The exercises in this problem set regard distribution theory and Fourier theory. They can be completed with material covered up to the end of Week 2 (Lecture 4).

**Problem 1**: Let $\phi \in C_c^\infty(U)$ where $U \subset \mathbb{R}^n$ is open. For $h \in \mathbb{R}^n$, let

$$\tau_h \phi(x) := \phi(x - h).$$

Note that $\tau_h \phi \in C_c^\infty(U)$ as well if $h$ is sufficiently small.

(a) For $t > 0$, let

$$\phi_{h,t} = \frac{\phi - \tau_h \phi}{t}.$$ 

Show, if $t > 0$ is sufficiently small, that $\phi_{h,t} \in C_c^\infty(U)$ as well, and that

$$\phi_{h,t} \to h \cdot \nabla \phi$$

as $t \to 0^+$ in the topology of $C_c^\infty(U)$.

(b) Let $u \in D'(\mathbb{R}^n)$. For $h \in \mathbb{R}^n$, let $\tau_h u$ be the distribution defined by

$$(\tau_h u, \phi) := (u, \tau_{-h} \phi)$$

for $\phi \in C_c^\infty(\mathbb{R}^n)$.

Show that if $u \in C_c^\infty(\mathbb{R}^n)$, then this definition agrees with the definition in (1). Moreover, for any $u \in D'(\mathbb{R}^n)$, if we let

$$u_{h,t} = \frac{u - \tau_{h} u}{t}$$

for $h \in \mathbb{R}^n$ and $t > 0$, then show that $u_{h,t} \to h \cdot \nabla u$ in the sense of distributions (i.e. in the topology of $D'(\mathbb{R}^n)$) as $t \to 0^+$.

**Problem 2**: For $a \in \mathbb{C}$ with Re $a > -1$, define $\chi^a_+ : \mathbb{R} \to \mathbb{R}$ by

$$\chi^a_+(x) = \begin{cases} \frac{x^a}{\Gamma(a+1)} & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

where $\Gamma$ is the gamma function (this is a meromorphic function whose poles are at 0, $-1$, $-2$, ..., and it satisfies $\Gamma(1) = 1$ and $\Gamma(z+1) = z\Gamma(z)$ for $z$ away from the poles). Recall as well that $x^a$ is defined as $e^{a \ln(x)}$ for $x > 0$ and $a \in \mathbb{C}$, where $\ln(x)$ is real-valued for $x > 0$.

(a) Show that $\chi^a_+ \in L^1_{loc}(\mathbb{R})$ and $\chi^a_+ = \frac{d}{dx} \chi^{a+1}_+ \frac{1}{1}$ for any Re $a > -1$. 

(b) For \( n \in \mathbb{N} \), define \( \chi^a_{+,n} \) for \( \text{Re} \, a > -n \) as follows: define \( \chi^a_{+,1} = \chi^a_+ \) as above, and if \( \chi^a_{+,n-1} \) is defined for all \( \text{Re} \, a > -n + 1 \), define
\[
\chi^a_{+,n} := \frac{d}{dx} \chi^a_{+,n-1} \quad \text{for} \quad a > -n.
\]
For any \( a \in \mathbb{C} \), show that \( \chi^a_{+,n} = \chi^a_{+,n'} \) for any \( n, n' > -\text{Re} \, a \). Thus, for any \( a \in \mathbb{C} \), we can define
\[
(2) \quad \chi^a_+ := \chi^a_{+,n} \quad \text{for any} \quad n > -\text{Re} \, a.
\]
Note for \( a \) with \( \text{Re} \, a > -1 \) that this agrees with the original definition.

(c) Let \( a \in \mathbb{R} \). Show that \( \chi^a_+ \) (as defined in (2)) is homogeneous of degree \( a \).

(d) For any \( \phi \in C^\infty_c(\mathbb{R}) \), let \( f_\phi : \mathbb{C} \to \mathbb{C} \) be the function
\[
f_\phi(a) = (\chi^a_+, \phi)
\]
where \( \chi^a_+ \) is defined in (2). Show that \( f_\phi \) is an entire analytic function.

(e) For any \( k \in \mathbb{N} \), show that
\[
\chi^{-k-1}_+ = \delta^{(k)},
\]
where \( \delta^{(k)} \) is the \( k \)th distributional derivative of the Dirac delta.

(f) If \( 0 \notin \supp \phi \) and \( a \) is not a negative integer, show that
\[
(\chi^a_+, \phi) = \int_0^\infty \frac{1}{\Gamma(a+1)} x^a \phi(x) \, dx.
\]
Here, we interpret \( \frac{1}{\Gamma(a+1)} \) as the entire analytic function which agrees with \( 1/\Gamma \) away from the poles of \( \Gamma \) (note in particular that this number equals 0 when \( a \) is a negative integer). In particular, if \( a \) is not a negative integer, then away from 0 we have that \( \chi^a_+ \) is given by the same formula as before, while if \( a \) is a negative integer the pairing gives zero, i.e. \( \chi^a_+ \) is supported only at 0 when \( a \) is a negative integer.

**Problem 3:** This problem concerns extending homogeneous distributions on \( \mathbb{R}^n \setminus \{0\} \) to distributions on \( \mathbb{R}^n \). See Section 3.2, particularly Theorem 3.2.3, of Hörmander’s *The Analysis of Linear Partial Differential Operators Vol. 1* [Hör90].

In this problem, \( a \) denotes a real number.

(a) Fix \( u \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\}) \) and \( \phi \in C^\infty_c(\mathbb{R}^n \setminus \{0\}) \). Consider the function \( f(t) = (u, \phi(tx)) \) for \( t \in \mathbb{R} \). Show that
\[
f'(1) = \left( u, \sum_{i=1}^n x_i \partial_i \phi \right).
\]

(b) Suppose that \( u \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\}) \) is homogeneous of degree \( a \). Show that
\[
\left( u, \sum_{i=1}^n x_i \partial_i \phi \right) + (a + n)(u, \phi) = 0
\]
for any \( \phi \in C^\infty_c(\mathbb{R}^n \setminus \{0\}) \).
(c) Suppose $\psi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ satisfies
\[ \int_0^\infty r^{a+n-1}\psi(r\omega)\,dr = 0 \quad \text{for all } \omega \in S^{n-1}. \]
Let $\phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be defined in polar coordinates by
\[ \phi(r\omega) = r^{-(a+n)} \int_0^r s^{a+n-1}\psi(s\omega)\,ds. \]
Show that $\phi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, and $\sum_{i=1}^n x_i \partial_i \phi + (a+n)\phi = \psi$. Conclude that $(u, \psi) = 0$ for all distributions $u \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ homogeneous of degree $a$.

We now assume either $a > -n$ or $a \notin \mathbb{Z}$. We now show that distributions on $\mathbb{R}^n \setminus \{0\}$ which are homogeneous of degree $a$ where can be extended to $\mathbb{R}^n$. Let $x_+^n = \Gamma(a+1)\chi_+^a$, where $\chi_+^a$ is defined in [2]. Note that $x_+^n$ restricted to $\mathbb{R}\setminus \{0\}$ equals $x^n$ for $x > 0$ and $0$ for $x < 0$.

(d) For $\phi \in C_c^\infty(\mathbb{R}^n)$, define
\[ (R_a \phi)(y) = (x_+^{a+n-1}, \phi(xy))_\mathbb{R} \quad \text{for } y \in \mathbb{R}^n \setminus \{0\}. \]
Show that $R_a \phi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ and that $R_a \phi$ is homogeneous of degree $-n - a$.

(e) Suppose $\phi_1, \phi_2 \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, and $R_a \phi_1 = R_a \phi_2$ on $\mathbb{R}^n \setminus \{0\}$. Show that $(u, \phi_1) = (u, \phi_2)$ for all distributions $u \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ homogeneous of degree $a$.

(f) Let $\psi \in C_c^\infty((0, \infty))$ satisfy
\[ \int_0^\infty \psi(t)\,dt = 1. \]
For $\phi \in C_c^\infty(\mathbb{R}^n)$ (not necessarily supported away from zero), show that $\psi(|x|)R_a \phi(x) \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$, and $R_a \psi(|x|)R_a \phi(x) = (R_a \phi)(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Conclude that if $\phi$ is supported away from zero, then
\[ (u, \psi(|x|)R_a \phi) = (u, \phi) \]
for all distributions $u \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ homogeneous of degree $a$.

(g) Given $u \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ homogeneous of degree $a$, let $\tilde{u}$ be the distribution on $\mathbb{R}^n$ defined by
\[ (\tilde{u}, \phi) := (u, \psi(|x|)R_a \phi). \]
Show that $\tilde{u}$ is indeed a distribution on $\mathbb{R}^n$ whose restriction to $\mathbb{R}^n \setminus \{0\}$ is $u$.

**Problem 4:** Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$, and for $\epsilon > 0$ let $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$.

(a) Show that $\varphi_\epsilon$ converges to $\delta$ in the sense of distributions as $\epsilon \to 0^+$.

(b) Show that $|\varphi_\epsilon|^2$ does not converge in the sense of distributions.

**Upshot:** There does not exist an operator $\mathcal{D}'(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ which extends the notion of pointwise multiplication (say defined initially on $C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n)$) which is also continuous on $\mathcal{D}'(\mathbb{R}^n) \times \mathcal{D}'(\mathbb{R}^n)$, since we can find a sequence $\varphi_n$ in $C_c^\infty(\mathbb{R}^n)$ converging to $\delta$ in $\mathcal{D}'(\mathbb{R}^n)$ such that $\varphi_n \cdot \varphi_n$ does not converge.
Problem 5:
(a) Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function such that $\nabla f(x) \neq 0$ when $f(x) = 0$. Note then that $f^{-1}(0)$ is a smooth codimension 1 submanifold of $\mathbb{R}^n$, i.e. a hypersurface. Show that

$$\delta(f) = \frac{dS}{|\nabla f|},$$

where $dS$ is the Euclidean surface measure on the $f^{-1}(0)$.

(b) With $f$ as in part (a), let $\Omega = \{ f > 0 \}$, and let $u = 1_\Omega$ be the indicator function of $\Omega$. Let $\phi_j \in C^\infty_c(\mathbb{R}^n)$, $1 \leq j \leq n$, and $\phi = (\phi_1, \ldots, \phi_n)$. Show that

$$\sum_{j=1}^n (\partial_j u, \phi_j) = (\delta(f), \phi \cdot \nabla f).$$

Problem 6: Let $a : \mathbb{R}^{n+1}_{\xi, \tau}\{0\} \to \mathbb{C}$ and $H : \mathbb{R}^{n+1}_{x,t} \to \mathbb{R}$ be defined by

$$a(\xi, \tau) = \frac{1}{|\xi|^2 + i\tau}$$

and

$$H(x, t) = \begin{cases} 
(4\pi t)^{-n/2}e^{-|x|^2/4t} & t > 0 \\
0 & t \leq 0
\end{cases}.$$ 

Here, $|.|$ is the norm on $\mathbb{R}^n$.

(a) Show that $a$ and $H$ are both locally integrable on $\mathbb{R}^{n+1}$, and furthermore that they define tempered distributions, i.e. $a \in S'(\mathbb{R}^{n+1}_{\xi, \tau})$ and $H \in S'(\mathbb{R}^{n+1}_{x,t})$.

(b) Show that the Fourier transform of $H$ equals $a$.

Problem 7:
(a) Let $H(x) = 1_{\{x > 0\}}$ be the Heaviside function. Show that the Fourier transform of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-\epsilon x}H(x)$ (where $\epsilon > 0$) is given by

$$\hat{f}(\xi) = \frac{1}{i\xi + \epsilon}.$$ 

(b) Conversely, show by direct computation (i.e. without invoking the Fourier Inversion formula and part (a)) that the inverse Fourier transform of $g(\xi) = \frac{1}{i\xi + \epsilon}$ is $f(x)$ defined in part (a). (Hint: First show that if $g_R(\xi) = 1_{[-R, R]}(\xi)g(\xi)$, then $g_R \to g$ in $S'(\mathbb{R})$. To compute the inverse Fourier transform of $g_R$, you need to evaluate the integral

$$\frac{1}{2\pi} \int_{-R}^R \frac{e^{ix\xi}}{i\xi + \epsilon} d\xi.$$ 

This integral, or rather its limit/asymptotics as $R \to \infty$, can be evaluated using contour integration, by taking a semicircular contour with the semicircle lying either in the upper or lower half plane, chosen appropriately so that the exponential factor $e^{ix\xi}$ is exponentially decaying and not growing in that half plane.)
Problem 8: Recall that the distributions \((x \pm i0)^{-1}\) on \(\mathbb{R}\) are defined by
\[
\phi \mapsto \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \frac{\phi(x)}{x \pm i\epsilon} \, dx.
\]
(a) Show that the limit on the RHS does indeed exist for any \(\phi \in C_c^\infty(\mathbb{R})\).
(b) Show that
\[(x - i0)^{-1} - (x + i0)^{-1} = 2\pi i \delta_0.
\]
(c) Compute the Fourier transforms of \((x \pm i0)^{-1}\).

Problem 9: For \(n \geq 3\), consider the distribution
\[u = \delta \left( x_n^2 - \sum_{i=1}^{n-1} x_i^2 \right).\]
This is a distribution of order \(-2\) well-defined on \(\mathbb{R}^n \setminus \{0\}\), so by Problem 3 there is a unique way to extend this to a homogeneous distribution of order \(-2\) defined on \(\mathbb{R}^n\) when \(n \geq 3\).
Show that \(u \in \mathcal{S}'(\mathbb{R}^n)\), and compute its Fourier transform in the case \(n = 4\).

Problem 10: The purpose of this problem is to derive the stationary phase lemma
for the particular quadratic phase function \((x, y) \mapsto x \cdot y\) on \(\mathbb{R}^{2n}\) using the Fourier inversion formula. See Theorem 7.7.3 of Hörmander’s *The Analysis of Linear Partial Differential Operators* Vol. 1 [Hör90] for the most general version of quadratic stationary phase, as well as generalizations to non-quadratic phase functions.

For \(n \in \mathbb{N}\), we view \(\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n\), with coordinates \((x, y)\) where \(x, y \in \mathbb{R}^n\).
(a) Let \(\phi, \psi \in C_c^\infty(\mathbb{R}^n)\). For \(\lambda > 0\), consider the integral
\[I(\lambda) = \int_{\mathbb{R}^{2n}} e^{i\lambda x \cdot y} \phi(x) \psi(y) \, dx \, dy.
\]
Show that \(\lambda^n I(\lambda) \to (2\pi)^n \phi(0) \psi(0)\) as \(\lambda \to +\infty\). Furthermore, show that if \(\phi \in C_c^\infty(\mathbb{R}^n)\) is constant in a neighborhood of the origin, then
\[\lambda^n I(\lambda) - (2\pi)^n \phi(0) \psi(0) = O(\lambda^{-N})\]
as \(\lambda \to +\infty\) for any \(N > 0\). (*Hint:* show that \(I(\lambda)\) equals
\[\frac{1}{\lambda^n} \int_{\mathbb{R}^n} \phi(-\xi/\lambda) \hat{\psi}(\xi) \, d\xi
\]
by making the substitution \(\xi = -\lambda x,\))
(b) Suppose \(\phi, \psi \in C_c^\infty(\mathbb{R}^n)\) are constant in a neighborhood of the origin. For multi-indices \(\alpha, \beta\), let
\[I_{\alpha,\beta}(\lambda) = \int_{\mathbb{R}^{2n}} e^{i\lambda x \cdot y} x^\alpha y^\beta \phi(x) \psi(y) \, dx \, dy.
\]
Show that if \(\alpha \neq \beta\), then \(I_{\alpha,\beta}(\lambda) = O(\lambda^{-N})\) for all \(N > 0\). In addition, show that
\[i^{-|\alpha|} \lambda^{|\alpha|+n} I_{\alpha,\alpha}(\lambda) - (2\pi)^n \alpha! \phi(0) \psi(0) = O(\lambda^{-N})\]
for all $N > 0$, where $\alpha! = \alpha_1! \alpha_2! \ldots \alpha_n!$ if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

(c) Suppose that $f(x, y) = x^\alpha y^\beta g(x, y)$ for some multi-indices $\alpha$ and $\beta$ and for some $g \in C_c^\infty(\mathbb{R}^{2n})$. Show that

$$\int_{\mathbb{R}^{2n}} e^{i\lambda x \cdot y} f(x, y) \, dx \, dy = O(\lambda^{-n-\max(|\alpha|, |\beta|)})$$

as $\lambda \to \infty$.

(d) Now let $f$ be any function in $C_c^\infty(\mathbb{R}^{2n})$, and let

$$I_f(\lambda) = \int_{\mathbb{R}^{2n}} e^{i\lambda x \cdot y} f(x, y) \, dx \, dy.$$

Show that $I_f(\lambda)$ admits the asymptotic expansion

$$I_f(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^n \sum_{\alpha} i^{|\alpha|} \partial_x^\alpha \partial_y^\alpha f(0) \frac{\alpha!}{\alpha!} \lambda^{-|\alpha|},$$

in the sense that for any integer $N > 0$ we have

$$\left| I_f(\lambda) - \left(\frac{2\pi}{\lambda}\right)^n \sum_{|\alpha| < N} i^{|\alpha|} \partial_x^\alpha \partial_y^\alpha f(0) \frac{\alpha!}{\alpha!} \lambda^{-|\alpha|} \right| = O(\lambda^{-n-N}).$$

**Hint:** Use Taylor’s theorem, which states that for any $f \in C^\infty(\mathbb{R}^m)$ and any $N > 0$ we have

$$f(z) = \sum_{|\gamma| < N} \frac{\partial^\gamma f(0)}{\gamma!} z^\gamma + \sum_{|\gamma| = N} R_\gamma f(z) z^\gamma$$

for some choice of functions $R_\gamma f$ which are also smooth. Also note that since in our case we have $f \in C_c^\infty(\mathbb{R}^{2n})$, there exist $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ which are identically one near the origin such that $f(x, y) = f(x, y)\phi(x)\psi(y)$.

**REFERENCES**