Economic Theory **18**, 711–744 (2001)

Economic Theory

Contests where there is variation in the marginal productivity of effort*

Nirvikar Singh and Donald Wittman

Economics Department, University of California, Santa Cruz, CA 95064, USA
(e-mail: wittman@cats.ucsc.edu)

Received: July 30, 1998; revised version: August 7, 2000

Summary. We provide a characterization of participants' behavior in a contest or tournament where the marginal productivity of effort varies across contestants and individual productivity is private information. We then consider the optimal design of such a contest.

We first analyze contestant behavior for the usual type of contest, where the highest output wins. Abilities need not be independently distributed. We demonstrate that there is a unique symmetric equilibrium output function, that output is increasing in ability, and that marginal effort is increasing in ability, while effort decreases when the cost of effort increases.

Next we consider the case where the highest output need not win, with independently distributed abilities. We analyze the contest designer's decisions in choosing contest rules optimal from her perspective. We show that the output produced, probability of winning, and contest designer's expected revenue are generally increasing in contestants' ability. We examine the relationship between the marginal cost of producing output and marginal utility per dollar of the net award for winning.

Keywords and Phrases: Optimal contest, Contest design, Asymmetric information, Correlated abilities.

JEL Classification Numbers: D3, D8, J3.

* We acknowledge the helpful comments of seminar audiences at UC Santa Cruz and the Econometric Society Winter Meeting, and the financial support of the UCSC Academic Senate and the Division of Social Sciences. We would like to thank William Brock for helpful suggestions, and an anonymous referee for detailed comments that helped us improve the paper substantially.
Correspondence to: D. Wittman

1 Introduction

Contests may involve a bonus or free vacation for the top salesperson, a patent for the first firm to invent a new product or process, a place in a prestigious graduate school for the best undergraduate students, a defense contract for a successful lobbyist, and so on. The goal of our analysis is two fold. We first provide a quite general characterization of participants' behavior in a contest or tournament where the marginal productivity of effort varies across contestants and individual productivity is private information. We then consider the optimal design of such contests.

Contests are a ubiquitous feature of economic life and, as a consequence, have received considerable attention. They are also closely related to auctions (and we use some methods from auction theory in our analysis). We will point out the relationship of our work to this larger literature in the concluding remarks, after the details of our model have been presented, as this will allow for a clearer comparison. For now, we will note some key differences.

As early as 1979, Holt [12] had developed a fairly general model of auctions that encompassed all-pay auctions as well as contests. His model, unlike ours, assumed independence and he did not develop the results much further than existence. In the 1980's there were a number of papers on contests that used very specific functional forms (for example, O'Keefe, Viscusi and Zeckhauser [22]). Building on Holt [12] and Weber [27], the mid 1990's have seen a number of papers on all-pay-auctions with affiliated values and private information (see for example, Amann and Leininger [2] and Krishna and Morgan [14]). A key difference between their papers and ours is that in their work the cost of bidding is linear, while in our paper contests involve non-linear cost of effort.

None of the above papers deal with contest or auction design, an important part of our paper. There is a large literature on auction design (Myerson [19] being seminal), including a small subset on designing auctions with non-linear utilities (see, for example, Maskin and Riley [16]), and very few papers that deal with the design of contests. Articles on contest design include work by Lazear and Rosen [15], Green and Stokey [10], Nalebuff and Stiglitz [21], and Glazer and Hassin [9]. The first three however assume that individuals have identical abilities or unknown abilities before the contestant exerts effort. The paper by Glazer and Hassin is more similar in spirit to ours, but it assumes a particular cost function, a uniform distribution of abilities and that the highest output always wins. In contrast, our paper considers quite general cost and distribution functions, and (in the latter part) whether the highest output wins is part of the design process.

The structure of our paper is as follows. In Section 2, we describe the setup of the model and the notation and assumptions used. In Section 3, we analyze contestant behavior for the usual type of contest, where the highest output always wins. We demonstrate that there is a unique symmetric equilibrium output function and that output is increasing in ability (Proposition 1). We also show that under some additional conditions marginal effort is increasing in ability (Proposition 2) and that effort decreases when the cost of effort increases (Proposition 3).

We turn to the question of the properties of the optimal contest in Section 4. We no longer assume that the highest output wins. For example, the highest output might be given a lottery ticket. We first characterize the contest designer's problem as a problem in optimal control. We show that output and probability of winning are continuous in ability (Proposition 6), and that it is optimal for the organizer to discourage low ability participants from producing and entering the contest (Proposition 7). Proposition 7 also shows that under the conditions imposed, the output produced and probability of winning are increasing in ability (whenever this probability is between zero and one), paralleling the results of Section 3. Proposition 8 demonstrates that for an open interval of types an optimal contest uses the rule that the highest output wins.

2 Model and notation

There are n contestants, indexed by $i = 1, \ldots n$. θ_i is contestant i's ability, and e_i is contestant i's effort. Ability and effort are continuous variables. Also θ_i and e_i are private information. Because of symmetry, we drop the subscripts when talking about any one contestant. The index of contestant types (abilities) is scaled so that $\theta \in [0, 1]$, while $e \in [0, \infty]$. The production function for output Q is given by $Q(e, \theta)$. Q is observable after all contestants have committed their effort levels. We assume $Q_1 > 0$, $Q_2 > 0$ for $e > 0$ for all θ, e, and $Q(0, \theta) = 0$, where subscripts denote partial derivatives. Hence higher effort and higher ability each increase output. There is no uncertainty that affects output. To facilitate analysis without sacrificing economic plausibility, we will make the following assumptions:

A1. $Q_{12} \geq 0$ – complementarity of inputs; $Q_{11} \leq 0$ – non-increasing returns to effort; and $Q_{22} \leq 0$ – non-increasing returns to ability. At least one of the three inequalities must be strict. Along with $Q_1, Q_2 > 0$, (A1) implies quasiconcavity.

A. Output and effort

A contestant chooses an effort function $e(\theta)$. This is equivalent, given our assumptions, to choosing an output function $q(\theta)$, where

$$q(\theta) \equiv Q(e(\theta), \theta) \ .$$

Note that $e(\theta)$ increasing is sufficient but not necessary for $q(\theta)$ increasing. For example, if $e(\theta)$ is differentiable, $q' = Q_1 e' + Q_2$. So $q' > 0 \Leftrightarrow e' > -Q_2/Q_1$.

The cost of effort for a participant is assumed to be $C(e)$, where $C' > 0, C'', C''' \geq 0, C(0) = 0$. Hence there is a positive, nondecreasing marginal cost of effort.

Since we will work with the output function, it is convenient to establish the following functional relationships implied by the model.

Output $q = Q(e, \theta)$. Since $Q_1 > 0$, this function can be inverted to define $e = R(q, \theta)$, where R is the input requirement function.

Hence $q \equiv Q(R(q, \theta), \theta)$.

Differentiating this last identity, we obtain

$$R_1 = \frac{1}{Q_1} > 0 \ , \quad R_2 = -\frac{Q_2}{Q_1} \leq 0$$

For $e > 0$, the latter inequality is strict.

Hence, with a given ability, effort increases as the required output increases; and with a given output, effort decreases as the contestant's ability increases.

B. The cost function

The function $R(q, \theta)$ allows us to define the cost function as a function of q and θ. Hence,

$$D(q, \theta) \equiv C(R(q, \theta)) \ ,$$

where $D_1 = C'R_1 > 0$ and $D_2 = C'R_2 < 0$ for $e > 0$. Thus $D(q, \theta)$ is the cost for type θ of producing output q. Given A1, $D_{12} < 0$ and $D_{11} \geq 0$. We now describe each of these characteristics in words. Formal derivations are in Appendices 1 and 2.

i. $D_{12} < 0$

The marginal cost of output decreases with ability; that is, $D_{12} < 0$. $D_{12} < 0$ is a single crossing property and a critical relationship in our model. It is therefore nice to know that our assumptions are stronger than necessary to produce a negative cross derivative. As shown in Appendix 2

$$D_{12} = C''R_1R_2 + C'R_{12} = C'' \left(\frac{-Q_2}{Q_1^2} \right) - C' \left[\frac{Q_{12}Q_1 - Q_{11}Q_2}{Q_1^3} \right] < 0$$

Therefore part of A1 ($Q_{11} \leq 0$) can be relaxed while preserving the analysis. Since $Q_1, Q_2 > 0$, all that is required is that

$$\frac{Q_{12}}{Q_2} > \frac{Q_{11}}{Q_1} \ , \quad \text{or equivalently} \quad \frac{\partial \ln Q_2}{\partial \ln e} > \frac{\partial \ln Q_1}{\partial \ln e}$$

i.e., the marginal product of ability is more elastic with respect to effort than is the marginal product of effort.

ii. $D_{11} \geq 0$

$D_{11} \geq 0$ is implied by nonincreasing returns to effort and nondecreasing marginal cost of effort.

C. Distribution of abilities

We now consider the distribution of abilities.

Let $f(\theta_1, \theta_2, ...\theta_n) > 0$ be the joint permutation symmetric density function.

Defining $x_i = \max\{\theta_j\}$ for $j \neq i$, let $F^*(x_i|y)$ be the conditional distribution of x_i given $\theta_i = y$. For example, with independence, $F^*(x_i|y) = F^{n-1}(x_i)$. In the following, using symmetry, we drop the subscript "i" on x.[1] We assume that $F_x^*(x|y) > 0$ for all x and $y \in [0, 1]$. Note that this assumption rules out perfect correlation, which would imply a degenerate conditional distribution.

A2. $\dfrac{F_x^*(x|\theta)}{D_1(q,\theta)}$ is strictly increasing in θ for all x, q.

This is seemingly a strong restriction, but note that if the θ_i are independent, then F_x^* is independent of θ, and A2 is implied by $D_{12} < 0$. Alternatively, since we are assuming differentiability, differentiating the expression in A2 yields

$$\frac{\partial}{\partial\theta}\left[\frac{F_x^*(x|\theta)}{D_1(q,\theta)}\right] = \frac{F_{xy}^* D_1 - F_x^* D_{12}}{D_1^2}$$

Hence A2 is equivalent to $\dfrac{F_{xy}^*}{F_x^*} > \dfrac{D_{12}}{D_1}$

The right hand side of this last inequality is negative, so that the condition allows F_{xy}^* to be negative but not "too negative". Finally, note that A2 applies also if q is functionally related to x, since the condition is assumed for all x, q. If it holds for all x, q, it must hold for $q = q(x)$.

Each contestant i has a strategy space S_i, which we identify below. On the basis of the n-tuple of strategies $(s_1, ..., s_n)$, the contest assigns i a probability of winning denoted by $H_i(s_1, ..., s_n)$, where $\sum_{i=1}^n H_i(s_1, ..., s_n) \leq 1$. Given payments of "$a$" and "$b$" respectively for winning and losing, contestant i's expected utility is given by $H_i(s_1, ..., s_n)U(a) + [1 - H_i(s_1, ..., s_n)]U(b) - D(q_i, \theta_i)$

Because $q_i = Q(e_i, \theta_i)$, as noted earlier, a choice of effort by a contestant is equivalent to a choice of the corresponding output, i.e., $q_i(\theta_i) = Q(e_i(\theta_i), \theta_i)$. Since we can restrict attention to direct revelation mechanisms (see, for example, Dasgupta, Hammond and Maskin [7]), we can treat the strategy space as the space of possible parameters, i.e., [0, 1]. Hence the probability function is given by $H_i(\theta_1, ..., \theta_n)$.

Let $\hat{F}(\theta_{-i}|\theta_i)$ be the conditional distribution of θ_{-i} (the vector of abilities excluding θ_i) given θ_i.

Let $G(x|\theta_i) = \int_{\theta_{-i}} H_i(x, \theta_{-i}) d\hat{F}(\theta_{-i}|\theta_i)$.

Therefore $G(x|\theta_i)$ is the probability that contestant i wins the contest when he has ability θ_i and acts like ('reports') ability x. By symmetry, G requires no subscript. If abilities are independent, then $G(x|\theta_i)$ reduces to $G(x)$. In either

[1] A slight abuse of notation is involved here. Later, we use x for the contestant's "reported" ability in a direct revelation game. However, as we shall see, in a highest output wins contest, the winning report is also the order statistic. The notation permits us to avoid introducing yet another letter in our derivatives.

case we refer to the function G as the reduced form probability of winning. In working with the reduced form, we can drop the subscript "i" on θ without ambiguity.

As an example of the reduced form probability function, consider the case where the highest output wins and where the equilibrium output is increasing for $\theta \geq \theta^0 \in [0, 1]$ and zero otherwise. In this case,

$$G(\theta|\theta) = \begin{cases} 0 & \theta < \theta^0 \\ F^*(\theta|\theta) & \theta \geq \theta^0 \end{cases}$$

Here θ^0 may be determined by a minimum output criterion; this is detailed in Section 3. In general, $G(\theta|\theta)$ may reflect other rules for determining who wins, possibly established by the contest designer – this issue is tackled in Section 4.

D. Award structure and participation constraint

If a contestant wins, he gets an award "a" with associated utility $U(a)$.[2] Otherwise he gets b (which may be negative if there is a participation fee) with associated utility $U(b)$. Using the reduced form, a contestant's expected utility from participating and choosing strategy x is

$$G(x|\theta)U(a) + [1 - G(x|\theta)]U(b) - D(q(x), \theta) .$$

In the case where b is exogenously given, we may scale utility so that $U(b) = 0$, and the expected utility expression simplifies to

$$G(x|\theta)U(a) - D(q(x), \theta)$$

In this case, the participation constraint with truth telling is given by

$$G(\theta|\theta)U(a) - D(q(\theta), \theta) \geq \bar{u} ,$$

where $\bar{u}$ is the expected utility of the participant in the next-best activity.

Depending on $\bar{u}$ relative to $U(a)$ and $U(b)$, and on $G(\theta|\theta)$, it is conceivable that some individuals will formally participate in the contest, but not put in any effort in equilibrium. Intuitively, this could happen if the payment for losing, b, is attractive, or if the alternative, $\bar{u}$, is unattractive.[3]

[2] Note that in this paper, for simplicity, we assume that there is at most one winner.

[3] A final issue to consider is the possibility that contestants may randomize over effort or the consequent output. In this case, the characterization of the equilibrium output function refers to the properties of a deterministic selection from the equilibrium random output function. For simplicity, we shall therefore ignore the possibility of randomization, since no particular insight is lost.

3 Contestant behavior for a standard contest with correlated abilities

In this section, we examine the properties of a standard, highest output wins contest, but with abilities that may be correlated. The latter possibility underlies the novelty of our results.

As noted, we assume that the highest output of those participating gets 'a' and that U is scaled so that $U(b) = 0$. The contest organizer is permitted to qualify the highest-output-wins criterion by specifying a minimum output, q^0, for awarding the prize. For the minimum output criterion to bite, it must be the case that some possible ability level individual is indifferent between participating or not participating at this level (i.e. at the cost necessary to produce q^0). Hence $U(a)G(\theta^0|\theta^0) - D(q^0, \theta^0) = \bar{u}$ for some $\theta^0 \in [0, 1]$

Proposition 1. If A1 and A2 hold, then

(1) Those for whom $\theta < \theta^0$ will not provide any effort;
(2) There is a unique symmetric equilibrium output function $q(\theta)$ for $\theta \geq \theta^0$.[4]
(3) $q(\theta)$ is increasing and differentiable.

Proof. We will begin with the existence of $q(\theta)$, then derive its properties.

(a) Existence of $q(\theta)$

We prove existence by construction. If an equilibrium with $q(\theta)$increasing does exist, then, for each θ, the probability that a contestant with ability θ has of winning must be

$$G(\theta|\theta) = \begin{cases} 0 & \theta < \theta^0 \\ F^*(\theta|\theta) & \theta \geq \theta^0 \end{cases}$$

Furthermore, if $q(\theta)$ is an equilibrium output function, then, for each $\theta \geq \theta^0$, $x = \theta$ maximizes

$$V(x, \theta) \equiv U(a)G(x|\theta) - D(q(x), \theta)$$

This is a consequence of the revelation principle, which says that the contest organizer can do no better than get each ability level to "tell the truth," i.e. to choose the output level consistent with the ability level.

Assuming that $q(\theta)$ is differentiable, and differentiating, we have

$$\frac{\partial V(x, \theta)}{\partial x} = U(a)G_x(x|\theta) - D_1(q(x), \theta) \cdot q'(x)$$

For $\theta \geq \theta^0, G(x|y) = F^*(x|y)$. Suppose that $q'(x) = \dfrac{U(a)^* F_x^*(x|x)}{D_1(q(x), x)}$. Then for $\theta \geq \theta^0$

[4] In two-player contests with independence, there are no asymmetric equilibria (see Appendix 6).

$$\begin{aligned}\frac{\partial V(x,\theta)}{\partial x} &= U(a)F_x^*(x|\theta) - D_1(q(x),\theta)\cdot\frac{U(a)^*F_x^*(x|x)}{D_1(q(x),x)}\\ &= U(a)D_1(q(x),\theta)\left[\frac{F_x^*(x|\theta)}{D_1(q(x),\theta)} - \frac{F_x^*(x|x)}{D_1(q(x),x)}\right]\end{aligned}$$

Recall A2 which implies $\dfrac{F_x^*(x|\theta)}{D_1(q(x),\theta)}$ is strictly increasing in θ.

If $\theta > x$, then $\dfrac{\partial V(x,\theta)}{\partial x} > 0$ and there is an incentive to increase x.

If $\theta < x$, then $\dfrac{\partial V(x,\theta)}{\partial x} < 0$ and there is an incentive to decrease x.

So $x = \theta$ yields a global maximum of $V(x,\theta)$.

Therefore, if $q(\theta)$ is defined by the differential equation and boundary conditions

$$\begin{cases} q'(\theta) = \dfrac{U(a)F_x^*(\theta|\theta)}{D_1(q(\theta),\theta)} \\ q(\theta^0) = q^0 \text{ where } U(a)G(\theta^0|\theta^0) - D(q^0,\theta^0) = \overline{u} \end{cases}$$

we have

$$\frac{\partial}{\partial x}V(x,\theta) \lesseqgtr 0 \quad \text{as} \quad x \gtreqless \theta; x,\theta \in [0,1].$$

Note that the differential equation is obtained from $\dfrac{\partial V}{\partial x}(\theta,\theta) = 0$. By construction, therefore, we have established existence.

(b) $q(\theta)$ is nondecreasing in equilibrium

Suppose that $q(\theta^2) \leq q(\theta^1)$ for $\theta^2 > \theta^1$.

By definition of equilibrium, for a contestant with ability θ^2,

$$U(a)F^*(\theta^2|\theta^2) - D(q^2,\theta^2) \geq U(a)F^*(\theta^1|\theta^2) - D(q^1,\theta^2)$$

and for a contestant with ability θ^1,

$$U(a)F^*(\theta^2|\theta^1) - D(q^2,\theta^1) \leq U(a)F^*(\theta^1|\theta^1) - D(q^1,\theta^1)$$

Equivalently,

$$D(q^2,\theta^2) - D(q^1,\theta^2) \leq U(a)F^*(\theta^2|\theta^2) - U(a)F^*(\theta^1|\theta^2)$$

and

$$U(a)F^*(\theta^2|\theta^1) - U(a)F^*(\theta^1|\theta^1) \leq D(q^2,\theta^1) - D(q^1,\theta^1)$$

Equivalently, we have:

$$\frac{F^*(\theta^2|\theta^2) - F^*(\theta^1|\theta^2)}{D(q^2,\theta^2) - D(q^1,\theta^2)} \leq \frac{1}{U(a)} \quad \text{and}$$
$$\frac{F^*(\theta^2|\theta^1) - F^*(\theta^1|\theta^1)}{D(q^2,\theta^1) - D(q^1,\theta^1)} \geq \frac{1}{U(a)}$$

Note that the denominator is negative in both cases and that when we divided through we reversed the inequalities. Hence

$$\frac{F^*(\theta^2|\theta^2) - F^*(\theta^1|\theta^2)}{D(q^2,\theta^2) - D(q^1,\theta^2)} \leq \frac{F^*(\theta^2|\theta^1) - F^*(\theta^1|\theta^1)}{D(q^2,\theta^1) - D(q^1,\theta^1)}$$

But this violates the fact that $\dfrac{F_x^*(x|\theta)}{D_1(q(x),\theta)}$ is strictly increasing in θ.

(c) $q(\theta)$ is strictly increasing

Suppose $q(\theta) = \hat{q}$ on the interval $[\theta^1, \theta^2]$, with $q(\theta) < \hat{q}$ for $\theta < \theta^1$ and $q(\theta) > \hat{q}$ for $\theta > \theta^2$. Since the highest output wins, and ties are broken by lottery, the contestant with ability θ^2 who produces $q(\theta^2) = \hat{q}$ has a measurable probability of sharing the award a with one or more other contestants. But if this contestant produces $\hat{q} + \epsilon$, for any $\epsilon > 0$ all these ties will be broken. Thus an infinitesimal increase in effort and output leads to a discrete gain in his probability of winning. Consequently, $q(\theta^2)$ cannot be the equilibrium output for θ^2, a contradiction.

Hence, for $\theta \geq \theta^0$, $q(\theta)$ is strictly increasing.

(d) $q(\theta)$ is continuous

For $\theta \geq \theta^0$, $q(\theta)$ is strictly increasing, and so is $G(\theta|\theta)$.

If $q(\theta)$ is not continuous, then there exists $\theta^* \geq \theta^0$ with

$$\limsup_{\theta<\theta^*} q(\theta) < \liminf_{\theta>\theta^*} q(\theta) \qquad \text{(using } q \text{ strictly increasing)}$$

But for $\epsilon > 0$ sufficiently small, an output of $\limsup q(\theta) - \epsilon$ has a probability of winning that is arbitrarily close to that of an output of $\liminf q(\theta) + \epsilon$, given the form of $G(\theta|\theta)$. But this is impossible since $\limsup q(\theta)$ is strictly less than $\liminf q(\theta)$, so no one would produce $\liminf q(\theta) + \epsilon$.

Hence $q(\theta)$ is continuous.

(e) $q(\theta)$ is differentiable for $\theta > \theta^0$

For any $\Delta\theta$,

$$U(a)F^*(\theta|\theta) - D(q(\theta),\theta) \geq U(a)F^*(\theta + \Delta\theta|\theta) - D(q(\theta + \Delta\theta),\theta)$$

and

$$U(a)F^*(\theta+\Delta\theta|\theta+\Delta\theta) - D(q(\theta+\Delta\theta),\theta+\Delta\theta)$$
$$\geq U(a)F^*(\theta|\theta+\Delta\theta) - D(q(\theta),\theta+\Delta\theta)\ ,$$

from truth telling or incentive compatibility.

Hence, invoking the mean value theorem, we obtain

$$U(a)[F^*(\theta|\theta) - F^*(\theta+\Delta\theta|\theta)] \geq -D_1(q^*,\theta)[q(\theta+\Delta\theta) - q(\theta)]$$

and

$$U(a)[F^*(\theta+\Delta\theta|\theta+\Delta\theta) - F^*(\theta|\theta+\Delta\theta)] \geq -D_1(q^{**},\theta+\Delta\theta)[q(\theta) - q(\theta+\Delta\theta)]$$

where both q^* and q^{**} are between $q(\theta)$ and $q(\theta+\Delta\theta)$.

Hence,

$$\frac{U(a)[F^*(\theta+\Delta\theta|\theta+\Delta\theta) - F^*(\theta|\theta+\Delta\theta)]}{\Delta\theta[D_1(q^{**},\theta+\Delta\theta)]}$$
$$\geq \frac{q(\theta+\Delta\theta)-q(\theta)}{\Delta\theta}$$
$$\geq \frac{U(a)[F^*(\theta+\Delta\theta|\theta) - F^*(\theta|\theta)]}{\Delta\theta[D_1(q^*,\theta)]}$$

Since $q(\theta)$ is continuous, the left and right-most terms of this double inequality tend to

$$\frac{U(a)F^*_x(\theta|\theta)}{D_1(q(\theta),\theta)} \quad \text{as} \quad \Delta\theta \to 0\ .$$

Hence $q(\theta)$ is differentiable for $\theta > \theta^0$.

(f) $q(\theta)$ is unique

Since $q(\theta)$ is strictly increasing and differentiable, it satisfies the first order condition for $\max_x\ V(x,\theta)$, for $\theta > \theta^0$. This is from (i) of the proof.

Suppose that $\bar{q}(\theta)$ also does so, but $\bar{q}(\theta^0) < q^0$. Then for $\alpha > 0$ sufficiently small, $\bar{q}(\theta^0+\alpha) < q^0$. Thus a contestant with ability $\theta^0+\alpha$ receives $\bar{u}$. But if he produces q^0, his expected payoff is

$$u(a)F^*(\theta^0|\theta^0) - D(q^0,\theta^0+\alpha) > \bar{u}$$

since $D_2 < 0$. So $\bar{q}(\theta) < q^0$ cannot be an equilibrium. Similarly, $\bar{q}(\theta^0) > q^0$ is ruled out. Hence $\bar{q}(\theta^0) = q^0$.

Thus $\bar{q}(\theta)$ and $q(\theta)$ coincide for all θ. Since the slope of q is given by the differential equation, this establishes uniqueness. Q.E.D.

Remark 1. If the contest designer sets q^0 so low that even the least able individuals ($\theta = 0$) would strictly prefer to produce that output, there is still a unique equilibrium $q(\theta)$ with boundary condition $U(a)G(0|0) - D(q(0),0) = \bar{u}$.

Remark 2. Independence of types is a special case of the above. With independence, assumption A2 can be dropped. The crucial condition is then $D_{12} < 0$. The meaning of this condition was discussed in Section 2.B. The case of independence is also considered in Appendix 6.

We next consider the marginal effect of ability, θ, on marginal output, q'. In the following proposition we show that under quite general conditions $q'(\theta)$ increases as θ increases. That is, not only does output increase with ability, but also output increases at an increasing rate. The proposition assumes that $F^*_{xx}(\theta|\theta) + F^*_{xy}(\theta|\theta) \geq 0$.

It is worthwhile discussing this assumption before we proceed with the proof. At first sight it might appear that there is little reason for the sign of this second derivative to be one way or the other, but we must remember that θ appears twice in $F^*(\theta|\theta)$ and the net effect is likely to be strongly positive if there is only weak positive correlation between the θ_i. The intuition can be best understood by considering a situation where the θ_i are independent and identically distributed with c.d.f. $F(\theta)$ and $F'(\theta) > 0$. Then $F^*(\theta|\theta) = F^{n-1}(\theta)$, $F^*_x(\theta|\theta) = F^{n-2}(\theta)F'(\theta)$, and $F^*_{xx}(\theta|\theta) = F^{n-2}(\theta)F''(\theta) + F^{n-3}(\theta)[F']^2$. (Since the θ are independent, $F^*_{xy} = 0$.) The last expression before the parenthetical statement is positive if $\dfrac{F''}{F'} \geq -\dfrac{1}{F}$. For F close to zero, this allows F'' to be very negative. As F approaches 1, F'' can still be negative as long as its absolute value is not greater than F' (equivalently, the elasticity of F' is greater than -1).

Proposition 2. Assume A1, A2, and that $F^*_{xx}(\theta|\theta) + F^*_{xy}(\theta|\theta) \geq 0$ and $D_{11} = 0$, then $q'(\theta)$ increases with θ for $\theta \geq \theta^0$.

Proof. Taking the derivative with respect to θ of the differential equation, $q'(\theta) = \dfrac{U(a)F^*_x(\theta|\theta)}{D_1(q(\theta),\theta)}$, yields

$$\begin{aligned} q''(\theta) &= \frac{U(a)\left[F^*_{xx}(\theta|\theta) + F^*_{xy}(\theta|\theta)\right]}{D_1} - \frac{U(a)F^*_x(\theta|\theta)}{[D_1]^2}[D_{11}q'(\theta) + D_{12}] \\ &= \frac{U(a)[F^*_{xx} + F^*_{xy}]}{D_1} - \frac{U(a)F^*_x}{[D_1]^2}\left[D_{11}\frac{U(a)F^*_x}{D_1} + D_{12}\right] \end{aligned}$$

By assumption, $F^*_{xx} + F^*_{xy} \geq 0$, and $F^*_x > 0$. Recall that $D_1 > 0$ and that $D_{12} < 0$. Therefore when $D_{11} = 0$, the whole expression is positive. Q.E.D.

D_{11} is zero if the marginal cost of output is constant. It is clear that Proposition 2 (output increases at an increasing rate) will continue to hold if D_{11} is positive as long as D_{11} is “relatively” small and the expression in the rightmost brackets is non-positive (or not too positive). Insight into the proof is generated by multiplying the q' equation by D_1. Then $q'(\theta)D_1(q(\theta),\theta) = U(a)F^*_x(\theta|\theta)$ or the marginal cost of increased output due to an increase in ability is equal to the marginal benefit of increased effort due to an increase in ability. Given our assumptions, an increase in ability increases the marginal probability of winning, and, at the same time, decreases the marginal disutility of output. Therefore the marginal increase in output, $q'(\theta)$, must increase.

We next consider the effect of an exogenous increase in cost on every contestant's output function. To do that we temporarily redefine C and D, to include a shift parameter t. Let the cost of effort be $C(e,t)$ where $C(0,t) = 0$, $C_1 > 0$, and $C_{11} \geq 0$. Thus the cost function has the same properties as before, only now there is an explicit shift parameter. $D(q,\theta,t) \equiv C(R(q,\theta,t))$. As before $D_1 > 0$, D_2, $D_{12} < 0$, etc.

Therefore the differential equation and boundary condition are respectively:

$$q_\theta(\theta,t) = \frac{U(a)F_x^*(\theta|\theta)}{D_1(q(\theta),\theta,t)}$$

$q(\theta^0,t) = q^0$, where $U(a)G(\theta^0|\theta^0) - D(q^0,\theta^0,t) = \bar{u}$.

Proposition 3. Given A1 and A2, if $D_{1t} \geq 0$ (that is, an increase in t increases the marginal cost of output) and $D_t > 0$ (an increase in t increases the total cost of the output), then an increase in t will result in (1) a larger θ needed to induce more than zero effort, and (2) a lower effort for all θ beyond that point. That is, the effort function will move to the right.

Proof. We first find the effect of a change in t on the differential equation.

$$q_{\theta t}(\theta,t) = -\frac{U(a)F_x^*(\theta|\theta)}{[D_1(q(\theta),\theta,t)]^2}[D_{11}q_t + D_{1t}]$$

(a) Suppose that $q_t > 0$ at some θ. Then by the above equation, $q_{\theta t} < 0$ at that θ since all the other terms on the right side are positive and the expression is preceded by a negative sign.

(b) But at θ^0, $q_t < 0$. This can demonstrated by looking at the total differential of the boundary condition. Taking the total differential of the boundary condition when there is an exogenous change in t, we get:

$$[U(a)G' - D_2]d\theta^0 = D_t dt.$$

Since G', $-D_2$, and D_t are greater than 0, then $d\theta^0$ is greater than 0 and q^0 shifts to the right. Thus $q_t(\theta^0,t) < 0$, which proves (1).

(c) We have supposed that $q_t > 0$ at some θ. By the continuity of $q_t(\theta,t)$, there exists some θ at which $q_t = 0$.

(d) At this point, $q_{\theta t} \leq 0$, and just to the right of that point $q_{\theta t} < 0$. This contradicts (a).

Hence $q_t < 0$ everywhere, proves (2). Q.E.D.

Remark 3. We can also note possible effects on the slope of the effort function, q_θ, as the cost parameter increases. If D_{1t} is small enough, then $q_{\theta t} \leq 0$; if D_{1t} is large enough, then $q_{\theta t} \geq 0$ (see Figure 1).

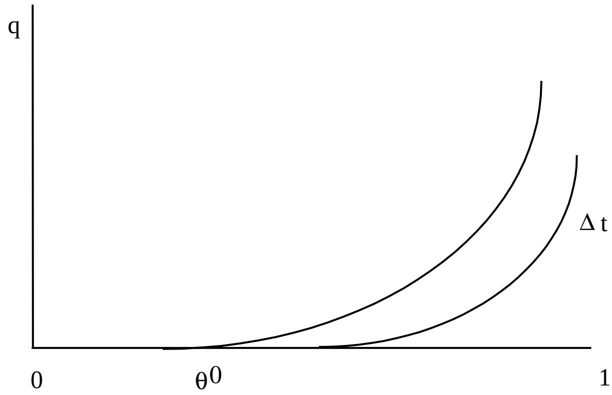

Figure 1

4. Optimal contests with independent abilities

In this section, we allow the awards, a and b, as well as the contest rules that govern the probability of winning function $G(x|\theta)$ to be determined by the contest organizer prior to the contest.

Unlike Section 3, the highest output need not win. For example, the person with the highest output might be eligible for a random drawing of the prize, rather than getting the prize for certain, and the probability of winning the random drawing may vary with output.

Because the highest output does not necessarily win even if the participation constraint is satisfied, our assumptions regarding F^* are no longer sufficient to derive the results of Section 3. In fact, we can no longer use our general distribution, F^*, and are forced to assume independence of the θ_i. In turn, independence means that $G(x|\theta)$ can be written as $G(x)$.

The θ_i are assumed to be *iid* with distribution $F(\theta)$. We assume that $f(\theta) \equiv F'(\theta) > 0$ for all $\theta \in [0, 1]$. Note that we use F instead of F^* to distinguish it from our earlier more general assumption regarding the distribution of θ.

We also make the following assumptions regarding the production of output:

A3. $Q_{111} \leq 0$; $\quad Q_{121} \geq 0$; $\quad$ and $\quad Q_{122} \leq 0$.

These assumptions imply that $D_{112} \leq 0$ (that is, the effect of ability on marginal cost of output is not diminished at higher output levels), and $D_{122} \geq 0$. See Appendices 1 and 2 for the derivations. Recall that $D_{12} < 0$, that is, the marginal cost of output decreases with ability. Then $D_{122} \geq 0$ states that this effect is subject to nonincreasing returns.

Finally, we make the following assumption to replace A2:

A2′. $\dfrac{F''}{F'} \geq \dfrac{D_{122}}{D_{12}}$,

where the right hand side is nonpositive, since $D_{12} < 0$ and $D_{122} \geq 0$. This condition provides a lower bound on the rate at which the probability density of the distribution of abilities can change.

A. Contestant behavior under these more general contest rules

We need to establish certain results under these more general contest rules.

Let $V(x,\theta) \equiv U(a)G(x) + U(b)[1 - G(x)] - D(q(x),\theta)$

As truth-telling constitutes an equilibrium, the maximized expected utility is

$$V(\theta,\theta) = \max_x V(x,\theta)$$

Proposition 4. The maximized expected utility of a contestant is (a) increasing and (b) continuous in ability. That is, $V(x,x)$ is (a) increasing and (b) continuous.

Proposition 5. If $D_{12} < 0$, then $q' \geq 0$ implies that the first order condition of the contestant is sufficient for truth-telling in equilibrium.

The results in Propositions 4 and 5 are fairly standard, and the proofs are relegated to Appendix 3.

We are now ready to consider the contest designer's problem. The contest designer's objective is to maximize expected revenue subject to participation and incentive constraints.[5] We consider the case where the designer has control over a, b, q^0, and $G(\theta)$.

B. The control problem

The contest designer's expected revenue from each contestant, assuming that output is the numeraire, is given by

$$\int \left[q(\theta) - aG(\theta) - b\left\{1 - G(\theta)\right\}\right] dF(\theta) \tag{1}$$

The contest designer chooses $G(\theta)$, $q(\theta)$, a and b (we assume $a > b$ to avoid an indeterminancy, where "winning" and "losing" are interchanged) to maximize (1) subject to the contestants' incentive constraints,

$$V(\theta,\theta) = \max_x V(x,\theta)\ , \tag{2}$$

the nonparticipation option,[6]

$$V(\theta,\theta) \geq 0\ , \tag{3}$$

and the constraint that G must be derived via

$$G(\theta_i) = \int_{\theta_{-i}} H_i(\theta_i,\theta_{-i}) \prod_{j \neq i} dF(\theta_j) \tag{4}$$

[5] The contest designer may have other objectives (e.g., a close race). See Singh and Wittman [24] for a further analysis.

[6] Since $U(b)$ is no longer scaled to zero, we simplify notation by now scaling $\bar{u}$ to be zero.

from the symmetric probability functions $H_1, \ldots, H_n$ satisfying $\sum_{i=1}^{n} H_i$ $(s_1, \ldots, s_n) \leq 1$.

Obviously, because G is itself the probability of winning, it must satisfy

$$0 \leq G \leq 1\ . \tag{5}$$

Following Maskin and Riley [16] or Matthews [17] (Theorem 7 in M & R), we require the following condition:

$$\int_y^1 G(s)dF(s) \leq \int_y^1 F^{n-1}(s)dF(s), 0 \leq y \leq 1 \tag{6}$$

Because of the non-linearity of the effort function and the consequent risk aversion of the contestant, this condition is necessary for our reduced form approach. It allows us to reduce the joint distribution to a marginal distribution.[7] If $G(s)$ is nondecreasing, (6) is sufficient for the H_j's to exist such that (4) holds.

The control problem for the contest designer is set up as follows. Choosing $G(\theta)$, $q(\theta)$, a and b also involves choosing

$$V(\theta, \theta) = U(a)G(\theta) + U(b)[1 - G(\theta)] - D(q(\theta), \theta)\ . \tag{7}$$

The objective is to maximize (1) subject to (2), (3), (5), (6) and (7). We will replace (2) by the first order condition.

$$\frac{d}{d\theta} V(\theta, \theta) = -D_2(q(\theta), \theta) \tag{8}$$

This is valid if $q(\theta)$ is nondecreasing, and that will be established at the optimum.

To convert (6) to standard form, define

$$Y = \int_\theta^1 [G(x) - F^{n-1}(x)]dF(x)$$

Then (6) becomes

$$Y \leq 0\ , \tag{9}$$

where

$$\frac{dY}{d\theta} = (F^{n-1}(\theta) - G)F'(\theta) \tag{10}$$

So the initial control problem is to maximize (1) subject to (3), (5), (7), (8), (9) and (10).

The goal is to show that there is a continuous solution to this problem, and to obtain a condition ensuring that G and q are everywhere nondecreasing.

The Hamiltonian for the control problem is

[7] The Matthews result is for probability measures in general; so it applies to contests as well as auctions.

$$
\begin{aligned}
L &= [q - aG - b(1-G)]F' - \lambda D_2 + \mu(F^{n-1} - G)F' \\
&\quad + \nu[U(a)G + U(b)(1-G) - D(q,\theta) - V] \\
&\quad + \alpha G + \beta(1-G) - \gamma Y + \phi V
\end{aligned}
$$

where λ and μ are the costate variables for (8) and (10), ν is the Lagrange multiplier for (7) and $\alpha, \beta, \gamma, \phi$ are the Lagrange multipliers for the inequality constraints (5), (9) and (3).

The first order (necessary) conditions from the maximum principle are

$$\frac{\partial L}{\partial G} = (-a + b)F' - \mu F' + \nu[U(a) - U(b)] + \alpha - \beta = 0 \tag{11}$$

$$\frac{\partial L}{\partial q} = F' - \lambda D_{12} - \nu D_1 = 0 \tag{12}$$

$$\int [-GF' + \nu U'(a)G]d\theta = 0 \tag{13}$$

$$\int [-(1-G)F' + \nu U'(b)(1-G)]d\theta = 0 \tag{14}$$

Equations (13) and (14) are obtained by differentiating the integral of the Hamiltonian with respect to a and b, since a and b are constants independent of θ.

C. Continuity

In this section we will demonstrate that q and G are continuous. Before doing so we will make use of the following lemma:

Lemma 1. (a) $\mu \geq 0$; (b) $\lambda \leq 0$ for $0 < G$.

Proof. From the maximum principle, the costate variables, λ and μ are continuous and piecewise differentiable in θ with $\mu(0), \lambda(1) = 0$.[8]

By (10)

$$\mu' = -\frac{\partial L}{\partial Y} = \gamma \geq 0\ . \tag{15}$$

Therefore $\mu \geq 0$.

From the maximum principle, λ satisfies

$$(\lambda)' = -\frac{\partial L}{\partial V} = \nu - \phi\ . \tag{16}$$

[8] The continuity of μ and λ follow indirectly since the state variable is bounded. The argument is due to Matthews [17].

From (8), V is increasing whenever $D_2 < 0$, which is true whenever $q > 0$, since $D_2 = -\frac{C'Q_2}{Q_1}$, and $Q_2 = 0$ only if $q = 0$, since then $e = 0$.

If anything is to be produced at all, since V is continuous (part b of Proposition 4), it follows that there exists θ^0, $0 \leq \theta^0 < 1$ such that

$$V = 0 \quad \text{only if} \quad \theta \leq \theta^0 \tag{17}$$

Hence, for $\theta > \theta^0, V > 0$ and $\phi = 0$ by complementary slackness.

Thus, for $\theta > \theta^0$, using (16) and (17),

$$(\lambda)' = \nu \quad \text{for} \quad \theta > \theta^0 \,.$$

We will now demonstrate that ν is positive and therefore λ' is positive. We have already demonstrated that $\mu \geq 0$. For $0 < G$, α is zero and (11) reduces to

$$(b - a - \mu)F' + [U(a) - U(b)]\nu - \beta = 0 \tag{18}$$

This in turn implies that

$$\nu = \frac{(a - b + \mu)F' + \beta}{U(a) - U(b)}$$

By assumption $a > b$ and $F' > 0$; and it has been demonstrated that μ is nonnegative. Also $\beta \geq 0$. Hence, $\nu > 0$, $\lambda' > 0$, and $\lambda \leq 0$. Q.E.D

Proposition 6. Given the above control problem, $q(\theta)$ and $G(\theta)$ are continuous.

Proof. From (7) and since V is continuous, if q is continuous, G must also be continuous. Therefore, suppose q is not continuous. In particular, let there be an upward discontinuity from the right at some θ, say θ^*. Let $q^* \equiv \lim_{\theta \downarrow \theta^*} q(\theta) > q(\theta^*)$.

From (11), (12), (16) and (17), if $0 < G < 1$, then

$$0 > \frac{\lambda(\theta^*)}{F'(\theta^*)} = \frac{[U(a) - U(b)] - [a - b + \mu(\theta^*)]D_1(q(\theta^*), \theta^*)}{[U(a) - U(b)]D_{12}(q(\theta^*), \theta^*)} \tag{19}$$

Since λ and F' are continuous, we can replace $q(\theta^*)$ by q^* in (19) without affecting the equality.

$D_{11} > 0$ for $e > 0$; therefore $D_1(q^*, \theta^*) > D_1(q(\theta^*), \theta^*) > 0$.

$D_{112} \leq 0$; therefore $D_{12}(q^*, \theta^*) \leq D_{12}(q(\theta^*), \theta^*) < 0$.

These inequalities imply

$$\frac{[U(a) - U(b)] - [a - b + \mu(\theta^*)]D_1(q(\theta^*), \theta^*)}{[U(a) - U(b)]D_{12}(q(\theta^*), \theta^*)} < \frac{[U(a) - U(b)] - [a - b + \mu(\theta^*)]D_1(q^*, \theta^*)}{[U(a) - U(b)]D_{12}(q^*, \theta^*)} < 0$$

This contradicts the equality argument after (19). Thus q cannot have an upward discontinuity from the right. A similar argument applies for an upward discontinuity from the left. By similar arguments, it cannot have a downward discontinuity. Hence q and G are continuous if $0 < G < 1$.

It remains to be shown that there cannot be discontinuities such that G is 0 or 1 on one side of the point of discontinuity. Suppose as before that q has an upward discontinuity from the right at θ^* such that $G(\theta^*) > 0$, but $G^* \equiv \lim_{\theta \downarrow \theta^*} G(\theta) = 1$. Also, let $\beta^* \equiv \lim_{\theta \downarrow \theta^*} \beta(\theta)$. A problem will arise only if $\beta^* > 0$. Thus assume that this is the case.

Following previous steps for the case $0 < G < 1$, we have, by continuity of λ/F',

$$\frac{[U(a) - U(b)] - [a - b + \mu(\theta^*)]D_1(q(\theta^*), \theta^*) - \beta(\theta^*)D_1/F'}{[U(a) - U(b)]D_{12}(q(\theta^*), \theta^*)} = \frac{[U(a) - U(b)] - [a - b + \mu(\theta^*)]D_1(q^*, \theta^*) - \beta^* D_1/F'}{[U(a) - U(b)]D_{12}(q^*, \theta^*)} \leq 0 \quad (20)$$

Note that the inequality is weak to cover the case $\lambda(1) = 0$. Using $\beta(\theta^*) = 0$ and the previous inequality, this implies

$$-\frac{\beta^* D_1(q^*, \theta^*)/F'(\theta^*)}{[U(a) - U(b)]D_{12}(q^*, \theta^*)} \leq 0 \quad (21)$$

But this contradicts $\beta^* > 0$. Hence the postulated discontinuity cannot exist.

Now suppose that q has an upward discontinuity from the right at θ^* such that $G(\theta^*) = 0$, but $G^* \equiv \lim_{\theta \downarrow \theta^*} G(\theta)$ such that $0 < G^* < 1$. Also, let $\alpha^* \equiv \lim_{\theta \downarrow \theta^*} \alpha(\theta)$. Then $\alpha^* = 0$ but $\alpha(\theta^*) \geq 0$.

A discontinuity arises only if $\alpha(\theta^*) > 0$. Hence assume that this is the case.

Following previous steps we obtain

$$-\frac{\alpha(\theta^*)D_1(q(\theta^*), \theta^*)/F'(\theta^*)}{[U(a) - U(b)]D_{12}(q^*(\theta^*), \theta^*)} \leq 0$$

which again gives a contradiction.

Discontinuities from the left and downward discontinuities may be ruled out similarly. Thus we establish that q and G are continuous everywhere. Q.E.D.

D. Monotonicity

Proposition 7. If $0 < G < 1$ in equilibrium, sufficient conditions for the equilibrium output and probability of winning to be increasing in ability (given the basic model assumptions) are:

(i) $D_{11} \geq 0$, which is implied by non-increasing returns to effort;
(ii) $D_{112} \leq 0$, i.e., the effect of ability on marginal cost of output is non-decreasing in output;

(iii) $\frac{F''}{F'} \geq \frac{D_{122}}{D_{12}}$, i.e., the probability density of ability cannot diminish too rapidly as ability increases ($A2'$).

Proof.

Step 1: V is differentiable almost everywhere (demonstrated in proof of Proposition 5). Differentiating (7), $V' = u(a)G' - u(b)G' - D_1 q' - D_2$.

Using (8), we have $[u(a) - u(b)]G' = D_1 q'$

$D_1 > 0$ and $a > b$, so that $u(a) - u(b) > 0$; it follows that $q' > 0 \Leftrightarrow G' > 0$.

If (9) is binding on an interval $[\theta^1, \theta^2]$, then $G = F^{n-1}$ on the interval. Hence, $G' = (n-1)F^{n-2}F' > 0$ on such an interval. Therefore $q' > 0$.

This establishes the proposition for ranges over which (9) is binding.

Step 2: From the proof of Proposition 6, $\lambda(\theta) < 0$, $\lambda' > 0$ are established whenever $0 < G < 1$.

Step 3: Whenever (9) is not binding, $\gamma = 0$. Hence $\mu' = \gamma = 0$, in this case, and $\mu(\theta)$ is a constant, which we will denote by $\bar{\mu}$.

From (12) and (18), with (9) not binding and $0 < G < 1$, we have

$$(b - a - \bar{\mu})F' + [U(a) - U(b)]\frac{(F' - \lambda D_{12})}{D_1} = 0 \tag{22}$$

where $\lambda < 0$, $\lambda' > 0$. Equivalently,

$$\frac{(a-b) + \bar{\mu}}{U(a) - U(b)} = \frac{1}{D_1} - \frac{\lambda D_{12}}{F' D_1} > 0 \tag{23}$$

Differentiating (23) with respect to θ gives

$$\begin{aligned} &-\frac{1}{(D_1)^2}(D_{11}q' + D_{12}) - \frac{\lambda' D_{12}}{F' D_1} + \frac{\lambda D_{12} F''}{(F')^2 D_1} \\ &-\frac{\lambda}{F' D_1}(D_{112}q' + D_{122}) + \frac{\lambda D_{12}}{F'(D_1^2)}(D_{11}q' + D_{12}) = 0 \end{aligned}$$

Collecting terms,

$$\begin{aligned} &-q'\left[\frac{D_{11}}{(D_1)^2}\left(1 - \frac{\lambda D_{12}}{F'}\right) + \frac{\lambda D_{112}}{F' D_1}\right] \\ &= \frac{D_{12}}{(D_1)^2} + \frac{\lambda' D_{12}}{F' D_1} - \frac{\lambda}{F'}\left[\left(\frac{D_{12}}{D_1}\right)^2 - \frac{D_{122}}{D_1} + \frac{D_{12}F''}{D_1 F'}\right] \end{aligned} \tag{24}$$

Also, from (12), since $\lambda' = \nu$ under these conditions,

$$\begin{aligned} F' - \lambda D_{12} - \lambda' D_1 = 0 \\ \text{or} \quad \lambda' = (F' - \lambda D_{12})/D_1 \end{aligned}$$

Substituting this in (24),

$$-q'\left[\frac{D_{11}}{(D_1)^2}\left(1-\frac{\lambda D_{12}}{F'}\right)+\frac{\lambda D_{112}}{F'D_1}\right]$$
$$=\frac{D_{12}}{(D_1)^2}+\frac{D_{12}}{(D_1)^2}-\frac{\lambda(D_{12})^2}{F'(D_1)^2}-\frac{\lambda}{F'}\left[\left(\frac{D_{12}}{D_1}\right)^2-\frac{D_{122}}{D_1}+\frac{D_{12}F''}{D_1F'}\right]$$
$$=\frac{2D_{12}}{(D_1)^2}\left[1-\frac{\lambda D_{12}}{F'}\right]+\frac{\lambda}{F'D_1}\left[D_{122}-\frac{D_{12}F''}{F'}\right] \tag{25}$$

Now $1-\lambda D_{12}/F' > 0$ by (21). Also $D_{11} \geq 0$; $D_{112} \leq 0$; $D_1 > 0$, and $\lambda' > 0$. Therefore, the term multiplying $-q'$ on the left hand side of (25) is positive.

On the right hand side of (25) the expression in the first brackets is positive. Since $D_{12} < 0$, the first term is negative. Since $\lambda < 0$, the second term is nonpositive if

$$D_{122}-\frac{D_{12}F''}{F'} \geq 0 \quad \text{or} \quad \frac{F''}{F'} \geq \frac{D_{122}}{D_{12}} \quad \text{which is true by assumption.} \tag{26}$$

Therefore $q' > 0$. Q.E.D.

The basic intuition behind Proposition 7 is that if contestants do not get a big positive jolt from increased ability, equilibrium effort and probability of winning will increase when contestant ability increases. Suppose to the contrary. Then increased ability might drastically decrease the probability that someone is above the contestant so the contestant might decrease effort; or if there were increasing returns to effort the person could slack off if ability increased because the person would again be so far ahead. Once the higher ability contestant works harder it makes sense for the contest designer to award her more by increasing the probability of highest output winning.

E. The values of G

In this section we establish when the equilibrium probability of winning is zero or one, and whether (9) must be binding over some range of abilities. Propositions 8A and 8B show that the probability of winning can be zero only for an interval of the lowest abilities and that it can only be one at the highest ability. These characteristics are similar to the highest-output-wins case. Proposition 8C shows that the optimal contest probability of winning is identical to that in the highest-output-wins case for an open interval of abilities. These results therefore compare the optimal probability to that in the restricted highest-output-wins case.

Proposition 8A. $G = 0$ if and only if θ is in the interval $[0, \theta^0]$ where $1 > \theta^0 > 0$.

Proof. See Appendix 4.

Proposition 8B. $G = 1$ is only possible at $\theta = 1$.

Proof. See Appendix 4.

Proposition 8C. (9) must be binding on some open interval of θ. That is, the highest output does win with certainty on an open interval.

Proof. From (11)

$$\nu = \frac{a - b + \mu}{U(a) - U(b)} F' + \frac{\beta - \alpha}{U(a) - U(b)} \tag{27}$$

Suppose that (9) is never binding. Then $\gamma = 0$ everywhere and $\mu(\theta) = 0$ everywhere. From Propositions 8a and 8b, $0 < G < 1$ for $1 > \theta > \theta^0$. Therefore, β, $\alpha = 0$. As a consequence, for $0 < G < 1$,

$$\nu = \frac{a - b}{U(a) - U(b)} F'.$$

Substituting this into (13), we obtain

$$\int_{\theta^0}^{1} \left[-1 + U'(a) \frac{a - b}{U(a) - U(b)} \right] GF' d\theta = 0 \, .$$

By strict concavity of U, with $a > b$, we have $U'(a) < \dfrac{U(a) - U(b)}{a - b}$.

Thus the expression in brackets is negative, and the above inequality cannot hold, a contradiction. Q.E.D.

F. Contestant utility and contest organizer profit

We next turn our attention toward the cost of effort to the contestant and to the profit of the contest designer. We show that even the most able contestant does not receive first-best incentives, while those less able are also subject to the usual informational inefficiency. Finally, we confirm the plausible result that the contest designer's expected revenue is increasing in contestant ability.

Proposition 9.

(a) For $\theta = 1$, $D_1(q(\theta), \theta) = \dfrac{U(a) - U(b)}{a - b + \mu}$; that is, the marginal cost of producing output is equal to the adjusted marginal utility per dollar of the net award for winning.

(b) For $\theta < 1$, if $0 < G < 1$, then $D_1(q(\theta), \theta) < \dfrac{U(a) - U(b)}{a - b + \mu}$; that is, the marginal cost of producing output is less than the adjusted marginal utility per dollar of the net award for winning.

(c) The contest designer's expected revenue is increasing in θ.

Proof. See Appendix 5.

Note that the adjustment to the marginal utility of winning reflects the additional constraint that the contest designer faces in our formulation. This constraint is binding over some interval, by Proposition 8C, so that $\mu(1) > 0$, and the most able person also does not receive first-best incentives. The nature of the adjustment to marginal utility is that of a further cost to the contest designer, since the denominator can be thought of as $(a + \mu) - b$. For all contestants below the most able, there is the usual informational inefficiency, since a fortiori, $D_1 < [U(a) - U(b)]/(a - b)$, i.e., effort is underexpended, since the marginal cost of output is less than the marginal benefit of producing it.

For the contest designer, greater ability increases profits even though the higher ability person has a higher expected award. This is because the increase in the expected award is less than the increase in productivity – some of the increased productivity does not have to be compensated for since it is due to higher ability rather than greater effort.

6. Concluding remarks

We have provided a quite general characterization of contests where effort and ability are private information and the marginal cost of effort is increasing. This not only characterizes those situations where the person who has made the most sales gets a free vacation to Hawaii, but other situations as well. For example, in patent races firms expend resources on research in order to be first. In trials, both sides hire lawyers and expert witnesses in order to win. Defense contractors may contribute to campaigns hoping that their firm will receive a government contract and students may work harder as undergraduates in order to get into prestigious graduate schools.

In this paper, we focus on those situations where the marginal productivity of effort may be affected by fluctuations in individual abilities. A particular individual knows his own ability, or learns it before making his effort decision, but does not know the ability of his rivals. The person who decides the award for winning the contest (the contest designer or organizer) does not observe contestants' abilities. Both the participants and the contest organizer are assumed to have consistent prior distributions on ability.

We demonstrate the existence of a contestants' equilibrium in the above model, and show that equilibrium output will be increasing in ability. We then show the solution to the contest designer's problem.

In our opening remarks we mentioned some differences between our paper and previous work. Now that our model has been presented, it is useful to outline some more detailed differences.

The results of Section 3 are more general than Holt [12] because we do not assume independent types. Krishna and Morgan [14] and Amann and Leininger [2] replace independence with affiliation, all within the context of an all-pay auction. In their model, therefore, the cost is just the bid itself, i.e., linear. Translating this into the terminology of our model, the bid is equivalent to our q so that in their model the marginal cost of effort (expenditure on the bid) is constant and identical for all contestants. With special functional forms, however, our models can be made isomorphic.[9]

A major difference is that they assume that the highest output always wins, so they do not consider contest design.

There is also a very large literature on all-pay auctions with complete information. See for example, Baye, Kovenock and de Vries [4,5] and the citations, therein. Complete information creates a mixed strategy equilibrium and is a very different exercise from modeling a contest with private information. Dixit [8] considers agents with different abilities, but again this is with complete information.

Skaperdas and Gan [25] consider symmetric "contest success functions" where the contestant's abilities are the same and probability of winning is solely a function of effort. The authors postulate several plausible contestant success functions but do not derive them from first principles. They concentrate on the behavior of the contestants rather than the issue of contest design. Dasgupta and Nti [6] consider the contest designer's choice of a contest success function. Taylor [26] considers research tournaments in the context of multiperiod decision-making. In each period, firms decide whether to invest in research, a 0-1 decision. The difference between his and our model is highlighted if we restrict our attention to a single period tournament. Then, unlike our model, the research productivity of each firm is the same and all firms would choose to undertake the same amount of investment. So his model has a different focus – it is a model of search.

Much of the work that investigates the contest designer's problem has assumed that abilities are identical or unknown before the contestant exerts effort. Lazear and Rosen [15] assume that individuals provide effort before they know their abilities (they also assume that a Nash equilibrium exists and that the probability of each contestant winning is one half); Green and Stokey [10] assume that everyone has the same ability, but that different contestants receive different signals about the common ability factor; and Nalebuff and Stiglitz [21] assume that the contestants observe the common ability factor, but commit to effort before the particular ability factor is known. They exploit the fact that the Nash equilibrium will be symmetric and that the probability of each contestant winning will be 1/2 (this easy way out is not possible in the contests that we explore, where participants observe their private ability signal before undertaking action).

[9] They assume affiliated values which implies certain relationships between the distribution of F^* and its density (e.g., the hazard rate is non-decreasing in θ). These conditions are considerably different from our assumptions. In addition, Krishna and Morgan utilize a condition on the product of the conditional expected utility and the hazard rate. This is different from, but roughly analogous to our assumption A2. Overlap occurs when $D = q(x)/s(\theta)$. Then one can multiply through by $s(\theta)$ and transform the contest into an all pay auction. Under such circumstances, assumption A2 is simplified as q is no longer part of the equation and $D_{12}/D_1 = -s'(\theta)/s(\theta)$. If the θ are also independent, then the all pay auction and a contest are similar; however, the conditions for the optimal contest design still differ.

Other researchers have modeled contests where abilities are private information, but they employ very specific models. O'Keefe, Viscusi and Zeckhauser [22] assume that $Q(e,\theta) = \theta e$, the θ are independent and F either a uniform distribution or a triangular distribution, $D = e^2$, and that $U(a)$ is linear in a. They consider the contestant's equilibrium under these restrictive assumptions but not the contest designer's problem. Singh and Wittman [23] consider a two-player contest with two possible actions and two ability levels but where the abilities may be correlated. Glazer and Hassin [9] partially bridge the two groups as they deal with part of the designer's problem. They consider the case where $D = q/\theta$ and F is uniform (they also assume that $a + b$ is given, that everyone participates and that the highest output always wins). In contrast to these papers, our paper provides a considerably more general and complete characterization of the contest designer's problem when information about abilities is private.

We have discussed past research; we conclude with two possible avenues of future research. One involves changing the awards to the contestants. Award 'a' could go to the m players with the m highest outputs. And there could be more than two award levels. The award could be a share of the total output of the other $n - 1$ players, rather than a fixed amount. If participants were risk neutral, this should not change the mathematical structure very much. The other alterations also appear to make only minor changes in the model.

The other possibility is to consider different objective functions of the contest designer. In this paper we analyzed the case where output value is the sum of the players' outputs. In a two-player game, the contest designer maximizes $\Pi = E[q^1+q^2]-a-b$ subject to the participation and other constraints (where q^i is the output of participant i, and E is the expected value operator). Alternatively, value may be based only on the output of the weakest player. This is a maximin, Leontief or weakest-link production function. It characterizes some sport contests (where quality of the weaker player and closeness counts). The contest designer then maximizes $S = E[\min\{q^1, q^2\}] - a - b$. Another possible production function is that only the winner's output matters to the contest designer. This is a maximax utility function and may characterize track and field records, patent races, and best shot production functions. Under these conditions, the contest designer maximizes $R = E[\max\{q^1, q^2\}] - a - b$. It will be interesting to explore how these alternative objectives change the optimal contest design.

Appendix 1: signing the derivatives of *R*

The derivatives of R are obtained by differentiating the identity $q \equiv Q(R(q,\theta),\theta)$.

1. $R_{12} < 0$

We first take the derivative of the identity with respect to q.

$$1 = Q_1(R(q,\theta),\theta)R_1(q,\theta)$$

We then take the derivative of this expression with respect to θ.

$$0 = Q_{11}R_2R_1 + Q_{12}R_1 + Q_1R_{12}$$

Equivalently,

$$-Q_{12}R_1 - Q_{11}R_2R_1 = Q_1R_{12}$$
$$-\frac{Q_{12}}{Q_1} + \frac{Q_{11}Q_2}{Q_1^2} = Q_1R_{12}$$
$$R_{12} = \frac{-Q_{12}Q_1 + Q_{11}Q_2}{Q_1^3} < 0 \quad \text{for} \quad Q_{12} > 0 \quad \text{and} \quad Q_{11} \leq 0$$

2. $R_{11} \geq 0$

Again taking the first derivative of the identity with respect to q, we get:

$$1 = Q_1(R(q,\theta),\theta)R_1(q,\theta)$$

Taking the derivative of this expression with respect to q yields:

$$0 = Q_{11}R_1^2 + Q_1R_{11}$$

Equivalently, $R_{11} = -\dfrac{Q_{11}R_1^2}{Q_1} \geq 0$ for $Q_{11} \leq 0$.

3. $R_{22} \geq 0$

The derivatives of R are obtained by differentiating the identity $q \equiv Q(R(q,\theta),\theta)$ with respect to θ.

$$\begin{aligned}
0 &= Q_1R_2 + Q_2 \\
0 &= Q_{11}R_2^2 + Q_1R_{22} + 2Q_{21}R_2 + Q_{22} \\
-R_{22} &= \frac{Q_{11}R_2^2}{Q_1} + \frac{2Q_{21}R_2}{Q_1} + \frac{Q_{22}}{Q_1} \\
R_{22} &= -\left[\frac{Q_{22}Q_1^2 - 2Q_{12}Q_1Q_2 + Q_{11}Q_2^2}{(Q_1)^3}\right] \geq 0 \quad \text{for} \quad Q_{11}, Q_{22} \leq 0 \quad \text{and} \\
Q_{12} &> 0\text{ . If } e > 0\text{ , then } Q_2 > 0\text{ , and, consequently, } R_{22} > 0\text{ .}
\end{aligned}$$

4. $R_{121} \leq 0$

Previously, in deriving R_{12}, we established that:

$$0 = Q_{11}R_2R_1 + Q_{12}R_1 + Q_1R_{12}$$

Taking the derivative with respect to q we get:

$$0 = Q_{111}R_2R_1^2 + 2Q_{11}R_{21}R_1 + Q_{11}R_2R_{11} + Q_{121}R_1^2 + Q_{12}R_{11} + Q_1R_{121} \ .$$

Equivalently,

$$\begin{aligned} R_{121} &= -Q_{111}R_2R_1^2/Q_1 - 2Q_{11}R_{21}R_1/Q_1 - Q_{11}R_2R_{11}/Q_1 \\ &\quad -Q_{121}R_1^2/Q_1 - Q_{12}R_{11}/Q_1 \leq 0 \ . \end{aligned}$$

Because $Q_1, Q_{12}, R_1 > 0; R_{11} \geq 0$; and $R_2, Q_{11} \leq 0$; and $R_{12} < 0$, the 2nd, 3rd, and 5th terms are all less than or equal to 0. If $Q_{111} \leq 0$ and $Q_{121} \geq 0$, as we have assumed, then $R_{121} \leq 0$.

5. $R_{122} \geq 0$

We start with the second derivative of the identity $q \equiv Q(R(q,\theta),\theta)$ obtained in deriving R_{12}:

$$0 = Q_{11}R_2R_1 + Q_{12}R_1 + Q_1R_{12}$$

Taking the derivative of this expression with respect to θ we get:

$$\begin{aligned} 0 &= Q_{111}R_2^2R_1 + Q_{112}R_2R_1 + Q_{11}R_{22}R_1 + Q_{11}R_2R_{12} + Q_{121}R_1R_2 \\ &\quad +Q_{122}R_1 + R_{12}Q_{12} + Q_{11}R_2R_{12} + Q_{12}R_{12} + Q_1R_{122} \end{aligned}$$

Equivalently,

$$\begin{aligned} R_{122} &= -\frac{Q_{111}R_2^2R_1}{Q_1} - \frac{Q_{112}R_2R_1}{Q_1} - \frac{Q_{11}R_{22}R_1}{Q_1} - \frac{Q_{11}R_2R_{12}}{Q_1} \\ &\quad -\frac{Q_{121}R_1R_2}{Q_1} - \frac{Q_{122}R_1}{Q_1} - \frac{R_{12}Q_{12}}{Q_1} - \frac{Q_{11}R_2R_{12}}{Q_1} - \frac{Q_{12}R_{12}}{Q_1} \geq 0 \ . \end{aligned}$$

$R_{122} \geq 0$ for $Q_1, Q_{12} > 0$; $Q_{121} \geq 0$; and $Q_{11}, Q_{122}, Q_{111} \leq 0$, as we have assumed, and $R_2 \leq 0$; $R_{12} < 0$, $R_1 > 0$ as we have shown.

Appendix 2: signing the derivatives of *D*

In this appendix we sign the derivatives of D. The proofs rely on the signs of the derivatives of R which were derived in Appendix 1. Thus the derivatives of D are based on primitive assumptions regarding Q.

i. $D_{12} < 0$

In order to understand why this derivative is negative, it is useful to express D in terms of the cost and production functions. Taking the derivative of $D_1 = C'R_1$ with respect to θ, we have

$$D_{12} = C''R_1R_2 + C'R_{12} = C''\left(\frac{-Q_2}{Q_1^2}\right) - C'\left[\frac{Q_{12}Q_1 - Q_{11}Q_2}{Q_1^3}\right] < 0$$

since $R_1, C' > 0$; $R_{12} < 0$; $R_2 \leq 0$; and $C'' \geq 0$.

ii. $D_{22} \geq 0$

The cost of a given output decreases at a non-increasing rate as ability increases; that is, $D_{22} \geq 0$. To see this, differentiate D to obtain

$$\begin{aligned} D_2 &= C'R_2 = -\frac{C'Q_2}{Q_1} \\ D_{22} &= C''(R_2)^2 + C'R_{22} \\ &= C''\left(\frac{Q_2}{Q_1}\right)^2 - C'\left[\frac{Q_{22}Q_1^2 - 2Q_{12}Q_1Q_2 + Q_{11}Q_2^2}{(Q_1)^3}\right] \geq 0 \end{aligned}$$

since $C'' \geq 0$; $C' > 0$; and $R_{22} \geq 0$. Note that $D_{22} > 0$ for $e > 0$.

iii. $D_{11} \geq 0$

$D_{11} \geq 0$ is implied by diminishing returns to equilibrium effort. This is immediately apparent if we take a look at the underlying cost function. Taking the derivative of $D_1 = C'R_1$ with respect to q yields:

$$D_{11} = C'R_{11} + C''R_1^2 = -\frac{C'Q_{11}}{Q_1^3} + \frac{C''}{Q_1^2} \geq 0$$

since $C' > 0$; $C'' \geq 0$ and $R_{11} \geq 0$.

iv. $D_{112} \leq 0$

When we discuss the contest designer's problem, we will need $D_{112} \leq 0$. That is, the effect of ability on marginal cost of output is not diminished at higher output levels. This is an appealing characteristic. Taking the derivative of D_{12} with respect to q, we get

$$D_{112} = C''R_2R_{11} + 2C''R_1R_{12} + C'R_{112} + C'''R_2R_1^2 \leq 0$$

since $R_1 > 0$; $R_{11} \geq 0$; $R_{12} < 0$; $R_2, R_{121} \leq 0$; $C' > 0$; and $C'', C''' \geq 0$.

v. $D_{122} \geq 0$

Taking the derivative of $D_{12} = C''R_1R_2 + C'R_{12}$ with respect to θ, yields:

$$D_{122} = C'''R_1R_2^2 + 2C''R_2R_{12} + C''R_1R_{22} + C'R_{221} \geq 0$$

since $R_1, C' > 0$; C'', C''', $R_{122} \geq 0$; $R_2 \leq 0$; and $R_{12} < 0$.

Recall that $D_{12} < 0$, which states that the marginal cost of output decreases with ability. Then $D_{122} \geq 0$ states that this effect is subject to nonincreasing returns.

Appendix 3: contestant behavior in the generalized contest

In this appendix, we establish some properties of the expected utility of a contestant who has made his optimal effort/output choice.

Proposition 4. The maximized expected utility of a contestant is (a) increasing and (b) continuous in ability.

Proof.
(a) $V(x,x)$ is increasing
For this general case, let

$$V(x,\theta) \equiv U(a)G(x) + U(b)[1 - G(x)] - D(q(x),\theta)$$

As truth-telling constitutes an equilibrium, the maximized expected utility is

$$\begin{aligned} V(\theta,\theta) &= \max_x V(x,\theta) \\ V(x,\theta) - V(x,x) &= -[D(q(x),\theta) - D(q(x),x)] \end{aligned}$$

Since $D_2 < 0$, if $\theta > x$, then $D(q(x),\theta - D(q(x),x) < 0$.
Hence $V(x,\theta) > V(x,x)$ for $\theta > x$
Also, $V(\theta,\theta) \geq V(x,\theta)$ since $V(\theta,\theta) = \max_x V(x,\theta)$
Combining these two inequalities yields:
$V(\theta,\theta) > V(x,x)$ for $\theta > x$. That is, $V(x,x)$ is increasing.

(b) $V(x,x)$ is continuous

$$\begin{aligned} \text{Also, } V(\theta,\theta) - V(\theta,x) &= [D(q(\theta),x) - D(q(\theta),\theta)] \\ &= -\int_x^\theta D_2(q(\theta),\alpha)d\alpha \\ &\leq -D_2(q(\theta),x)(\theta - x) \end{aligned}$$

for $D_2 \leq 0$ and $D_{22} \geq 0$. Since $V(x,x) \geq V(\theta,x)$ and $V(x,x)$ is increasing,

$$0 \leq V(\theta,\theta) - V(x,x) \leq -D_2(q(\theta),x)(\theta - x)$$

Hence $V(x,x)$ is continuous. Q.E.D.

Proposition 5. If $D_{12} < 0$, then $q' \geq 0$ implies that the first order condition of the contestant is sufficient for truth-telling in equilibrium.

Proof. Truth-telling implies that, for all x and $\theta, \theta \in \arg\min_x [V(x,x) - V(\theta,x)]$.

By definition, $V(\theta,x)$ is a differentiable function of x. Also, since $V(x,x)$ is continuous and increasing, it is differentiable almost everywhere.

Hence, almost everywhere, the first order condition for truth-telling is

$$\frac{dV}{dx}(x,x) - V_2(\theta,x) = 0 \quad \text{at} \quad x = \theta \quad \text{or} \quad \frac{d}{d\theta}V(\theta,\theta) = -D_2(q(\theta),\theta)$$

(since $V_2(\theta,x) = -D_2(q(\theta),x)$)

To establish sufficiency as well as necessity, we use the continuity of V and integrate to obtain

$$V(\theta_2,\theta_2) - V(\theta_1,\theta_1) = \int_{\theta_1}^{\theta_2} -D_2(q(\theta),\theta)d\theta$$

We can establish that $-D_2(q(x),\theta)$ is nondecreasing in x, as follows:

We have $\frac{\partial}{\partial x}[-D_2(q(x),\theta)] = -D_{12}q'(x)$

Hence if $D_{12} < 0$ and $q' \geq 0$, $-D_2(q(x),\theta)$ is nondecreasing in x. Therefore,

$$V(\theta_2,\theta_2) - V(\theta_1,\theta_1) \geq \int_{\theta_1}^{\theta_2} -D_2(q(\theta_1),\theta)d\theta = V(\theta_1,\theta_2) - V(\theta_1,\theta_1)$$

Hence $V(\theta_2,\theta_2) \geq V(\theta_1,\theta_2)$

Therefore, if $D_{12} < 0$, then $q' \geq 0$ implies that the first order condition is sufficient as well as necessary for truth-telling. Q.E.D.

Remark. Unlike the auction problem, the above properties of the contestant's expected utility do not directly require G to be nondecreasing. However, for G to derive from a family of probability functions $H_i(s)$, where $s$is the strategy vector, Gnondecreasing is used, so there is no relaxation. Of course, in the highest-output-wins contest, G is nondecreasing because of its particular form.

Appendix 4: Characteristics of *G*

In this Appendix we establish when the equilibrium probability of winning is zero or one.

Proposition 8A. $G = 0$ if and only if θ is in the interval $[0,\theta^0]$ where $1 > \theta^0 > 0$.

Proof. Since G is increasing, it follows that $G = 0$ only on an interval of the form $[0,\theta^0]$.

To establish that $\theta^0 > 0$, we proceed as follows:

From (11)

$$\nu = \frac{a-b+\mu}{U(a)-U(b)}F' + \frac{\beta-\alpha}{U(a)-U(b)} \tag{A4.1}$$

Substituting (A4.1) into (14) we obtain

$$\int_0^1 \left[\frac{a-b+\mu}{U(a)-U(b)}u'(b) - 1\right](1-G)F'd\theta + \int_0^1 \frac{\beta-\alpha}{U(a)-U(b)}u'(b)(1-G)d\theta = 0 \tag{A4.2}$$

Since u is strictly concave and $a > b$,

$$u'(b) > \frac{u(a) - u(b)}{a - b + \mu} \tag{A4.3}$$

Hence, $\frac{a - b + \mu}{u(a) - u(b)} u'(b) - 1 > 0$.

Hence, the first term in (A4.2) is positive, and the second term must be negative. This can only hold if $\beta - \alpha < 0$ over some range, which requires $\alpha > 0$ over some range, i.e. $G = 0$ for some θ.

Clearly $\theta^0 < 1$ if there is to be any contest at all, since otherwise nothing is produced. Q.E.D.

Proposition 8B. $G = 1$ is only possible at $\theta = 1$.

Proof. We have already established that G is nondecreasing. Therefore, $G = 1$ is only possible on an interval of the form $[\theta^1, 1]$. However if $\theta^1 < 1$, then G would violate (9) on such an interval.

Appendix 5: contestant utility and organizer profit

Proposition 9.

(a) For $\theta = 1$, $D_1(q(\theta), \theta) = \frac{U(a) - U(b)}{a - b + \mu}$; that is, the marginal cost of producing output is equal to the adjusted marginal utility per dollar of the net award for winning.

(b) For $\theta < 1$, if $0 < G < 1$, then $D_1(q(\theta), \theta) < \frac{U(a) - U(b)}{a - b + \mu}$; that is, the marginal cost of producing output is less than the adjusted marginal utility per dollar of the net award for winning.

(c) The contest designer's expected revenue is increasing in θ.

Proof of (a). By the maximum principle $\lambda(1) = 0$. Therefore by (12) $\nu = \frac{F'}{D_1}$ at $\theta = 1$.

Substituting this into (11) and rearranging, we get

$$\left(-a + b - \mu + \frac{[U(a) - U(b)]}{D_1}\right) F' = \beta - \alpha \tag{A5.1}$$

If $0 < G < 1$ at $\theta = 1$, then $\beta = \alpha = 0$ and $D_1(q(\theta), \theta) = \frac{U(a) - U(b)}{a - b + \mu}$.

If $G(1) = 1$, $\beta(1)$ equals 0 by the continuity arguments in Proposition 6, since $\beta = 0$, for $G < 1$ and $G(\theta) < 1$ for θ arbitrarily close to 1 by Proposition 8B.

Proof of (b). Making use of the right hand side equality in (19) we get

$$\begin{aligned}\frac{\lambda(\theta)}{F'(\theta)} &= \frac{[U(a)-U(b)]-[a-b+\mu]D_1(q(\theta),\theta)}{[U(a)-U(b)]D_{12}(q(\theta),\theta)} \\ &= \frac{1-\frac{a-b+\mu}{U(a)-U(b)}D_1(q(\theta),\theta)}{D_{12}(q(\theta),\theta)}\end{aligned} \tag{A5.2}$$

$D_{12} < 0$ and $\lambda < 0$. Therefore for $\theta^0 < \theta < 1$,

$$1-\frac{a-b+\mu}{U(a)-U(b)}D_1(q(\theta),\theta) > 0\,. \quad \text{Equivalently,}$$

$$D_1(q(\theta),\theta) < \frac{U(a)-U(b)}{a-b+\mu}\,. \tag{A5.3}$$

Proof of (c). For a given θ, the expected profit of the contest designer from any individual equals 0 for $\theta < \theta^0$. Since the individual expends no effort, there is no production and there is no award. For $\theta \geq \theta^0$, expected profit equals

$$q(\theta) - aG(\theta) - b[1-G(\theta)]. \tag{A5.4}$$

The derivative of (A5.2) with respect to θ is

$$q'(\theta) - (a-b)G'(\theta). \tag{A5.5}$$

But by step 1, $G' = \dfrac{D_1 q'}{U(a)-U(b)}$. Therefore (A5.5) equals

$$q'\left[1-\frac{a-b}{U(a)-U(b)}D_1\right] \tag{A5.6}$$

We have already shown that the expression in brackets is greater than 0 and that $q' > 0$. Therefore, profits are increasing in θ. Q.E.D.

Appendix 6: proof that the equilibrium is unique

In this appendix we demonstrate that there are no asymmetric equilibria in a two-person contest with independent types.[10] Because this issue is not central to our investigation, we make some simplifying assumptions to speed the analysis along.

Proposition 10. If there are two contestants, $U(b) = 0$, and θ is *iid* with distribution, F, then there is a unique equilibrium.

Proof. We will denote contestant 1 and 2's abilities by superscripts 1 and 2, respectively. Output of contestant i is thus $q^i(\theta^i) = Q(e^i(\theta^i), \theta^i)$. We have already

[10] Nalebuff and Riley [20] demonstrated that there were a continuum of equilibria in a two-person war of attrition when private information was *iid*; Amann and Leininger [1] demonstrated that there were no asymmetric equilibria in two-person all-pay auctions where private information was *iid*.

shown in Section 2 of the main text that Q_1, Q_2, and $e^{i\prime} > 0$. Therefore $q^{i\prime}(\theta^i) = Q_1 e^{i\prime} + Q_2 > 0$ and we can invert q.

Let the inverse function of q^i be $y^i(q^i) = \theta^i$.
Thus for example $y^2\left(q^1\left(\theta^1\right)\right)$ is that θ^2 needed to produce output q^1 when contestant 2's effort function is e^2.

$$\begin{aligned}\text{Player 1 maximizes } W^1 &= F\left(y^2\left(q^1(\theta^1)\right)\right)U(A) - D\left(q^1(\theta^1), \theta^1\right)\\ \text{Player 2 maximizes } W^2 &= F\left(y^1\left(q^2(\theta^2)\right)\right)U(A) - D\left(q^2(\theta^2), \theta^2\right)\end{aligned}$$

The first order conditions are:

$$\begin{aligned}W^1_{q1} &= U(A)F'\left(y^2\left(q^1(\theta^1)\right)\right)y^{2\prime}\left(q^1(\theta^1)\right) - D_1\left(q^1(\theta^1), y^1\left(q^1(\theta^1)\right)\right) = 0\\ W^2_{q2} &= U(A)F'\left(y^1\left(q^2(\theta^2)\right)\right)y^{1\prime}\left(q^2(\theta^2)\right) - D_1\left(q^2(\theta^2), y^2\left(q^2(\theta^2)\right)\right) = 0\end{aligned}$$

Let $K(\theta^1) = y^2\left(q^1(\theta^1)\right)$.
Then $K'(\theta^1) = y^{2\prime}\left(q^1(\theta^1)\right)q^{1\prime}(\theta^1)$.

We arrange the first of the first order conditions as follows:

$$y^{2\prime}\left(q^1(\theta^1)\right) = \frac{D_1\left(q^1(\theta^1), y^1(q^1(\theta^1))\right)}{U(A)F'\left(y^2\left(q^1(\theta^1)\right)\right)} = \frac{D_1\left(q^1(\theta^1), \theta^1\right)}{U(A)F'\left(K(\theta^1)\right)}$$

Equivalently,

$$K'(\theta^1) = \frac{D_1\left(q^1(\theta^1), \theta^1\right)q^{1\prime}(\theta^1)}{U(A)F'\left(K(\theta^1)\right)} \tag{A6.1}$$

Next we manipulate the second of the first order conditions. Our ultimate interest is in finding how the function varies as θ^1 varies. Therefore, instead of dealing with θ^2 as an exogenous variable, we treat it as a function of θ^1. Let $q^2 = q^1(\theta^1)$. Then $\theta^2 = y^2\left(q^1(\theta^1)\right)$. That is, we are tracking that θ^2 needed to achieve the same output, $q^1(\theta^1)$.

$$W^2_{q2} = U(A)F'\left(y^1(q^2)\right)y^{1\prime}(q^2) - D_1(q^2, \theta^2) = 0$$

Substituting $q^1(\theta^1)$ for q^2 and $y^2\left(q^1(\theta^1)\right)$ for θ^2, we get:

$$\begin{aligned}W^2_{q2} &= U(A)F'\left(y^1(q^1(\theta^1)\right)y^{1\prime}\left(q^1(\theta^1)\right) - D_1\left(q^1(\theta^1), y^2\left(q^1(\theta^1)\right)\right)\\ &= U(A)F'(\theta^1)y^{1\prime}\left(q^1(\theta^1)\right) - D_1\left(q^1(\theta^1), K(\theta^1)\right) = 0\ .\end{aligned}$$

Equivalently, $\dfrac{1}{y^{1\prime}\left(q^1(\theta^1)\right)} = \dfrac{U(A)F'(\theta^1)}{D_1\left(q^1(\theta^1), K(\theta^1)\right)}$
Recall that the property of inverse functions implies the following relationship:

$$q^{1\prime}(\theta^1) = \frac{1}{y^{1\prime}\left(q^1(\theta^1)\right)}\ .$$

Hence,

$$q^{1\prime}(\theta^1) = \frac{U(A)F'(\theta^1)}{D_1\left(q^1(\theta^1), K(\theta^1)\right)} \tag{A6.2}$$

Substituting into (A6.1), we get

$$K'(\theta^1) = \frac{D_1\left(q^1(\theta^1), \theta^1\right) F'(\theta^1)}{D_1\left(q^1(\theta^1), K(\theta^1)\right) F'\left(K(\theta^1)\right)} \tag{A6.3}$$

(A6.2) and (A6.3) constitute a pair of non-linear differential equations.

We next show that there is a fixed terminal point at $\theta = 1$. $y^{2\prime} > 0$. In equilibrium $e^1(1) = e^2(1)$. Otherwise the person with strictly greater effort would be wasting effort to no avail. Therefore, $K(1) = y^2(q^1(1)) = 1$ in equilibrium and as a consequence $K'(1) = 1$ in equilibrium.

We need to show that there is a unique solution to (A2) and (A3). Given our assumptions on F and D_1, it is clear that these equations are continuous and satisfy a Lipschitz condition in some neighborhood of the fixed terminal point. This implies that at most one solution exists (see Hurewicz, Theorem 3, p. 28). We have already shown that there is a symmetric equilibrium. Thus there are no asymmetric equilibria. Q.E.D.

References

1. Amann, E., Leininger, W.: Asymmetric all-pay auctions with incomplete information. Games and Economic Behavior **14**, 1–18 (1996)
2. Amann, E., Leininger, W.: Expected revenue of all-pay and first price sealed bid auctions with affiliated signals. Journal of Economics **61**, 273–279 (1995)
3. Anderson, S., Goeree, J., Holt, C.: Rent dissipation in an all-pay auction with bounded rationality. University of Virginia Working Paper (1997)
4. Baye, M., Kovenik, D., De Vries, C.: Rigging the lobbying process: an application of the all-pay auction. American Economic Review **83**, 289–294 (1993)
5. Baye, M., Kovenik, D., De Vries, C.: The all-pay auction with complete information. Economic Theory **8**, 291–305 (1996)
6. Dasgupta, A., Nti, K.: Designing an optimal contest. European Journal of Political Economy **14**, 687–603 (1998)
7. Dasgupta, P., Hammond, P., Maskin, E.: The implementation social choice rules: some general results on incentive compatability. Review of Economic Studies **46**, 185–216 (1979)
8. Dixit, A.: Strategic behavior in contests. American Economic Review **89**, 891–898 (1987)
9. Glazer, A., Hassin, R.: Optimal contests. Economic Inquiry **26**, 133–144 (1988)
10. Green, J., Stokey, N.: A comparison of tournaments and contracts. Journal of Political Economy **91**, 349–364 (1983)
11. Holmström, B.: Moral hazard in teams. Bell Journal of Economics **3**, 329–340 (1982)
12. Holt, C.: A theory of signalling auctions: University of Minnesota Working Paper 79–110 (1979)
13. Hurewicz, W.: Lectures on ordinary differential equations. Cambridge: MIT Press 1958
14. Krishna, V., Morgan J.: An analysis of the war of attrition and the all-pay auction. Journal of Economic Theory **72**, 343–362 (1997)
15. Lazear, E., Rosen, S.: Rank order tournaments as optimal labor contracts. Journal of Political Economy **89**, 841–864 (1981)
16. Maskin, E., Riley, J.: Optimal auctions with risk averse buyers. Econometrica **52**, 1473–1518 (1984)
17. Matthews, S.: On the implementability of reduced form auctions. Econometrica **52**, 1519–1522 (1984)

18. Mookherjee, D.,: Optimal incentive schemes with many agents. Review of Economic Studies **51**, 433–446 (1984)
19. Myerson, R.B.: Optimal auction design. Mathematics of Operations Research **6**, 58–73 (1981)
20. Nalebuff, B., Riley, J.: Asymmetric equilibria in the war of attrition. Journal of Theoretical Biology **113**, 517–527 (1983)
21. Nalebuff, B., Stiglitz, J.: Prizes and incentives. Bell Journal of Economics **14**, 21–43 (1983)
22. O'Keefe, M. Kip Viscusi, W., Zeckhauser, R.: Economic contests: comparative reward schemes. Journal of Labor Economics **2**, 27–56 (1984)
23. Singh, N., Wittman, D.: Economic contests with incomplete information: Management Science **34**, 528–540 (1988)
24. Singh, N., Wittman, D.: Contest design and the objective of the contest designer: sales promotions, sporting events, and patent races. In: Baye, M. (ed.) Advances in applied microeconomics: contests. Stamford CT: JAI Press 1998
25. Skaperdas, S., Gan, L.: Risk aversion in contests. Economic Journal **105**, 951–962 (1995)
26. Taylor, C.: Digging for golden carrots: an analysis of research tournaments: American Economic Review **85**, 872–890 (1995)
27. Weber, R.: Auctions and competitive bidding. Proceedings of Symposia in Applied Mathematics **14**, 143–170 (1985)