumptions of part (a), so that in particular the power series defines a nonzero smooth solution $u(t, x)$ of the heat equation which nonetheless satisfies $u(0, x) = 0$ for all $x \in \mathbb{R}$. (This shows the Cauchy problem for the heat equation does not have a unique solution among the space of all smooth functions.)
(c) **Bonus:** Show that $|u(t,x)| \leq Ce^{c|x|^{2a/(a-1)}}$ for some $C, c > 0$. (Note that the exponent of $|x|$ is always larger than 2 and tends towards 2 as $a \to +\infty$; this shows that the growth condition $|u(t,x)| \leq Ce^{c|x|^2}$ needed to guarantee uniqueness is sharp in the exponent of $|x|$ in the exponential.)

**Problem 7:** Let $U$ be either $\mathbb{R}^n$ or a bounded open set. Let $u \in H^2(U)$, and if $U$ is a bounded open set, make the additional assumption that $u \in H^1_0(U)$ as well. Let

$$\|D^2 u\|_{L^2(U)}^2 := \sum_{i,j=1}^n \|\partial_i \partial_j u\|_{L^2(U)}^2.$$  

Show that $\|D^2 u\|_{L^2(U)} = \|\Delta u\|_{L^2(U)}$.

**Problem 8:** Let $U \subset \mathbb{R}^n$ be a bounded open set. For $u, v \in H^1_0(U)$, and $a^{ij}, b^i, c \in L^\infty(U)$ ($1 \leq i \leq n, 1 \leq j \leq n$), with $a^{ij}$ satisfying

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i \xi_j \geq \theta |\xi|^2$$

for all $\xi \in \mathbb{R}^n$ for some $\theta > 0$ (with the estimate holding uniformly for all $x$), let

$$B[u,v] = \int_U \sum_{i,j=1}^n a^{ij}(x)\partial_i u(x)\partial_j v(x) + \sum_{i=1}^n b^i(x)\partial_i u(x)v(x) + c(x)u(x)v(x) \, dx.$$  

Show the upper and lower bounds

$$|B[u,v]| \leq \alpha \|u\|_{H^1_0(U)} \|v\|_{H^1_0(U)}$$

and

$$B[u,u] \geq \beta \|u\|_{H^1_0(U)}^2 - \gamma \|u\|_{L^2(U)}^2$$

for some $\alpha, \beta > 0$ and $\gamma \geq 0$. Express $\alpha$, $\beta$, and $\gamma$ in terms of relevant estimates on the coefficients $a^{ij}$, $b^i$, and $c$.

**Problem 9:** Let

$$L = -\sum_{i,j=1}^n a^{ij}(x)\partial_i \partial_j + \sum_{i=1}^n b^i(x)\partial_i + c(x)$$

where $a^{ij}, b^i, c \in C^\infty(\mathbb{R}^n)$, and $a^{ij}$ satisfies elliptic estimates

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i \xi_j \geq \theta |\xi|^2, \quad \theta > 0$$

uniformly on $\mathbb{R}^n$. Suppose $u \in L^2(\mathbb{R}^{n+1})$ is compactly supported, say with support in $(0,T) \times U$ for some bounded open set $U$. Let

$$f = \partial_t u + Lu.$$
Suppose $f$, initially well-defined as a distribution in $\mathcal{D}'((0, T) \times U)$, is in fact in $L^2((0, T) \times U)$.

Fix $\rho \in C_c^\infty(\mathbb{R})$ with $\rho \geq 0$ and $\int_\mathbb{R} \rho = 1$, and let $\rho_\epsilon(s) = \epsilon^{-1} \rho(s/\epsilon)$. Viewing $u$ as a function in $L^2((0, T); L^2(U))$, let $u_\epsilon = u *_t \rho_\epsilon$ be the time convolution of $u$ with $\rho_\epsilon$, i.e.

$$u_\epsilon(t) = \int_\mathbb{R} \rho_\epsilon(s) u(t - s) \, ds.$$ 

Note that $u_\epsilon$ is also supported in $(0, T) \times U$ if $\epsilon$ is sufficiently small; furthermore $u_\epsilon \to u$ as $\epsilon \to 0^+$ in $L^2((0, T) \times U)$.

(a) Show that $u_\epsilon$ is a weak solution (as defined in class) to the problem

$$\partial_t u_\epsilon - Lu_\epsilon = f_\epsilon \text{ in } (0, T) \times U, \quad u_\epsilon(0, x) = 0 \text{ on } U$$

for all sufficiently small $\epsilon > 0$, where $f_\epsilon = f *_t \rho_\epsilon$. (Hint: the main difficulty is showing that $u_\epsilon \in L^2((0, T]; H^1_0(U))$, in particular that $u_\epsilon(t)$ has $H^1$ regularity for (almost) every $t$. To do so, you may use elliptic estimates, such as those in [Eva10] Section 6.3; you may take for granted that all constants in the elliptic estimates are continuous with respect to the $C^\infty$ topology on the coefficients in question.)

(b) Show that

$$\|u_\epsilon\|_{H^1((0, T) \times U)}$$

is uniformly bounded as $\epsilon \to 0^+$.

(c) Conclude that $u$ is also in $H^1((0, T) \times U)$, and that $u$ is a weak solution to

$$\partial_t u - Lu = f \text{ in } (0, T) \times U, \quad u(0, x) = 0 \text{ on } U.$$ 

(In particular, $u$ enjoys all of the regularity estimates derived in class.)

**Problem 10**: This problem regards deriving the expansion for heat kernels on compact manifolds $M$ by finding a sequence $u_j \in C^\infty(M \times M)$ such that

$$(\partial_t - (\Delta_g)_x) \left( p_0(t, x, y) \sum_{j=0}^k t^j u_j(x, y) \right) \in p_0 t^k C^\infty([0, \infty) \times M \times M).$$

You may take for granted that there exists $\epsilon > 0$ (known as the “injectivity radius”) such that, for every $y \in M$, the map

$$(0, \epsilon) \times \mathbb{S}^{n-1} \to M, \quad (r, \omega) \mapsto \exp_y(r\omega)$$

is a diffeomorphism between $(0, \epsilon) \times \mathbb{S}^{n-1}$ and a punctured neighborhood of $y$ in $M$; in particular $(r, \omega)$ provide “geodesic polar coordinates” on $M$ near $y$. Furthermore, we have $d_g(\exp_y(r\omega), y) = r$, and under these geodesic polar coordinates we have

$$\det g(r, \theta) = r^{2(n-1)} D(\exp_y(r, \omega), y)$$
for all $0 < r < \epsilon$, where $D(x,y) \in C^\infty(M \times M)$ and $D(y,y) = 1$ for all $y \in M$, with $(\Delta_g)_{x}D(y,y) = -\frac{1}{3}S(y)$ where $S$ is the scalar curvature at $y$, and the Laplace-Beltrami operator takes the form

$$\Delta_g = \partial_r^2 + \left( \frac{\partial_r(\sqrt{D})}{D} + \frac{n-1}{r} \right) \partial_r + \Delta_{g^{n-1}_{y}}^\epsilon,$$

where $\Delta_{g^{n-1}_{y}}^\epsilon$ is the Laplace-Beltrami operator corresponding to the metric induced on the geodesic sphere of radius $r$ centered at $y$ (in particular it annihilates any function depending on $r$ only). (If you have some background in differential geometry, you are welcome to verify these facts as well.)

Let

$$p_0(t, x, y) = \frac{1}{(4\pi t)^{n/2}} e^{-d_g(x,y)^2/(4t)}.$$

Note that under geodesic polar coordinates we have $p_0 = (4\pi t)^{-n/2} e^{-r^2/(4t)}$.

(a) Show that if $v \in C^\infty(M \times M)$, then for any $j \geq 0$ we have

$$\left( \partial_t - (\Delta_g)_{x} \right) \left( p_0 t^j v(x,y) \right) \in \left( t^j - 1 \right) \left( \left( j + \frac{r}{2} \frac{\partial_r D}{D} \right) v + r \partial_r v \right) - t^j (\Delta_g)_{x} v \right] p_0.$$

Conclude that if $u_j \in C^\infty(M \times M)$ satisfy the recursive equations

$$\left( j + \frac{r}{2} \frac{\partial_r D}{D} \right) u_j + r \partial_r u_j = \Delta u_{j-1}$$

for $j \geq 0$ (where by convention we set $u_{-1} \equiv 0$), then

$$(\partial_t - (\Delta_g)_{x}) \left( p_0(t, x, y) \alpha(d_g(x, y)) \sum_{j=0}^{k} t^j u_j(x, y) \right) \in p_0 t^k C^\infty([0, \infty) \times M \times M),$$

where $\alpha \in C^\infty_c(\mathbb{R})$ is supported in $r < \epsilon$ and is identically 1 on $r \leq \epsilon/2$.

(b) By solving the recursive equation above for $j = 0$, show that

$$u_0(x, y) = CD^{-1/2}(x, y)$$

for some constant $C$ for all $(x, y)$ with $d_g(x, y) < \epsilon$. Show that if we insist on the property

$$p_0(t, x, y) \alpha(d_g(x, y)) \sum_{j=0}^{k} t^j u_j(x, y) \rightarrow \delta_y(x)$$

as $t \rightarrow 0^+$, then (regardless of the choice of the other $u_j$) we must have $C = 1$.

(c) By solving the recursive equation above for $j = 1$, show that

$$u_1(y, y) = -\frac{1}{2} (\Delta_g)_{y} D(y, y) = \frac{1}{6} S(y).$$

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1In this expression, the $\partial_r$ derivatives are interpreted as acting on the left factor $x$, where the left factor is given geodesic polar coordinates centered at $y$. 
REFERENCES