Icosahedral Galois Representations and Modular Forms

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Contents

Abstract

In this paper, we establish some numerical evidence for Artin's conjecture in the icosahedral case. Specifically, we consider an extension of the rationals with Galois group A_5 and compute the Artin L-series associated to a natural three-dimensional representation. While Brauer's theorem guarantees the meromorphic continuation and functional equation of this L-series, we numerically check the analytic continuation. By carrying this out for some twists of this L-function as well, we provide evidence for the modularity of the three-dimensional representation via the Weil converse theorem.

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Chapter 1

Introduction

Before giving evidence for Artin's conjecture, we give some background information on Artin L-functions, and a precise statement of their conjectured functional equation. For this introduction, we follow closely the treatment in the survey article by Martinet [7]. Then we briefly discuss modular forms for GL_2, and state a conjecture which may be seen as a GL_2 case of Langland's program.

1.1 Artin L-functions

In 1923, Emil Artin gave his first definition of a new kind of L-series (c.f. [4]), associated to finite-dimensional Galois representations. Namely, given a finite Galois extension of number fields E/K with Galois group G, and a representation $\rho\colon G \to GL(V)$ of G on a finite-dimensional complex vector space V, he defined the L-function, when $Re(s) > 1$:

$$L(s,\rho)_{unr} = \prod_{\wp \ unramified} \frac{1}{det\,(1-\rho(\mathrm{Fr}_\wp)N(\wp)^{-s})}. \tag{1.1}$$

Here $\mathrm{Fr}_\wp$ denotes a Frobenius element associated to $\wp$ of K; note that since all such Frobenii are conjugate, the L-function above is well-defined. Motivated by the proof by Hecke of the functional equation for abelian L-functions in 1917, Artin conjectured the existence of a functional equation relating $L(s,\rho)$ and $L(s,\overline{\rho})$, where ρ denotes the complex conjugate of ρ. Moreover he conjectured that the above L-function extends to an entire function in the complex plane as long as ρ does not contain the trivial representation.

Later, in 1930, Artin defined local factors at the ramified primes, and at infinity, for a number of reasons. First, he wanted operations on Galois representations to correspond nicely to operations on the L-series. Namely, the L-function of the direct sum of two representations should equal the product of the two L-functions (this is satisfied by Artin's first definition of L-series however). Also, lifting a representation from a quotient, and inducing a representation from a subgroup should not change the L-function. Finally, Artin wanted to define the L-function at these additional places in order to get a simple functional equation.

With this in mind, we give the local factors at the ramified primes as follows: at a ramified prime $\wp$ of K, we fix a prime P of E lying above $\wp$, and let D_P and I_P denote the decomposition group and the inertia group of P. Then the quotient D_P/I_P is isomorphic to the Galois group of the residue field extension, so we may define a Frobenius element $\mathrm{Fr}_\wp$ as an element of D_P/I_P. Now, let V^{I_P} denote the subspace of V fixed by inertia. Then we define the local L-factor at $\wp$ by:

$$L(s,\rho)_\wp = \frac{1}{det_{V^{I_P}}\,(1 - N(\wp)^{-s}\,\mathrm{Fr}_\wp)}. \tag{1.2}$$

Now, we define the total Artin L-function $L(s,\rho)$ to be the product of all the (finite) local factors that we have defined. It is defined initially in the right half plane $Re(s) > 1$, and satisfies the induction and lifting properties we mentioned before. For the purposes of discussing the functional equation, it remains to discuss the local factors at the infinite primes – composed of a series of Γ-factors and a constant which measures the ramification of ρ.

First, for the Γ-factors, let $\gamma(s) = \pi^{-s/2}\Gamma(s/2)$, and for each infinite place v of K we define local factors $\gamma_\rho{}^v(s)$ as follows: if v is complex, we define $\gamma_\rho{}^v(s) = (\gamma(s)\gamma(s+1))^{tr(\rho(1))}$. If v is real, then for each place w of E lying above v, there exists a decomposition group D_w of order 1 or 2. The generator is the analogue of the Frobenius element at the infinite prime v, so we call it Fr_v; it is well defined up to conjugacy class. Now every eigenvalue of $\rho(\mathrm{Fr}_v)$ is either 1 or -1 so let d_+ and d_- be the multiplicities of each of these eigenvalues. Then we put $\gamma_\rho{}^v(s) = \gamma(s)^{d_+}\gamma(s+1)^{d_-}$. Finally, we define the whole Γ-factor for ρ by

$$\gamma_\rho(s) = \prod_{v\ archimedean} \gamma_\rho{}^v(s). \tag{1.3}$$

Finally, we define the constant $A(\rho)$ which measures the ramification of the Galois representaion ρ. For each (finite) prime $\wp$ of K, choose a prime P

of E lying above $\wp$. Let G_i denote the sequence of ramification groups at $\wp$ (where G_0 is the inertia group) and let g_i be the order of G_i. Let V^{G_i} denote the subspace of V fixed by $\rho(G_i)$. Then we define the local conductor:

$$f(\rho, \wp) = \sum_{i=0}^{\infty} \frac{g_i}{g_0}(\dim(V) - \dim(V^{G_i})).$$

Then $f(\rho, \wp) = 0$ at almost every prime $\wp$ of K, and is in fact an integer, so we may define the Artin conductor by:

$$f(\rho) = \prod_{\wp} \wp^{f(\rho,\wp)}.$$

Then the Artin conductor is an ideal of K, and allows us to define the constant

$$A(\rho) = |d_K|^{tr(\rho(1))} N_{K/\mathbf{Q}}(f(\rho)). \tag{1.4}$$

Here d_K denotes the absolute discriminant of K.

We are now ready to define the enlarged L-function for $Re(s) > 1$:

$$\Lambda(s, \rho) = A(\rho)^{s/2} \gamma_\rho(s) L(s, \rho). \tag{1.5}$$

The following is an unproven conjecture, stated first by Artin (c.f. [4]):

Conjecture 1 *(Artin's Conjecture) If ρ does not contain the unit character, then the function $\Lambda(s, \rho)$ defined above for $Re(s) > 1$ extends to an analytic function of the complex plane, and satisfies the functional equation $\Lambda(1 - s, \rho) = W(\rho)\Lambda(s, \overline{\rho})$ for a complex constant $W(\rho)$ of absolute value 1.*

It was proven in 1947 by Brauer [1] that Λ has *meromorphic* continuation and satisfies the above functional equation. More specifically, Brauer shows that the character of any representation of a finite group G may be expressed as a linear combination (with integer coefficients) of induced characters from 1-dimensional representations of subgroups of G. This allows us to express any Artin L-function as a product of integer powers of abelian L-functions in the sense of Hecke, and implies that Λ extends to a meromorphic function on the complex plane.

If ρ is a two-dimensional Galois representation, then we can break down Artin's conjecture as follows: composing ρ with the natural projection $GL_2(\mathbf{C}) \to PGL_2(\mathbf{C})$ yields a projective representation ρ'. The image of ρ' is a finite subgroup of $PGL_2(\mathbf{C}) \cong SO(3, \mathbf{R})$, which by classical results is isomorphic to a cyclic group, a dihedral group, or one of the groups: A_4, S_4, or A_5. A_4, S_4, and A_5 may be realized as the groups of proper symmetries of the tetrahedron, the octahedron, and the icosahedron respectively.

There has been a tremendous amount of effort put into just the two-dimensional case of Artin's conjecture. The cyclic case is follows directly from Hecke's theory of abelian L-functions. The dihedral case also reduces without too much difficulty to the abelian theory. In the past few decades, the work of Langlands [5], and later of Tunnell [8], have put to rest the two solvable cases – the tetrahedral and octahedral cases – through advanced techniques using (among other things) the trace formula and converse theorems. In the current state of affairs, only the icosahedral case remains unproven.

Originating with the work of Buhler in his thesis (c.f. [2]), the icosahedral case of Artin's conjecture has been proven in a handful of examples. However, all of these examples have been cases where the icosahedral representation ρ was odd, i.e. the determinant $det \circ \rho$ corresponds to an odd Dirichlet character. (Alternatively, we may classify ρ as being even or odd based on whether $det(\rho(\sigma)) = \pm 1$ where σ is a complex conjugation). Therefore in this thesis we focus on the *even* icosahedral case. In this case very little is known, though Artin's conjecture, and the broad-sweeping Langlands program predicts a great deal.

1.2 Automorphic Forms on GL_2

In general, the Langlands program predicts a correspondence between Galois representations (actually representations of the larger Weil group) into $GL_n(\mathbf{C})$ and "automorphic cuspidal representations" of GL_n over our ground field. This correspondence moreover should respect natural operations on representations, such as tensor products, symmetric and alternating powers, etc... While it would go too far astray (and be far too much to include) to discuss the Langlands program in this degree of generality, we can discuss the Langlands program in the 2-dimensional case from the classical point of view.

We consider automorphic cuspidal representations of GL_2 classically as functions on the upper half-plane $\mathbf{H} = \{z \in \mathbf{C} | Im(z) > 0\}$. These functions come in two very distinct types, which we will describe in this section. Such a function f may be a holomorphic modular form, or a Maass form, which is not holomorphic, but is an eigenfunction of the Laplacian on $\mathbf{H}$. Holomorphic modular forms have been studied now for over a century, originating with the study of elliptic modular functions in the 19^{th} century. The study of Maass forms began in the 1949 paper of Maass [6]. We follow Bump's text [3] particularly in our treatment of holomorphic modular forms and Maass forms. Further information and proofs may be found there.

Let $\Gamma_0(D) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) | c \equiv 0(D) \right\}$. We define the non-Euclidean Laplacian on $\mathbf{H}$: $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$. Then we have the following definitions:

Definition 1 *A holomorphic modular form of weight k for $\Gamma_0(D)$ is a complex valued function f on $\mathbf{H}$ satisfying the following three conditions:*

- *f is holomorphic.*
- *For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D)$ we have $f(\gamma z) = (cz + d)^k f(z)$.*
- *f is holomorphic at the cusps of $\Gamma_0(D)$.*

Definition 2 *A Maass form (of weight 0) for $\Gamma_0(D)$ is a complex-valued function f on $\mathbf{H}$ satisfying the conditions:*

- *f is smooth.*
- *f is an eigenfunction of the Laplacian Δ.*
- *$f(\gamma z) = f(z)$ for all $\gamma \in \Gamma_0(D)$.*
- *f has at most polynomial growth at the cusps of $\Gamma_0(D)$.*

We let $\mathcal{M}(k, D)$ denote the space of holomorphic modular forms of weight k for $\Gamma_0(D)$. Similarly, we let $\mathcal{M}_\Delta(D, \lambda)$ denote the space of Maass forms for $\Gamma_0(D)$ with eigenvalue λ. To understand these spaces of modular forms better, note that $\Gamma_0(D)$ contains the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, so that any f in $\mathcal{M}(k, D)$ or $\mathcal{M}_\Delta(D, \lambda)$ has a Fourier expansion:

$$f(z) = \sum_{r=-\infty}^{\infty} a_r(y) e^{2\pi i r x}. \tag{1.6}$$

Now if f is a holomorphic modular form, then the coefficients $a_r(y)$ have the form $a_r(y) = a_r e^{-2\pi r y}$, for constants a_r and $a_r = 0$ when $r < 0$. By moving a cusp c for $\Gamma_0(D)$ to infinity, we may obtain other Fourier expansions of f with coefficients $a_{c,r}$ for each cusp. If $a_{c,0} = 0$ for all cusps c then we say that f is a cusp form. We denote the subspace of $\mathcal{M}(k, D)$ consisting of cusp forms by $\mathcal{S}(k, D)$. For every holomorphic modular form f, we define the Dirichlet series $L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$. Then we define the enlarged L-function by:

$$\Lambda(s, f) = (2\pi)^{-s}\Gamma(s)L(s, f).$$

If f is a Maass form, $f \in \mathcal{M}_\Delta(D, \lambda)$, then by the differential equation which f satisfies, and our given growth condition, we arrive at $a_r(y) = a_r\sqrt{y}K_v(2\pi|r|y)$ where K_v denotes the K-Bessel function, and $\lambda = \frac{1}{4} - v^2$. Moreover, either $a_r = a_{-r}$ or $a_r = -a_{-r}$, according to which we say that f is even or odd, respectively. We say that f is a Maass cusp form if $a_0 = 0$ and similarly for every other cusp. We denote the subspace of cusp forms of $\mathcal{M}_\Delta(D, \lambda)$ by $\mathcal{S}_\Delta(D, \lambda)$. For every Maass form, we again associate a Dirichlet series: $L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$. If ϵ is 1 or -1, depending on whether f is even or odd, we define the enlarged L-function by:

$$\Lambda(s, f) = \frac{1}{2}\pi^{-s}\Gamma\left(\frac{s+\epsilon+v}{2}\right)\Gamma\left(\frac{s+\epsilon-v}{2}\right)L(s, f).$$

It is an exercise in calculus (and one that is done in Bump's text [3]) that when f is a holomorphic modular form or a Maass form, we have the formula:

$$\int_0^\infty f(iy)y^{s-1/2}\frac{dy}{y} = \Lambda(s, f). \tag{1.7}$$

We are now able to precisely state the conjectured relationship between 2-dimensional Galois representations and modular forms. It is the purpose of this thesis to establish evidence for this conjecture in the icosahedral case.

Conjecture 2 *Suppose ρ is a continuous 2-dimensional complex representation of the Galois group $Gal(\bar{\mathbf{Q}}/\mathbf{Q})$, with Artin conductor D. Let $\Lambda(s, \rho)$ denote the enlarged Artin L-function associated to ρ as in the previous section. Then if ρ is odd (respectively even) there exists a holomorphic cusp form f of weight 1 (respectively a Maass cusp form of eigenvalue $1/4$) with respect to the congruence subgroup $\Gamma_0(D)$ satisfying $\Lambda(s, \rho) = \Lambda(s, f)$.*

The analytic continuation and functional equation of $\Lambda(s, f)$ for holomorphic cusp forms or Maass cusp forms f is well known, so that the above conjecture implies the 2-dimensional case of Artin's conjecture. However, we wish to verify the above conjecture by checking Artin's conjecture in a series of cases. We proceed as follows: given a two-dimensional projective Galois representation $\bar{\rho} : Gal(\bar{\mathbf{Q}}/\mathbf{Q}) \to PGL_2(\mathbf{C})$, a lifting of $\bar{\rho}$ is a representation $\rho : Gal(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL_2(\mathbf{C})$ such that $\pi \circ \rho = \bar{\rho}$ where π denotes the projection from $GL_2(\mathbf{C})$ to $PGL_2(\mathbf{C})$. We say that such a projective representation $\bar{\rho}$ is modular if every lifting ρ satisfies our Conjecture 2.

Now every such projective representation $\bar{\rho}$ yields a representation $\alpha : Gal(\bar{\mathbf{Q}}/\mathbf{Q}) \to GL_3(\mathbf{R})$ via the maps: $PGL_2(\mathbf{C}) \stackrel{\simeq}{\to} SO(3, \mathbf{R}) \hookrightarrow GL_3(\mathbf{R})$. Then for every lifting ρ of $\bar{\rho}$, we see that the adjoint square lift $Ad(\rho) = \alpha$. It is a deep result, essentially proven by Flicker using trace formula techniques, that when this adjoint square lift α is modular in the sense of Langlands, then ρ itself is modular. Thus we give evidence for our Conjecture 2 by checking Artin's conjecture numerically for this adjoint square representation $Ad(\rho)$ which is not too difficult to compute, and some of its twists. By the converse theorems of Weil and Jacquet, this implies the modularity of $Ad(\rho)$, which by the result just stated, implies the modularity of ρ.

Chapter 2

The Artin L-Function of an A_5 Extension

We fix a totally real A_5 extension E of $\mathbf{Q}$, defined as the splitting field of a quintic polynomial g with integer coefficients. In our tests, we have used the polynomial $g(x) = x^5 + 5x^4 - 7x^3 - 11x^2 + 10x + 3$, which was taken from the tables in the back of [2]. We fix a three-dimensional representation $\rho : Gal(E/Q) \to GL_3(C)$. The Artin L-function $L(s, \rho)$ is defined as an Euler product, as in the introduction. It is the purpose of this chapter to describe the computation of these local factors in our A_5 case.

2.1 Frobenius Elements in an A_5 Extension

It is very helpful, since we are considering the natural representaion ρ of A_5, to understand the group in terms of the symmetries of the icosahedron. A_5 is a simple group of order 60. It has five conjugacy classes, which we name according to their order: 1A, 2A, 3A, 5A, and 5B. 1A consists only of the identity element. 2A consists of 15 rotations by 180 degrees, one for each pair of opposite edges of the icosahedron. 3A consists of 20 rotations, two for each pair of opposite faces. 5A and 5B contain the 24 rotations about axes through the vertices of the icosahedron. The character table for A_5 is given below, where $u = \frac{1+\sqrt{5}}{2}$ and $v = \frac{1-\sqrt{5}}{2}$.

A_5	1A	2A	3A	5A	5B
χ_1	1	1	1	1	1
χ_3	3	-1	0	u	v
χ_3'	3	-1	0	v	u
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

We are most interested in the character $\chi_3 = tr(\rho)$. Now fix a prime p, and assume for now that p is unramified in E. To compute the local factor of the L-series $L(s, \rho)$, we need to know the conjugacy class of the Frobenius element Fr_p. Let E_v denote the completion of E at a prime lying above p, and let $k/\mathbf{F}_p$ denote the residue field extension. Recall that E was given as the splitting field of a quintic polynomial g, so that k is given as the splitting field over $\mathbf{F}_p$ of the reduced polynomial $\bar{g}$. Now, there are a number of ways in which $\bar{g}$ can factor over $\mathbf{F}_p$:

- $\bar{g}$ could split into five distinct linear factors, in which case $k = \mathbf{F}_p$ and Fr_p is the identity element.
- $\bar{g}$ could split into two quadratic and one linear factor, in which case we see that Fr_p has order 2 or 4. However, A_5 has no elements of order 4, so Fr_p has order 2.
- $\bar{g}$ could split into one cubic and two linear factors, in which case Fr_p has order 3.
- $\bar{g}$ could not split at all, in which case Fr_p has order 5.

It is not hard to see that these are the only options, by the structure of A_5. For instance, $\bar{g}$ cannot split into one quadratic and three linear factors, since otherwise Fr_p would act as a transposition on the roots of g. From the possible factorizations of $\bar{g}$, we have an algorithm to find the order of Fr_p for any unramified prime p. The order of Fr_p *almost* determines its conjugacy class; however, looking at the character table of A_5, we must be able to distinguish the conjugacy classes 5A and 5B in order to determine $\rho(\mathrm{Fr}_p)$.

If Fr_p has order 5, we follow Buhler in [2] in using an idea of Serre to tell whether Fr_p is in 5A or 5B. Since E has Galois group A_5, the discriminant of g is a perfect square: $g = D^2$. Since Fr_p has order 5, the completion E_v is an unramified extension of degree 5 over $\mathbf{Q}_p$. We have the following useful criterion:

Proposition 1 *Let y be a root of g in E_v. Then*

$$\prod_{0\le i<j\le 4}\left(\mathrm{Fr}_p{}^i(y)-\mathrm{Fr}_p{}^j(y)\right)=\pm D. \tag{2.1}$$

The sign of D determines whether Fr_p is in conjugacy class 5A or 5B.

Of course, if $p \neq 2$ then it suffices to consider the above formula in the residue field extension $k/\mathbf{F}_p$. In this extension, the Frobenius element Fr_p reduces to the usual Frobenius automorphism $x \mapsto x^p$. Thus for odd primes, the above product is not too difficult to compute, using fast algorithms for finite fields. However, when $p = 2$, we must be more careful. The above formula still holds, but reducing mod 2 renders it useless, since we can't distinguish D from $-D$ mod 2. Instead, we reduce mod 4, requiring us to analyze the action of Frobenius on E_v mod 4.

Let R_v be the valuation ring of E_v. Then, viewing $\tau = \mathrm{Fr}_2$ as the generator of $Gal(E_v/\mathbf{Q}_2)$, we have: $\tau(x) \equiv x^2 + 2\alpha(x) \pmod 4$ for $x \in R_v$, for some function α. Now since τ is an automorphism, we have $\tau(u+v) = \tau(u)+\tau(v)$, and $\tau(uv) = \tau(u)\tau(v)$, from which we may compute:

$$\alpha(u+v) = \alpha(u) + \alpha(v) - uv,$$

$$\alpha(uv) = \alpha(u)v^2 + \alpha(v)u^2.$$

Hence for any polynomial P with integer coefficients, we may compute inductively a unique polynomial $\tilde{P}$ such that $\alpha(P(u)) = \tilde{P}(\alpha(u))$ for all $u \in R_v$. Now, letting y be a root of g in E_v again, we see that $\tau(g(y)) = g(\tau(y)) = 0$. Hence $\alpha(g(y)) = 0$. Therefore $\alpha(y)$ is a root of $\tilde{g}$ in R_v. Hence we may effectively compute $\alpha(y)$ for any root y of g, which allows us to compute the action of Frobenius on $R_v/4R_v$.

We now have the ability to determine the conjugacy classes of Frobenii at all unramified primes. For convenience, we list the L-factors that will arise for each possible conjugacy class. In the following, ζ_5 denotes a fixed 5^{th} root of unity:

- $\mathrm{Fr}_p \in (1A) \Longrightarrow L(s,\rho)_p = (1-p^{-s})^{-3}$,
- $\mathrm{Fr}_p \in (2A) \Longrightarrow L(s,\rho)_p = (1-p^{-s})^{-1}(1+p^{-s})^{-2}$,
- $\mathrm{Fr}_p \in (3A) \Longrightarrow L(s,\rho)_p = (1-p^{-3s})^{-1}$,
- $\mathrm{Fr}_p \in (5A) \Longrightarrow L(s,\rho)_p = (1-p^{-s})^{-1}(1-\zeta_5 p^{-s})^{-1}(1-\zeta_5{}^{-1}p^{-s})^{-1}$,
- $\mathrm{Fr}_p \in (5B) \Longrightarrow L(s,\rho)_p = (1-p^{-s})^{-1}(1-\zeta_5{}^2 p^{-s})^{-1}(1-\zeta_5{}^{-2}p^{-s})^{-1}$.

2.2 The Ramified Primes

If E is an A_5 extension of the rationals, and p ramifies in this extension, then there are 19 possible types of ramification that may occur, as enumerated in Buhler [2]. Here we provide a table, derived from Buhler's, of the types of ramification, the resulting L-factors, and the local conductors for the natural 3-dimensional representation of A_5. The ramification groups are listed beginning with the decomposition group. Here C_n denotes the cyclic group of order n, D_n denotes the dihedral group of order $2n$ if $n > 2$, D_2 denotes the Klein group of 4 elements, and A_4 denotes the alternating group on 4 symbols.

Type	Ramification Groups	Conductor	L-Factor
1	C_5, C_5	p^2	$(1-p^{-s})^{-1}$
2	C_3, C_3	p^2	$(1-p^{-s})^{-1}$
3	C_2, C_2	p^2	$(1-p^{-s})^{-1}$
4	D_5, C_5	p^2	$(1+p^{-s})^{-1}$
5	D_3, C_3	p^2	$(1+p^{-s})^{-1}$
6	D_2, C_2	p^2	$(1+p^{-s})^{-1}$
7	C_5, C_5, C_5	p^4	$(1-p^{-s})^{-1}$
8	D_5, D_5, C_5	p^4	1
9	D_5, C_5, C_5	p^4	$(1+p^{-s})^{-1}$
10	C_3, C_3, C_3	p^4	$(1-p^{-s})^{-1}$
11	D_3, D_3, C_3	p^4	1
12	D_3, C_3, C_3	p^4	$(1+p^{-s})^{-1}$
13	D_3, D_3, C_3, C_3, C_3	p^6	1
14	C_2, C_2, C_2	p^4	$(1-p^{-s})^{-1}$
15	C_2, C_2, C_2, C_2	p^6	$(1-p^{-s})^{-1}$
16	D_2, C_2, C_2	p^4	$(1+p^{-s})^{-1}$
17	A_4, D_2, D_2	p^6	1
18	D_2, C_2, C_2, C_2	p^6	$(1+p^{-s})^{-1}$
19	D_2, D_2, D_2, C_2, C_2	p^8	1

Chapter 3

Numerical Tests for Analyticity

In this chapter, we derive a method to numerically test Artin's conjecture in specific cases. We fix an Artin L-series given by $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ for $Re(s) > 1$. Let $\gamma(s)$ denote the Γ-factors associated for this L-function, and let A, W denote the Artin conductor and root number, respectively. Let $\Lambda(s) = A^{s/2}\gamma(s)L(s)$ denote the enlarges L-function, so that $\Lambda(s) = W\Lambda(1-s)$.

3.1 An Approximate Functional Equation

We begin by assuming Artin's conjecture holds to derive an approximate functional equation that estimates $\Lambda\left(\frac{1}{2}\right)$. For notational convenience, let R_σ denote the line in the complex plane given by $Re(z) = \sigma$ (oriented upwards). We begin with a lemma:

Lemma 1 *Let $G(s)$ be a polynomial, and define:*

$$V(y) = \frac{1}{2\pi i}\int_{R_\sigma} y^{-s}\gamma(s+\frac{1}{2})G(s)\frac{ds}{s}.$$

Here σ can be any real number greater than 0 by Cauchy's theorem. Then $V(y)$ decays faster then any polynomial as $y \to \infty$, and as $y \to 0$, $V(y) = O(y^{-t})$, for t an arbitrarily small real number.

Proof: As y approaches infinity, we note that:

$$2\pi|V(y)| = \left|\int_{R_\sigma} y^{-s}\gamma(s+\frac{1}{2})G(s)\frac{ds}{s}\right| \tag{3.1}$$

$$\leq \int_{R_\sigma} |y^{-s}||\gamma(s+\frac{1}{2})||G(s)|\frac{ds}{|s|} \tag{3.2}$$

$$\leq y^{-\sigma}\int_{R_\sigma} |\gamma(s+\frac{1}{2})||G(s)|\frac{ds}{|s|}. \tag{3.3}$$

Now this last integral converges due to the rapid decrease of $\gamma(z)$ as $Im(z) \to \infty$. Hence $|V(y)| < y^{-\sigma}C$ for any $\sigma > 0$, so $V(y)$ decreases rapidly as $y \to \infty$. As y approaches zero, we follow the same approach, moving the contour close the imaginary axis to get our desired estimate. ♢

We may now apply this lemma to derive an approximate functional equation for Artin L-series:

Proposition 2 *Let X be a positive real number, and $G(s)$ be a polynomial satisfying $G(0) = 1$ and $G(s) = G(-s)$ for all s. Define the function $V(y)$ as in the lemma above. Then if Artin's conjecture holds for our Artin L-series, i.e. it extends to an entire function in the complex plane, then we have the following identity:*

$$\Lambda\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} A^{1/4}\frac{a_n}{\sqrt{n}}V\left(\frac{n}{X\sqrt{A}}\right) + \sum_{n=1}^{\infty} A^{1/4}\frac{a_n}{\sqrt{n}}V\left(\frac{nX}{\sqrt{A}}\right).$$

Proof: Consider the integral, where $0 < \sigma < \frac{1}{2}$:

$$I = \frac{1}{2\pi i}\int_{R_\sigma} \Lambda\left(s+\frac{1}{2}\right)X^sG(s)\frac{ds}{s}. \tag{3.4}$$

Since the Γ-factor $\gamma(z)$ decays rapidly as $Im(z) \to \infty$, and eveything else in the above integral has at most polynomial growth along our contour, the above integral converges uniformly. Therefore, substituting in $\Lambda(s) = A^{s/2}\gamma(s)\sum\frac{a_n}{n^s}$ and exchanging summation and integration yields:

$$I = \sum_{n=1}^{\infty}\frac{a_n}{\sqrt{n}}A^{1/4}\frac{1}{2\pi i}\int_{R_\sigma}\left(\frac{n}{X\sqrt{A}}\right)^{-s}\gamma\left(s+\frac{1}{2}\right)G(s)\frac{ds}{s} \tag{3.5}$$

$$= \sum_{n=1}^{\infty}\frac{a_n}{\sqrt{n}}A^{1/4}V\left(\frac{n}{X\sqrt{A}}\right). \tag{3.6}$$

Now we evaluate I again, but this time shift the contour of integration from R_σ to $R_{-\sigma}$. This picks up an extra term (and only one, by the assumed analyticity of the L-series) due to the pole of $\frac{1}{s}$ at $s = 0$. Since $G(0) = 1$ and $X^0 = 1$, we arrive at the expression:

$$I = \frac{1}{2\pi i}\int_{R_{-\sigma}} \Lambda\left(s+\frac{1}{2}\right)X^sG(s)\frac{ds}{s} + \Lambda\left(\frac{1}{2}\right).$$

We apply the functional equation of Λ, and the given identity $G(s) = G(-s)$ to get:

$$I = \frac{1}{2\pi i}\int_{R_{-\sigma}} \Lambda\left(\frac{1}{2} - s\right) X^s G(-s)\frac{ds}{s} + \Lambda\left(\frac{1}{2}\right).$$

Now making the substitution $w = -s$, we finally see that:

$$I = -\frac{1}{2\pi i}\int_{R_{\sigma}} \Lambda\left(w + \frac{1}{2}\right) X^{-w} G(w)\frac{dw}{w} + \Lambda\left(\frac{1}{2}\right) \quad (3.7)$$

$$= -\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}} A^{1/4} V\left(\frac{nX}{\sqrt{A}}\right) + \Lambda\left(\frac{1}{2}\right). \quad (3.8)$$

Putting the two expressions (3.6), (3.8) together, we arrive at the desired result. $\diamondsuit$

3.2 Detecting Deviations from Artin's Conjecture

Now let us assume the generalized Riemann hypothesis, and see what would happen if Artin's conjecture were false for our L-function. Then our L-function would have poles located on the line $Re(s) = 1/2$, say at points $\frac{1}{2} + i\tau$, for τ in some (most likely infinite) set of real numbers T. The approximate functional equation derived above no longer holds, but we may modify it with an error term based on T as follows:

Proposition 3 *With the same notation as the previous proposition, let us assume Artin's conjecture does not hold. Then with T as above, suppose that r_τ is the residue of $L(s)$ at the pole $\frac{1}{2} + i\tau$. We then have the following expression for $\Lambda(1/2)$:*

$$\Lambda\left(\frac{1}{2}\right) = \sum_{n=1}^{\infty} A^{1/4}\frac{a_n}{\sqrt{n}} V\left(\frac{n}{X\sqrt{A}}\right) + \sum_{n=1}^{\infty} A^{1/4}\frac{a_n}{\sqrt{n}} V\left(\frac{nX}{\sqrt{A}}\right) + E(X), \quad (3.9)$$

where the error term E is given by:

$$E(X) = A^{1/4}\sum_{\tau\in T}(X\sqrt{A})^{i\tau}\gamma\left(\frac{1}{2} + i\tau\right)\frac{G(i\tau)}{i\tau} r_\tau.$$

Proof: Following the proof of our previous proposition, note that when we slide the contour from R_σ to $R_{-\sigma}$, we pick up a new term for each pole of $L(s)$ as embodied in our error term above. To make this rigorous, we must only note that the rapid decay of our Γ-factor as τ approaches infinity guarantees the convergence of the sum for $E(X)$. $\diamondsuit$

Our previous two propositions give us a robust numerical test for Artin's conjecture in any specific case, that requires only about $\sqrt{A}$ terms of the L-series to apply. Namely, if Artin's conjecture holds, then our first proposition allows us to evaluate $\Lambda\left(\frac{1}{2}\right)$ numerically, by using any parameter X; the value computed for $\Lambda\left(\frac{1}{2}\right)$ will not depend on X. However, if Artin's conjecture fails, then the error term $E(X)$ depends very much on X, and therefore the sum:

$$\sum_{n=1}^{\infty} A^{1/4}\frac{a_n}{\sqrt{n}}V\left(\frac{n}{X\sqrt{A}}\right)+\sum_{n=1}^{\infty} A^{1/4}\frac{a_n}{\sqrt{n}}V\left(\frac{nX}{\sqrt{A}}\right)$$

will depend on X.

In the appendices, we have applied this numerical test to give strong evidence for Artin's conjecture for a three-dimensional Galois representation and some twists. As stated in the introduction, this gives evidence for the modularity of a family of two-dimensional icosahedral representations. While our evidence certainly doesn't approach a proof, it is essentially equivalent to verifying that there are no low-lying poles in our Artin L-function. These would almost without a doubt occur if Artin's conjecture failed. Thus we can safely say that a family of icosahedral Galois representations correspond to Maass forms.

Bibliography

[1] Richard Brauer. On Artin's L-series with general group characters. *Ann. of Math.*, 48(2):502–514, 1947.

[2] Joe P. Buhler. *Icosahedral Galois Representations.* Number 654 in Lecture Notes in Mathematics. Springer-Verlag, 1978.

[3] Daniel Bump. *Automorphic forms and representations*, volume 55 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, 1997.

[4] Serge Lang and John T. Tate, editors. *The collected papers of Emil Artin.* Addison-Wesley Publishing Co., Inc., 1965.

[5] Robert Langlands. *Base change for GL(2).* Ann. of Math. Studies, 96. Princeton Univ. Press, 1980.

[6] Hans Maass. Über eine neue art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgeichungen. *Math. Ann.*, 121:141–183, 1949.

[7] Jacques Martinet. Character theory and Artin L-functions. In *Algebraic number fields: L-functions and Galois properties*, pages 1–87, 1977.

[8] Jerrold Tunnell. Artin's conjecture for representations of octahedral type. *Bull. Amer. Math. Soc. (N.S.)*, 5(2):173–175, 1981.

Appendix A

Numerical Methods for Artin L-Functions

Here we include the code used to run numerical tests on Artin's conjecture in the icosahedral case. The included code was written for Maple V, release 5, and run on a Pentium II 350 MHz personal computer.

A.1 Computation of Frobenius Conjugacy Classes

In this section, we give procedures to determine the conjugacy class of Frobenii in an A_5 extension over $\mathbf{Q}$. There are five conjugacy classes in the group A_5, which we call $1A$, $2A$, $3A$, $5A$, and $5B$; $1A$ consists of the identity element, $2A$ consists of the elements of order 2, etc... It is not difficult to determine the order of a Frobenius element, but it is a little tricky to determine whether a Frobenius element of order 5 lies in $5A$ or $5B$.

We first initialize the necessary packages and functions.

```
> NoPrimes := 99000:
> Digits := 15:
> readlib(ifactors):
> with(linalg):
> readlib(GF):
```

The following procedure determines whether the Frobenius element associated to an odd prime p lies in conjugacy class $5A$ or $5B$ (given that it lies in one of these). To do this, it evaluates the product:

$$K = \prod_{0 \le i < j \le 4} \left({Fr_p}^i(y) - {Fr_p}^j(y) \right).$$

Then the conjugacy class is $5A$ or $5B$ depending on whether this product is equal to the discriminant D, or $-D$ respectively.

```
> Test5AB := proc(f::polynom(constant,y),p::prime)
> local GP,F,i,j,P,H,D,K;
> GP := GF(p,5,f):
> F[0] := GP[ConvertIn](y):
> for i from 1 to 4 do
>    F[i] := GP[`^`](F[i-1],p):
> od;
> P := GP[1];
> for i from 0 to 3 do
> for j from i+1 to 4 do
>    P := GP[`*`](GP[`-`](F[i],F[j]),P):
> od;
> od;
> D := Normal(sqrt(discrim(f,y))) mod p:
> K := GP[ConvertOut](P):
> if D = K then
>    1:
> elif D = -K mod p then
>    -1:
> else 0:
> fi;
> end:
```

The following procedures determine whether the Frobenius element associated to 2 lands in conjugacy class $5A$ or $5B$, again given that it lands in one of these. Essentially, we compute the same product as before, except we must work modulo 4 instead of mod p. The difficulty lies in figuring out how the Frobenius automorphism acts on the degree 5 unramified extension of $\mathbf{Q}_2$ up to congruence mod 4.

```
> ComputeG := proc(f::polynom(constant, y))
>    local Gpower,n,G,RP,i,a,ga,m;
>    global x;
>    alias(r=RootOf(f));
>    Gpower := proc(t::integer)
>       t*r^(2*t-2)*x;
>    end:
```

```
>    n := degree(f,y);
>    G := 0;
>    RP := 0;
>    for i from 0 to n do
>       a := coeff(f,y,i);
>       ga := ((a-a^2)/2)*r^(2*i) + a*Gpower(i);
>       m := a*r^i;
>       G := G + ga - m*RP;
>       RP := RP + m;
>    od;
>    G;
> end:
> FrobOfr := proc(f::polynom(constant,y))
>    local G,j,F;
>    G := ComputeG(f);
>    j := RootOf(G,x);
>    F := r^2 + 2*j;
> end:
> FrobProd := proc(f::polynom(constant,y))
>    local FR,F,j,i,a,P;
>    FR[1] := Expand(FrobOfr(f)) mod 4;
>    FR[2] := Expand(FR[1]^2) mod 4;
>    FR[3] := Expand(FR[1]^3) mod 4;
>    FR[4] := Expand(FR[1]^4) mod 4;
>    FR[0] := 1;
>    F[1] := FR[1];
>    for j from 2 to 4 do
>       F[j] := 0;
>       for i from 0 to 4 do
>          a := coeff(F[j-1],r,i);
>          F[j] := F[j] + a*FR[i];
>       od;
>    od;
>    F[0] := r;
>    P := 1;
>    for i from 0 to 3 do
>    for j from i+1 to 4 do
>       P := P * (F[i] - F[j]);
>    od;
>    od;
```

```
>    Normal(P) mod 4;
> end:
> Test5AB2 := proc(f::polynom(constant,y))
> local K,D;
> K := FrobProd(f):
> D := sqrt(discrim(f,y)) mod 4:
> if D = K then
>    1:
> elif D = -K mod 4 then
>    -1:
> else 0:
> fi;
> end:
```

This procedure computes the conjugacy class of the Frobenius element associated to a given prime p. First, it checks whether p is a ramified prime, and if so outputs 0. Then, if the polynomial f is irreducible mod p, we know that the Frobenius element is in a conjugacy class of 5-cycles. If p is not a ramified prime, the procedure tests whether f is irreducible, and if so, tests which conjugacy class of 5-cycles the Frobenius element is in. Finally, if p is not ramified, and f is not irreducible mod p, the procedure counts the roots of f mod p to decide which conjugacy class the Frobenius element lies in.

```
> FindConj := proc(p::prime, f::polynom(constant, y))
> local A,L,B;
> A := 0;
> if discrim(f,y) mod p = 0 then
>    A := 0;
> elif Irreduc(f) mod p then
>    if p > 2 then A := 5 * Test5AB(f,p);
>    else A := 5 * Test5AB2(f);
>    fi;
> else
>    L := Roots(f) mod p;
>    B := nops(L);
>    if B = 1 then A := 2;
>    elif B = 2 then A := 3;
>    elif B = 5 then A := 1;
>    else A := 0;
>    fi;
```

```
> fi;
> A;
> end:
> GenConj := proc(p :: integer, f :: polynom(constant, y))
> local A;
> if p = 2 then A := 0;
> elif isprime(p) then A := FindConj(p,f);
> else A := 0;
> fi;
> A;
> end:
```

Here we use our previous procedures to produce a list $c[i]$, where c is a number representing the conjugacy class in A5 of the Frobenius element of i when i is prime. The conjugacy classes of elements of order 2 and 3 are represented by values of 2 and 3, and the two conjugacy classes of order 5 are represented by 5 and -5. The list has elements for primes p up to a given n.

```
> ListConj := proc(f::polynom(constant, y), n::integer)
> local L,C,i;
> L := [seq(i,i=1..n)];
> C := map(GenConj,L,f);
> end:
```

These procedures save and load a list computed by ListConj:

```
> SaveCList := proc(f::polynom(constant, y),
                    n::integer, fn::string)
> local R;
> R := ListConj(f,n);
> writedata(fn,R,[integer]);
> end:
> LoadCList := proc(fn::string)
> local R;
> R := readdata(fn, [integer]);
> end:
```

Now we produce a list of the primes up to 100000 and their conjugacy classes, and save it to a file. The A5 extension of the rationals is given by the splitting field of the polynomial g below; g was taken from the table of A_5 quintics in [2].

```
> g := y^5 + 5*y^4 - 7*y^3 - 11*y^2 + 10*y + 3:
> SaveCList(g,100000,"c7947_100K.txt"):
> CL := LoadCList("c7947_100K.txt"):
```

A.2 Numerical Testing of Artin's Conjecture

In this section, we follow the tests described in chapter 3 of the thesis to test Artin's Conjecture for a given 3-dimensional representation, and one of its twists. To speed up computation, we compute some recurring coefficients in advance. There are a number of tasks to take care of individually:

A.2.1 Computation of the Coefficients of the Artin L-Function

We compute the coefficients of our L-series by computing the prime power coefficients, then extending multiplicatively to all positive integers. The following constants occur repeatedly in the coefficients, so we compute them in advance.

```
> c5a[0] := 1: c5a[1] := evalf((1+sqrt(5))/2):  c5a[2] := 1:
  c5a[3] := 0:  c5a[4] := 0:
> c5b[0] := 1: c5b[1] := evalf((1-sqrt(5))/2):
  c5b[2] := 1: c5b[3] := 0: c5b[4] := 0:
> c3[0] := 1: c3[1] := 0: c3[2] := 0:
```

The procedure "PrimePowerCoefficient" computes the coefficients of prime powers, as its name suggests.

```
> PrimePowerCoefficient := proc(p :: prime, n :: integer)
> local c,A;
> if p = 3 then A := (-1)^(n mod 2);
> elif p = 883 then A := 1;
> elif p > NoPrimes then A := 0;
> else
>    if (p = 2) then c := 5; else c := CL[p]; fi;
>    if c = 1 then A := (n+1)*(n+2)*(0.5);
>    elif c = 2 then A := 0.5 * (n+1+((n+1) mod 2))
                             * (-1)^(n mod 2);
>    elif c = 3 then A := c3[n mod 3];
>    elif c = 5 then A := c5a[n mod 5];
>    elif c = -5 then A := c5b[n mod 5];
```

```
>    fi;
> fi;
> A;
> end:
```

Now, since the coefficients of the Artin L-series are multiplicative, we use this procedure to compute the general n^{th} coefficient of our series.

```
> LCoefficient := proc(n :: integer)
> local F,Fn,P,i;
> F := ifactors(n);
> Fn := nops(F[2]);
> P := 1;
> if Fn = 0 then P := 1;
> elif (Fn = 1) and (n > NoPrimes) then P := 0;
> else
> for i from 1 to Fn do
>    P := P *  PrimePowerCoefficient(F[2][i][1],F[2][i][2]);
> od;
> fi;
> P;
> end:
```

A.2.2 Gamma-Factors and the Artin Conductor

The following procedures define the gamma-factor associated to our Artin L-series. The constants "dplus" and "dminus" determine what gamma-factor is used.

```
> gammapart := proc(s :: anything)
> evalf(Pi^(-s/2) * GAMMA(s/2));
> end:
> dplus := 0:
> dminus := 3:
> GammaFactor := proc(s :: anything)
> evalf((gammapart(s)^dplus) * (gammapart(s+1)^dminus));
> end:
```

We compute the Artin conductor for our representation:

```
> ArtinConductor := evalhf((3^2) * (883^2));
```

$$\text{ArtinConductor} := .7017201\ 10^7$$

A.2.3 Computation of the Non-Elementary Function V

Recall that in equation (3.1) we define a function $V(y)$ as a certain rapidly convergent integral. V is not an elementary function, however it is not very difficult to evaluate by using an optimized version of Simpson's rule. The following procedures are designed to prepare tables that allow for the rapid computation of $V(y)$ for any real $y > 0$.

```
> G := 1:
```

The function $V(y)$ may be considered as the inverse mellin transform of a certain function, computed by the procedure "Integrand."

```
> Integrand := proc(s :: anything)
> evalf(GammaFactor(1.5+s*I)*G/(1+s*I));
> end:
```

To numerically compute this inverse Mellin transform, we approximate the integral via Simpson's rule. In order to speed up computations, the interval sizes chosen are not equal. The following parameters determine the intervals used for Simpson's rule: "smin" and "smax" are the lower and upper ranges for the initial pass of Simpson's rule. "npoints" is the number of evenly spaced intervals used, each of length "spacing," between "smin" and "smax." The rapid decay of the gamma-factor makes this initial pass a decent approximation, but to get an even better one, we integrate outside the interval [smin, smax]. The variables "outsidelength," "npoints2," and "spacing2" describe the additional region integrated over by Simpson's rule. Though this procedure is slightly more complicated to program, it greatly improves the speed and accuracy of computation.

```
> smin := -5.0:
> smax := 5.0:
> npoints := 500:
> spacing := (smax-smin)/npoints:
> outsidelength := 5.0:
> npoints2 := 100:
> spacing2 := outsidelength/npoints2:
```

Since we must compute an integral every time we want to compute $V(y)$, we make tables which store the values of "Integrand" at appropriately spaced points. We have three different tables, to allow for variable spacing.

```
> MakeIntegrandTable1 := proc()
> local i,T;
> for i from 0 to npoints do
>    T[i] := Integrand(i*spacing + smin);
> od;
> T;
> end:
> MakeIntegrandTable2R := proc()
> local i,UR;
> for i from 0 to npoints2 do
>    UR[i] := Integrand(i*spacing2 + smax);
> od;
> UR;
> end:
> MakeIntegrandTable2L := proc()
> local i,UL;
> for i from 0 to npoints2 do
>    UL[i] := Integrand(smin - outsidelength + i*spacing2);
> od;
> UL;
> end:
> T := MakeIntegrandTable1():
> UR := MakeIntegrandTable2R():
> UL := MakeIntegrandTable2L():
```

Now, with the tables computed and stored in memory, we may compute $V(y)$ rapidly using the following procedures:

```
> Simps := proc(i :: integer, j :: integer, k :: integer)
> local a;
> if (i = 0) or (i = j) then
>    a := 1;
> else a := (2 * ((i/k) mod 2)) + 2;
> fi;
> end:
> V := proc(y :: float, N :: integer)
> local i, S1, S2, P, pm, P2R, P2L, pm2;
> S1 := 0; S2 := 0;
> P := evalf(y^(-(1 + I*smin)));
> pm := evalf(y^(-N*I*spacing));
```

```
> for i from 0 to npoints by N do
>    S1 := evalf(S1 + T[i]*P*Simps(i,npoints,N));
>    P := evalf(P * pm);
> od;
> S1 := Re(S1*spacing*N/3);
> P2R := evalf(y^(-(1 + I*smax)));
> P2L := evalf(y^(-(1 + I*(smin-outsidelength))));
> pm2 := evalf(y^(-I*spacing2));
> for i from 0 to npoints2 do
>    S2 := evalf(S2 + UR[i]*P2R*Simps(i,npoints2,1) +
                             UL[i]*P2L*Simps(i,npoints2,1));
>    P2R := evalf(P2R*pm2);
>    P2L := evalf(P2L*pm2);
> od;
> S2 := Re(S2*spacing2/3);
> evalf(S1+S2);
> end:
```

A.2.4 Put it to the Test!

Now we compute $\Lambda\left(\frac{1}{2}\right)$ via the sum in Proposition 2. The constant "Acc" controls the accuracy of our approximation. The constant "nmax" controls the number of terms used in each sum; the rapid decay of $V(y)$ allows us to use approximately the square root of the conductor as "nmax." There are two sums on the right hand side of Proposition 2 – the first one is computed by the procedure "Comp1" and the second one by "Comp2." The procedure "LambdaOneHalf (X)" adds them up, using the parameter X, twisting the L-series by a quadratic Dirichlet character given by "Twist."

```
> Acc := 1.0:
> Comp1 := proc(X :: integer)
> local S, n, term, RootCond, A, nmax, Charn, TwistCond;
> TwistCond := nops(Twist);
> A := ArtinConductor*(TwistCond^3);
> S := 0;
> RootCond := evalhf(sqrt(A));
> nmax := round(RootCond*Acc)*X;
> for n from 1 to nmax do
>    Charn := Twist[(n mod TwistCond) + 1];
>    if Charn <> 0 then
```

```
>       term := LCoefficient(n) * Charn * evalhf(n^(-0.5))
                * V(evalhf(n/(X*RootCond) ), 1);
>       S := evalhf(S + term);
>    fi;
> od;
> S := evalhf((A^0.25) * S);
> end:
> Comp2 := proc(X :: integer)
> local S, n, term, RootCond, A, nmax, Charn, TwistCond;
> TwistCond := nops(Twist);
> A := ArtinConductor*(TwistCond^3);
> S := 0;
> RootCond := evalhf(sqrt(A));
> nmax := round(RootCond*Acc/X);
> for n from 1 to nmax do
>    Charn := Twist[(n mod TwistCond) + 1];
>    if Charn <> 0 then
>       term := LCoefficient(n) * Charn * evalhf(n^(-0.5))
                * V(evalhf(n*X/RootCond), 1 );
>       S := evalhf(S + term);
>    fi;
> od;
> S := evalhf((A^0.25) * S);
> end:
> LambdaOneHalf := proc(X :: integer)
> local A;
> if X <> 1 then A := evalf(Comp1(X) + Comp2(X));
> else A := 2 * evalf(Comp1(X));
> fi;
> end:
```

Now we run the test; as seen below, LambdaOneHalf does not depend on its parameter X to within 5 or 6 decimal places, suggesting that Artin's conjecture holds for our L-series. Also, we run the test on our L-series twisted by the primitive quadratic character of conductor 4, suggesting Artin's conjecture holds for the twisted L-series as well.

```
> Twist := [1];
```

$$\text{Twist} := [1]$$

```
> LambdaOneHalf(1);
```

9742.04174772344

```
> LambdaOneHalf(2);
```

9742.04234774679

```
> LambdaOneHalf(3);
```

9742.04069431213

```
> LambdaOneHalf(4);
```

9742.04409727788

```
> Twist := [0,1,0,-1];
```

Twist := [0, 1, 0, -1]

```
> LambdaOneHalf(1);
```

1296.95005191828

```
> LambdaOneHalf(2);
```

1296.91907569532

```
> LambdaOneHalf(3);
```

1296.94178653574