ASSIGNMENT #4: Principal Component Analysis

Covariance and Principal Components

As we discussed in class, the principal components of a covariance matrix are a powerful way of separating the dominant co-varying parts of time series of observations made at different geographical locations. This assignment will help to solidify your ability to interpret the principal components.

Question 1:

In class, we used principal component analysis to explore and quantify how temperature variations at three major Australia cities (Sydney, Melbourne and Perth) have co-varied during

Figure 1: Map of Australia showing the location of Sydney, Melbourne, Perth, Alice Springs and Darwin.
during the last 60 years or so. In this assignment, we will extend the ideas of principal component analysis to study how the temperatures at five locations in Australia co-vary. The cities considered are Sydney, Melbourne, Perth, Alice Springs and Darwin, and the location of each city is shown in Fig. 1. Specifically, we will consider the anomalies (i.e. the seasonal cycle is removed) in the annually averaged daily maximum temperature at each location.

In keeping with the notation introduced in class, time series of temperature anomalies at Sydney will be denoted by \(S_i\), those at Melbourne by \(M_i\), those at Perth by \(P_i\), and those at Alice Springs and Darwin by \(A_i\) and \(D_i\) respectively. The temperatures at Sydney and Melbourne were made at the city observatories and date back to the mid-1800s. However, observations at Perth, Alice Springs and Darwin were made at airports which were not opened until the mid-1940s. Therefore in the following example, time series spanning the period 1947-2013 are considered in which case the subscript \(i = 1\) corresponds to 1947, \(i = 2\) to 1948, \(i = 3\) to 1949, and so on. The time series \(S_i\), \(M_i\), \(P_i\), \(A_i\) and \(D_i\) are shown in Fig. 2. At all five locations, the temperature anomalies show a pronounced upward trend (indicative of warming) punctuated by year-to-year variations.

\[\text{Figure 2: Time series of the annual mean daily maximum temperature anomalies at Sydney (black), Melbourne (red), Perth (blue), Alice (green) and Darwin (yellow) for the period 1947-2013. (Data are courtesy of the Australian Bureau of Meteorology).}\]

Since we now dealing with five separate time series of annual mean daily maximum temperatures, the associated covariance matrix will be a \(5\times5\) symmetric matrix given by:
\[
\mathbf{C} = \begin{pmatrix}
\sigma_S^2 & \text{cov}(M,S) & \text{cov}(P,S) & \text{cov}(A,S) & \text{cov}(D,S) \\
\text{cov}(S,M) & \sigma_M^2 & \text{cov}(P,M) & \text{cov}(A,M) & \text{cov}(D,M) \\
\text{cov}(S,P) & \text{cov}(M,P) & \sigma_P^2 & \text{cov}(A,P) & \text{cov}(D,P) \\
\text{cov}(S,A) & \text{cov}(M,A) & \text{cov}(P,A) & \sigma_A^2 & \text{cov}(D,A) \\
\text{cov}(S,D) & \text{cov}(M,D) & \text{cov}(P,D) & \text{cov}(A,D) & \sigma_D^2
\end{pmatrix}
\] (1)

where \( \sigma_S^2 \) is the temperature variance at Sydney, \( \sigma_M^2 \) is the temperature variance at Melbourne, \( \sigma_P^2 \) the temperature variance at Perth, etc. Similarly, \( \text{cov}(S,M) = \text{cov}(M,S) \) is the covariance between temperatures at Sydney and Melbourne, \( \text{cov}(D,P) = \text{cov}(P,D) \) is the covariance between temperatures at Darwin and Perth, \( \text{cov}(A,M) = \text{cov}(M,A) \) is the covariance between temperatures at Alice Springs and Melbourne, and so on.

Using the observed temperature time series at the five locations shown in Fig. 2, the covariance matrix becomes:

\[
\mathbf{C} = \begin{pmatrix}
0.31 & 0.22 & 0.19 & 0.30 & 0.11 \\
0.22 & 0.44 & 0.25 & 0.30 & 0.09 \\
0.19 & 0.25 & 0.60 & 0.19 & 0.09 \\
0.30 & 0.30 & 0.19 & 0.84 & 0.19 \\
0.11 & 0.09 & 0.09 & 0.19 & 0.17
\end{pmatrix}
\] (2)

(a) Based on the covariance matrix in (2), what is: (i) the temperature variance at Alice Springs (i.e. \( \sigma_A^2 \))? (ii) the temperature covariance between Sydney and Darwin, \( \text{cov}(S,D) \)? (iii) the temperature covariance between Alice Springs and Perth, \( \text{cov}(A,P) \)? (vi) the temperature covariance between Darwin and Sydney, \( \text{cov}(D,S) \)? In each case also include the units of the variance and covariance.

(b) The principal components of \( \mathbf{C} \) in equations (1) and (2) are defined by the eigenvector equation:

\[
\mathbf{C} \mathbf{e}_k = \lambda_k \mathbf{e}_k
\] (3)

where \( \mathbf{e}_k \) are the principal components, \( \lambda_k \) are the associated eigenvalues, and the subscript \( k \) has the value 1, 2, 3, 4 or 5. It is conventional to order the principal components in terms of descending size of the eigenvalues so that \( \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5 \). Using this convention, Table 1 shows the elements of each principal component \( \mathbf{e}_k \) and the corresponding eigenvalues.
<table>
<thead>
<tr>
<th>City</th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$e_3$</th>
<th>$e_4$</th>
<th>$e_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sydney</td>
<td>0.37</td>
<td>0.01</td>
<td>0.18</td>
<td>0.74</td>
<td>-0.53</td>
</tr>
<tr>
<td>Melbourne</td>
<td>0.43</td>
<td>0.19</td>
<td>0.81</td>
<td>-0.28</td>
<td>0.20</td>
</tr>
<tr>
<td>Perth</td>
<td>0.43</td>
<td>0.77</td>
<td>-0.45</td>
<td>-0.13</td>
<td>-0.01</td>
</tr>
<tr>
<td>Alice Springs</td>
<td>0.67</td>
<td>-0.60</td>
<td>-0.31</td>
<td>-0.30</td>
<td>-0.07</td>
</tr>
<tr>
<td>Darwin</td>
<td>0.20</td>
<td>-0.08</td>
<td>-0.11</td>
<td>0.51</td>
<td>0.82</td>
</tr>
</tbody>
</table>

Table 1: The values of the elements of each of the principal components of the covariance matrix (2) are listed along with the corresponding cities where the temperature observations were collected. Also shown are the eigenvalues $\lambda_k$ that correspond to each principal component.

Using the data in Table 1, compute the fraction of the total variance that is accounted for by each of the principal components.

(c) For each of the principal components $e_k$ listed in Table 1, *carefully* draw colored circles on the appropriate maps of Australia in Fig. 3 at the locations of the cities corresponding to each element of $e_k$. The city locations are shown in Fig. 1. Draw red circles for positive elements of each $e_k$ and blue circles for negative elements. In addition, draw each circle so that the circle diameter is proportional to the relative size of each element. Figure 4.6 from Module 4 shows an example of this type of diagram in the case where the principal components have only three elements.

(d) Briefly interpret, in your own words, the principal component patterns that you have drawn in Fig. 3 in terms of the co-variations of temperature measured at the five cities.

(e) Recall from Module 4, section 4.5, that the individual elements of the time series of the five cities can be arranged as the elements of a column vector $y_i$, in this case a 5×1 column vector, where:

$$y_i = \begin{pmatrix} S_i \\ M_i \\ P_i \\ A_i \\ D_i \end{pmatrix}.$$  \hfill (4)

Furthermore, the vector $y_i$ can be expressed as a unique combination of the principal components so that:

$$y_i = a_1e_1 + b_1e_2 + c_1e_3 + d_1e_4 + f_1e_5$$  \hfill (5)
where $a_i$, $b_i$, $c_i$, $d_i$ and $f_i$ are the principal component time series of the principal components $e_1$, $e_2$, $e_3$, $e_4$ and $e_5$ respectively. Figure 4 shows the principal component time series $a_1$ of the first principal component, $e_1$. In just a few words, give a physical interpretation of the time series in Fig. 4 in light of the pattern of colored circles that you have drawn for $e_1$ in Fig. 3.

**Figure 3**: Blank maps of Australia for your answers to part (c).
Figure 4: The principal component time series $a_t$ of the first principal component $e_1$.

Question 2:

Recall from class that principal components of long, satellite-derived time series are commonly used to identify important climate modes of variability. One such mode of variability discussed in Module 4, section 4.7.2, is the North Pacific Gyre Oscillation (NPGO). Figure 5 shows a leading principal component (aka Empirical Orthogonal Function; EOF) of sea surface height (SSH) in the North Pacific Ocean (as measured by earth orbiting satellite altimeters) that is associated with the NPGO.

Figure 5: The second principal component of NE Pacific SSH derived from satellite data. Areas colored red and blue are of opposite sign. (http://www.o3d.org/web_db_slides/npgo.png)
(a) In just a few words, describe how the sea surface height (SSH) covaries across the NE Pacific based on the principal component for SSH shown in Fig. 5.

(b) Draw a schematic of the circulation associated with the subtropical gyre and subpolar gyre in the region of the NE Pacific shown in Fig. 5.

(c) The SSH pattern in Fig. 5 can also be used as an indicator of variations in the ocean pressure below the surface. Suppose that the SSH stands anomalously high equatorward of 40°N and anomalously low poleward of 40°N (i.e. the anomaly in SSH would look like Fig. 5). This would correspond to the positive phase of the NPGO. Assuming that the surface ocean circulation is in geostrophic balance with the near surface pressure field associated with the changes in SSH, what influence would the *positive* phase of the NPGO have on the strength of the subtropical and subpolar gyre circulations?

(d) What influence would the *negative* phase of the NPGO have on strength of the subtropical and subpolar gyre circulations?