The Fourier-Jacobi Map and Small Representations

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Abstract

We study the “Fourier-Jacobi” functor on smooth representations of split, simple, simply-laced p-adic groups. This functor has been extensively studied on the symplectic group, where it provides the representation-theoretic analogue of the Fourier-Jacobi expansion of Siegel modular forms. Our applications are different from those studied classically with the symplectic group. In particular, we are able to describe the composition series of certain degenerate principal series. This includes the location of minimal and small (in the sense of the support of the local character expansion) representations as spherical subquotients.

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Dedicated to my grandparents,

Daniel and Lorraine Howard,

and my excellent teachers, especially

Alice Snodgrass

on the occasion of her retirement.

Chapter 1

Introduction

The theory of automorphic representations has developed from ideas about classical modular forms. Arithmetic information can often be retrieved from the Fourier coefficients of a classical modular form, and from the local factors of an automorphic representation. Especially, the Fourier coefficients of theta series contain valuable information about the arithmetic of quadratic forms. For the symplectic group, Siegel studied the theta series associated to quadratic forms in many variables over $\mathbb{Z}$. It was this study of theta series and Siegel's modular forms that motivated the study of singular modular forms – those whose Fourier coefficients vanish more often than usual. Singular modular forms seem very strongly connected to arithmetic, and it can be proven that their non-zero Fourier coefficients provide valuable information about integral quadratic forms. The representation theoretic analogue of singular modular forms are *small* automorphic representations, and these are studied locally in this paper. In this introduction, we provide some motivation from the classical theory of singular modular forms.

Let $\mathfrak{h}^n$ denote the Siegel upper half-space of degree n; in other words, $\mathfrak{h}^n$ is the space of symmetric n by n matrices $Z = X + iY$, where X, Y are real symmetric matrices, and Y is positive definite. Then the symplectic group $Sp_{2n}(\mathbb{R})$ acts on $\mathfrak{h}^n$ by linear fractional transformations. If $M \in Sp_{2n}(\mathbb{R})$ then we write:

$$M\langle Z\rangle = (AZ+B)(CZ+D)^{-1},$$

where M is the $2n$ by $2n$ matrix given in block form as:

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right).$$

Let $\Gamma = Sp_{2n}(\mathbb{Z})$ denote the full modular group. Then if $f\colon \mathfrak{h}^n \to \mathbb{C}$ is any function, and $M \in Sp_{2n}(\mathbb{R})$, we define the weight k action of M on f for $k \in \mathbb{Z}$ by:

$$f|_M^k(Z) = f(M\langle Z\rangle)|CZ+D|^{-k}.$$

This action can also be defined for $k \in \frac{1}{2}\mathbb{Z}$ in a well-known way; for the simplicity of this introduction, we refer the reader to the book of Andrianov [1]. From this action of $Sp_{2n}(\mathbb{R})$ on functions, we may define Siegel modular forms:

Definition 1 *A Siegel modular form of degree $n > 1$ and weight $k \in \frac{1}{2}\mathbb{Z}$ (and level 1) is a function $f\colon \mathfrak{h}^n \to \mathbb{C}$ that additionally satisfies:*

- *f is holomorphic.*
- *$f|_M^k = f$ for all $M \in \Gamma$.*

The space of such modular forms will hereafter be denoted by $\mathcal{M}(n,k)$.

The condition $n > 1$ makes any growth condition at the cusp unnecessary by a well known principle of Koecher.

Let $Q = MN$ denote the "Siegel parabolic" in $Sp_{2n}(\mathbb{R})$, i.e.,

$$Q = \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array} \right) \right\}, M = \left\{ \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array} \right) \right\}, N = \left\{ \left(\begin{array}{c|c} 1 & * \\ \hline 0 & 1 \end{array} \right) \right\}.$$

For $f \in \mathcal{M}(n,k)$, the invariance of f under the weight k action of the group N yields a Fourier expansion:

$$f(Z) = \sum_T a(T) e^{2\pi i \mathrm{tr}(TZ)}.$$

In the above expansion, T ranges over all semi-integral symmetric $n \times n$ matrices, i.e., those whose entries are in $\frac{1}{2}\mathbb{Z}$, except the diagonal whose entries must be in $\mathbb{Z}$.

Definition 2 *A Siegel modular form $f \in \mathcal{M}(n,k)$ is called singular if whenever a coefficient $a(T) \neq 0$, T is singular, i.e.,* $\det(T) = 0$.

Clearly, the phenomenon of singular modular forms is particular to $n > 1$; the only singular classical modular form is 0. Examples exist in abundance for $n > 1$, as was known to Siegel. Suppose $n > 1$ and $1 \leq r < n$. Let S be a semi-integral positive definite $r \times r$ matrix, which we may view as a quadratic form. For G a $r \times n$ matrix, we write $S[G] = {}^tGSG$. The theta function associated to S is defined by:

$$\theta_S(Z) = \sum_{G \in M_{r \times n}(\mathbb{Z})} e^{2\pi i \mathrm{tr}(S[G]Z)}.$$

Then θ_S is a singular Siegel modular form of weight $r/2$ and of degree n. The arithmetic significance of the functions θ_S is clear by definition – their coefficients count representations of quadratic forms by each other. Many important results about singular modular forms are due to Resnikoff [16] and Freitag [7]; we mention two of particular relevance here:

Theorem 1 *A modular form $f \in \mathcal{M}(n,k)$ is singular if and only if $k < n/2$.*

Theorem 2 *The subspace of $\mathcal{M}(n,k)$ of singular modular forms is spanned by suitable theta functions θ_S.*

We view the first of these theorems as a *recognition* theorem; one can recognize a singular modular form just by looking at its weight. If the weight is less than half of the degree, then the form is singular. We view the second theorem as a *construction* theorem; one may construct all singular modular forms from theta series. We do not spend any more time on the construction theorem now. Rather, we focus on the recognition theorem through the lens of representation theory, as studied by Howe in [11].

Let π_v denote an admissible representation of $Sp_{2n}(\mathbb{Q}_v)$ where v denotes any place of $\mathbb{Q}$. Let $\bar{Q} = M\bar{N}$ denote the opposite parabolic to the Siegel parabolic Q. Then characters of N correspond to elements of the opposite unipotent radical $\bar{N}$, and M orbits on the set of characters ϕ of N correspond to M orbits on $\bar{N}$. The action of M on $\bar{N}$ is just the action of GL_n on the space of n-ary quadratic forms over $\mathbb{Q}_v$. Corresponding to the GL_n-invariant rank of a quadratic form, every character ϕ of N has an associated rank, $0 \leq \mathrm{rank}(\phi) \leq n$. When $v = p$ is a non-archimedean place, we define:

Definition 3 *π_v is small if $(\pi_v)_{N,\phi} \neq 0$ implies that ϕ is singular, i.e., rank$(\phi) < n$. Here the subscript $(\pi_v)_{N,\phi}$ denotes the space of (twisted) co-invariants.*

Later, we say that such a π_v has small N-rank, to prevent any ambiguity about the choice of parabolic $Q = MN$. Smallness of π_v is a local representation-theoretic analogue of singularity of a modular form. There is a similar definition of smallness for v archimedean that we do not require here. The following result of Howe [11] makes an even stronger connection:

Theorem 3 *Let $f \in \mathcal{M}(n,k)$ be a Siegel modular form, and $\pi = \bigotimes_v \pi_v$ be the associated automorphic representation of Sp_{2n}. Then the following are equivalent:*

- *f is singular.*
- *Every local representation π_v is small (under which hypothesis, we say that the global representation π is small).*
- *There exists a place v such that π_v is small.*

From this standpoint, the recognition theorem can be interpreted as follows: a modular form is singular if and only if its representation at infinity is small (this corresponds to the singular weights $k < n/2$). However, it is also possible to recognize a singular modular form by looking at any place, not just the archimedean place. The implications are as follows:

- If one is able to construct a global automorphic representation in such a way that one local factor is small, the entire automorphic representation is small.
- If one understands precisely the small representations at the finite places, then one understands the arithmetic information encoded in small automorphic representations.

The first of these implications is frequently used in the construction of global small representations. Especially by looking at residues at poles of Eisenstein series, one is able to construct global small representations. We view these implications as motivation to study and construct the small representations of p-adic groups in greater generality (for other groups besides Sp_{2n}).

We will be interested in the smooth representations of a split, simple, simply-laced (type A-D-E) group G over a p-adic field k. Our methods rely on the existence of two different parabolic subgroups P and Q having the following properties: the unipotent radical of P is isomorphic to a Heisenberg group, and the unipotent radical of Q is abelian (like the Siegel parabolic of Sp_{2n}). Fix Levi decompositions $P = LH$ and $Q = MN$ of these parabolics over k. If G_1 denotes the derived subgroup of L, then the Fourier-Jacobi map we define will be a functor from the category of smooth representations of G to the category of smooth representations of G_1. The functorial properties of this map allow us to study representations inductively, by reducing representations of G to representations of the smaller group G_1.

This functor is only well-behaved on representations that are already somewhat small. For instance, admissibility is only preserved on a restricted subcategory of representations of G. However, the Fourier-Jacobi map is well-behaved on some degenerate principal series representations $I(s)$ induced from Q. This allows us to explicitly determine the N-rank of spherical subquotients of these principal series.

We outline the contents of this paper here: in the second chapter, we describe the structure theory of the parabolics P and Q as above. This includes a classification of the groups G which we can study, together with parabolics P and Q. Chapter 3 reviews the p-adic analogue of the Stone-von Neumann theorem, completely describing the representations of the Heisenberg groups over a p-adic field.

In Chapter 4, the main definitions and functorial properties of the Fourier-Jacobi map are discussed. In Chapter 5, we describe the effect of the Fourier-Jacobi map on degenerate principal series. Chapter 6 contains a description of the composition series of the degenerate principal series. Finally, in chapter 7, we study the effect of the Fourier-Jacobi map on the rank of representations, finding small representations in degenerate principal series. The appendix contains tables that describe the precise location of reducibility points of degenerate principal series, and thus by the results of this paper, the location of small representations.

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Chapter 2

Structure Theory

Fix a simple split Lie algebra $\mathfrak{g}$ over a field k of characteristic 0, with Cartan subalgebra $\mathfrak{t}$ and Borel subalgebra $\mathfrak{b}$. Suppose that $\mathfrak{g}$ is simply-laced (type A, D, or E). Let Φ be the set of roots for the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$. For any root $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ denote the corresponding one-dimensional root space. Then, viewing Φ as a subset of the dual space $\mathfrak{t}^\vee$, the vector space $E = \sum_{\alpha\in\Phi} \mathbb{Q}\alpha$ has dimension equal to the rank of $\mathfrak{g}$. The Weyl group W of $\mathfrak{g}$, generated by symmetries s_α, $\alpha \in \Phi$, is a finite group that acts irreducibly on E. There is a unique positive-definite W-invariant inner product $\langle \cdot, \cdot \rangle$ on E up to scaling; we will soon normalize this inner product.

The choice of Borel subalgebra $\mathfrak{b}$ corresponds to a partition of Φ into positive and negative roots: $\Phi = \Phi^+ \cup \Phi^-$. If

$$\Delta = \{\alpha_1, \ldots, \alpha_l\}$$

denotes the corresponding root basis for E, consisting of simple roots of Φ^+, then every $\alpha \in \Phi^+$ has a unique expression,

$$\alpha = \sum_{i=1}^{l} m_i(\alpha) \cdot \alpha_i,$$

where the $m_i(\alpha)$ are non-negative integers.

2.1 The Heisenberg Parabolic

Our study of the Heisenberg parabolic borrows heavily from the treatment in Gross and Wallach [10]. The highest root β in Φ^+ is characterized by $m_i(\beta) \geq m_i(\alpha)$ for all $\alpha \in \Phi^+$. We now normalize the inner product on E by setting $\langle \beta, \beta \rangle = 2$. Let w_β denote the corresponding element of the Weyl group of $\mathfrak{g}$, so that

$$w_\beta(\alpha) = \alpha - \langle \alpha, \beta \rangle \beta.$$

We recall the following proposition from [10]:

Proposition 1 *If $\alpha \in \Phi^+$, then $\langle \alpha, \beta \rangle \in \{0, 1, 2\}$. If $\langle \alpha, \beta \rangle = 2$ then $\alpha = \beta$. If $\Sigma^+ = \{\alpha \in \Phi^+ | \langle \alpha, \beta \rangle = 1\}$ then $\alpha \mapsto \beta - \alpha$ is a fixed-point free involution of Σ^+. The cardinality of Σ^+ is an even integer $2d$.*

The highest coroot $\beta^\vee = \langle \cdot, \beta \rangle$ corresponds to a standard parabolic subalgebra whose nilradical is of Heisenberg type. We define this Heisenberg parabolic,

$$\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi, \langle \alpha, \beta \rangle \geq 0} \mathfrak{g}_\alpha.$$

The Heisenberg parabolic has Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{h}$, where

$$\mathfrak{l} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi, \langle \alpha, \beta \rangle = 0} \mathfrak{g}_\alpha,$$

$$\mathfrak{h} = \mathfrak{g}_\beta \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha.$$

The dimension of $\mathfrak{h}$ is equal to $2d+1$, and we call d the structure constant of $\mathfrak{p}$. Define $\mathfrak{g}_1$ to be the derived subalgebra $[\mathfrak{l}, \mathfrak{l}]$.

Then the Weyl group W_L of $\mathfrak{l}$ is precisely the stabilizer of $\mathfrak{g}_\beta$. We recall the further structure of this parabolic from [10]:

Proposition 2 *The algebra $\mathfrak{h}$ is of Heisenberg type, with center $\mathfrak{z} = \mathfrak{g}_\beta$. The quotient $\mathfrak{s} = \mathfrak{h}/\mathfrak{z}$ is abelian of dimension $2d$, and the Lie bracket induces a non-degenerate symplectic pairing $\bigwedge^2 \mathfrak{s} \to \mathfrak{z}$. The elements of $\mathfrak{g}_1$ act trivially on $\mathfrak{g}_\beta$, and thus symplectically on the vector space $\mathfrak{s}$.*

2.2 Parabolics with Abelian Nilradical

Our study of parabolics with abelian nilradical follows the paper by Richardson, Röhrle, and Steinberg [17]. We assume here that $\mathfrak{g} \neq \mathfrak{s}l_2$. To any simple root $\tau \in \Delta$, we associate a maximal parabolic subalgebra $\mathfrak{q}$ by setting

$$\mathfrak{q} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi, m_\tau(\alpha) \geq 0} \mathfrak{g}_\alpha.$$

We fix a Levi decomposition, $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{n}$ with

$$\mathfrak{m} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi, m_\tau(\alpha) = 0} \mathfrak{g}_\alpha,$$

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi, m_\tau(\alpha) > 0} \mathfrak{g}_\alpha.$$

Define

$$\rho_\mathfrak{n} = \frac{1}{2} \sum_{\alpha \in \Phi, m_\tau(\alpha) > 0} \alpha.$$

Then we call $D = \langle \rho_\mathfrak{n}, \tau \rangle$ the structure constant of $\mathfrak{n}$. Now we have the following result from Richardson et al. [17]:

Proposition 3 *The algebra $\mathfrak{q}$ associated to a simple root τ has abelian nilradical if and only if the multiplicity $m_\tau(\beta)$ of τ in β is equal to 1.*

Hereafter, we fix τ with $m_\tau(\beta) = 1$, so that $\mathfrak{n}$ is abelian. Let Ψ^+ (respectively Ψ^0, Ψ^-) denote the set of roots α satisfying $m_\tau(\alpha) > 0$ (resp., $m_\tau(\alpha) = 0$, $m_\tau(\alpha) < 0$).

A sequence $(\beta_0, \ldots, \beta_{r-1})$ of roots is called orthogonal if it is pairwise orthogonal, i.e., $\langle \beta_i, \beta_j \rangle = 0$ for all $i \neq j$. Given an orthogonal sequence $(\beta_0, \ldots, \beta_{r-1})$, we say that r is the length of the sequence. If W_M denotes the Weyl group of $\mathfrak{m}$, then we have the following useful result from [17]:

Proposition 4 *For all $r \geq 1$, W_M acts transitively on the set of orthogonal sequences $(\beta_0, \ldots, \beta_{r-1})$ of length r.*

From this proposition, it makes sense to define the rank of $\mathfrak{n}$ to be the length of a maximal orthogonal sequence $(\beta_0, \ldots, \beta_{r-1})$ of roots. Every such maximal orthogonal sequence is conjugate via the action of W_M, and thus has the same length.

2.3 Interaction between Parabolics

The structural foundation of this paper is the interaction between the two parabolics $\mathfrak{p}$ and $\mathfrak{q}$. We recall that a complete polarization on the Heisenberg algebra $\mathfrak{h}$ is a vector space decomposition: $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{a}' \oplus \mathfrak{z}$, such that $\mathfrak{a}$ and $\mathfrak{a}'$ are isotropic, i.e.,

$$[\mathfrak{a}, \mathfrak{a}] = [\mathfrak{a}', \mathfrak{a}'] = 0.$$

Then $\mathfrak{a}$ and $\mathfrak{a}'$ have the same dimension d. We have the following important structural fact:

Proposition 5 *Our choice of a parabolic subalgebra $\mathfrak{q}$ with abelian unipotent radical allows us to canonically choose a complete polarization of $\mathfrak{h}$. Set $\Sigma^+_{\mathfrak{a}} = \Sigma^+ \cap \Psi^+$ and $\Sigma^+_{\mathfrak{a}'} = \Sigma^+ \cap \Psi^0$. Then we define*

$$\mathfrak{a} = \bigoplus_{\alpha \in \Sigma^+_{\mathfrak{a}}} \mathfrak{g}_\alpha,$$

$$\mathfrak{a}' = \bigoplus_{\alpha \in \Sigma^+_{\mathfrak{a}'}} \mathfrak{g}_\alpha.$$

This gives a complete polarization of $\mathfrak{h}$.

Proof: It is clear from the definition that $\mathfrak{a}$ and $\mathfrak{a}'$ are contained in $\mathfrak{h}$, and their intersection with $\mathfrak{z}$ is trivial. Since every root in Σ^+ is positive, and every positive root is contained in Ψ^+ or Ψ^0, we see that $\Sigma^+ = \Sigma^+_{\mathfrak{a}} \cup \Sigma^+_{\mathfrak{a}'}$. Corresponding to this, we have

$$\mathfrak{a} \oplus \mathfrak{a}' \oplus \mathfrak{z} = \mathfrak{h}.$$

It is left to show that $\mathfrak{a}$ and $\mathfrak{a}'$ are isotropic. But recall that the nilradical $\mathfrak{n}$ is abelian, and $\mathfrak{a} \subset \mathfrak{n}$, so $\mathfrak{a}$ is isotropic. Since $\mathfrak{a}' = \mathfrak{m} \cap \mathfrak{h}$, we know that $[\mathfrak{a}', \mathfrak{a}'] \subset \mathfrak{m} \cap \mathfrak{z} = 0$. Hence $\mathfrak{a}'$ is isotropic.

□

Just as the choice of a root τ with $m_\tau(\beta) = 1$ yields additional structure on the Heisenberg parabolic, the highest root yields additional structure on the parabolic $\mathfrak{q}$.

Note that unless $\mathfrak{p}$ is contained in $\mathfrak{q}$, $\mathfrak{q}_1 = \mathfrak{q} \cap \mathfrak{g}_1$ is a maximal parabolic subgroup of $\mathfrak{g}_1$. $\mathfrak{g}_1$ is not necessarily simple, but there is a unique simple factor of $\mathfrak{g}_1$ which is not contained in $\mathfrak{q}$. Let β_1 denote the highest root of this simple factor, and $\mathfrak{p}_1 = \mathfrak{l}_1 \oplus \mathfrak{h}_1$ the corresponding Heisenberg parabolic. Then $\mathfrak{q}_1$ yields a complete polarization $\mathfrak{a}_1 \oplus \mathfrak{a}_1' \oplus \mathfrak{z}_1 = \mathfrak{h}_1$. Note that $\mathfrak{a}_1$ and $\mathfrak{z}_1$ are contained in $\mathfrak{q}_1$.

Continuing this process inductively yields a decomposition of $\mathfrak{n}$:

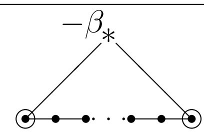
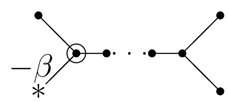
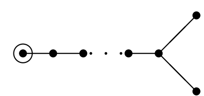
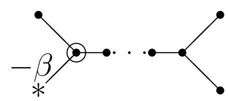
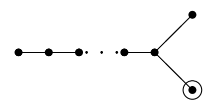
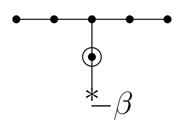
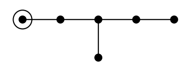
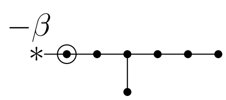
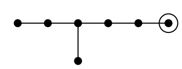
$$\mathfrak{n} = \bigoplus_{i=0}^{r-1} (\mathfrak{a}_i \oplus \mathfrak{z}_i),$$

$$\mathfrak{z}_i = \mathfrak{g}_{\beta_i}.$$

Here $\mathfrak{a}_0 = \mathfrak{a}$, and $\mathfrak{z}_0 = \mathfrak{z}$. The sequence $(\beta_0, \ldots, \beta_{r-1})$ is a maximal orthogonal sequence of roots. Define $d_i = \dim(\mathfrak{a}_i)$. Then $d = d_0$, and more generally d_i is the structure constant of a Heisenberg parabolic of $\mathfrak{g}_i$. Define $D_i = \langle \rho_{\mathfrak{n}_i}, \tau \rangle$, where $\rho_{\mathfrak{n}_i}$ denotes half the sum of the roots occurring in $\mathfrak{n}_i$.

2.4 Classification of Groups and Algebras

For future reference, we tabulate the possible algebras $\mathfrak{g}, \mathfrak{p}, \mathfrak{q}$ by listing the Cartan type of $\mathfrak{g}$, and circling the "deleted simple root(s)" associated to the parabolics $\mathfrak{p}$ and $\mathfrak{q}$.

Cartan type of $\mathfrak{g}$	Vertices for $\mathfrak{p}$ circled	Vertex for $\mathfrak{q}$ circled
A_n		Any maximal parabolic
D_n		
D_n		
E_6		
E_7		

Now, suppose that k is a finite extension of $\mathbb{Q}_p$ for a prime p. Let q denote the cardinality of the residue field of k. Let $\mathcal{O}_k$ denote the ring of integers of k, with uniformizer ϖ. Normalize the valuation on k by setting $v(\varpi) = 1$, and $|\varpi| = q^{-1}$. We fix the Haar measure on k which gives $\mathcal{O}_k$ measure 1.

Corresponding to the simply-laced simple Lie algebras classified here, let G denote a simply-laced, simply-connected, absolutely simple split group over k. Suppose moreover that $G \neq SL_2$. Let T be a maximal split k-torus of G, and B a k-Borel subgroup of G containing T. Let P and Q denote standard parabolic subgroups of G corresponding to a choice of subalgebras $\mathfrak{p}$ and $\mathfrak{q}$ as classified above. Fix Levi decompositions over k: $P = LH$, $Q = MN$, and set $G_1 = [L, L]$. The Heisenberg group H has canonical polarization: $H = AA'Z$, where Z is the center of H, and $AZ = H \cap N$. We say that the triple (G, P, Q) is terminal if P is contained in Q, or equivalently if the rank r of N is equal to 1. Note that we have a decomposition of N:

$$N = \prod_{i=0}^{r-1} A_i Z_i,$$

where $A = A_0$, $Z = Z_0$. We list the possible triples (G, P, Q) below for future reference:

Type A_n We consider the group $G = SL_{n+1}$, with $n > 1$. The Heisenberg parabolic P has unipotent radical H of dimension $2n - 1$. G_1 is isomorphic to SL_{n-1}. Every maximal parabolic has abelian unipotent radical. Thus $Q = MN$ may be any maximal parabolic, and (up to conjugacy) M may equal $S(GL_r \times GL_{n+1-r})$ for any $1 \leq r \leq \lfloor n/2 \rfloor$. The rank of N is equal to r. The triple (G, P, Q) is terminal if $r = 1$.

Type D_n Here there are two possibilities for the triple (G, P, Q). We have $G = Spin_{2n}(k)$, with $n > 3$. Then $G_1 = SL_2(k) \times Spin_{2n-4}(k)$. There are two choices for the parabolic Q with abelian unipotent radical. In the first case, we may choose $Q = MN$ with $M = CSpin_{2n-2}(k)$. The group $CSpin_{2m}$ fits into a short exact sequence:

$$1 \longrightarrow GL_1 \longrightarrow CSpin_{2m} \longrightarrow SO_{2m} \longrightarrow 1.$$

The most convenient construction of $CSpin_{2m}$ is via Clifford algebras; a good treatment can be found in the paper of Deligne, [5].

In the second case, we choose $Q = MN$ with M a double cover of $GL_n(k)$ coming from the square root of the determinant character.

Type E_6 We take $G = E_6(k)$ to be the simply connected simple group of type E_6, in which $G_1 = SL_6(k)$. There are two maximal parabolics with abelian unipotent radical; both have Levi factor M isomorphic to $CSpin_{10}(k)$.

Type E_7 We take $G = E_7(k)$ to be the usual split simply-connected group, in which $G_1 = Spin_{12}(k)$. Then $Q = MN$ with $M = CE_6(k)$ as its Levi factor. CE_6 is defined to be the quotient of $E_6 \times GL_1$ obtained by identifying the center $\mathbb{Z}/3\mathbb{Z}$ of E_6 with the subgroup μ_3 of third roots of unity in GL_1.

Let w_β denote the root reflection associated to β in the Weyl group of the pair G, T. Let σ_β denote a representative for w_β in the normalizer of T. Later, we will use the following structural result:

Proposition 6 *If $a \in A$, then $\sigma_\beta a \sigma_\beta^{-1} \in M$.*

Proof: The adjoint action of σ_β sends the root subgroup of $\alpha \in \Phi$ to that of:

$$w_\beta[\alpha] = \alpha - \langle \alpha, \beta \rangle \beta.$$

Hence if $\alpha \in \Sigma_{\mathfrak{a}}^+$, so $m_\tau(\alpha) = 1$, and $\langle \alpha, \beta \rangle = 1$, we have

$$\begin{aligned} m_\tau(w_\beta[\alpha]) &= m_\tau(\alpha - \langle \alpha, \beta \rangle \beta) \\ &= m_\tau(\alpha) - \langle \alpha, \beta \rangle m_\tau(\beta) \\ &= 1 - 1 \cdot 1 = 0. \end{aligned}$$

Thus $w_\beta[\alpha] \in \Psi^0$. The desired result follows directly.

□

Chapter 3

Representations of the Heisenberg group

In this section, we study the irreducible smooth representations of the Heisenberg group H of dimension $2d+1$ over k, and in particular the p-adic analogue of the Stone-von Neumann theorem. This appears countless times in the literature, with many different applications. In [12], one finds applications to dual reductive pairs and the Θ-distribution. One encounters a detailed treatment of the representation theory of H in the articles of Mœglin-Vigneras-Waldspurger [4], and van Dijk [19]. Here we review some mostly well-known results about the Heisenberg group and its representations. The proofs given are heavily influenced by existing literature, and personal communications with G. Savin.

Let $\underline{H}$ denote a smooth group scheme over $\mathcal{O}_k$, with center $\underline{Z}$, such that $H = \underline{H}(k)$ and $Z = \underline{Z}(k)$. Suppose $\underline{H}$ is moreover endowed with a complete polarization over $\mathcal{O}_k$: $\underline{H} = \underline{AA'Z}$. Let $L = \underline{A}(\mathcal{O}_k)$, $L' = \underline{A}'(\mathcal{O}_k)$, and $C = \underline{Z}(\mathcal{O}_k)$ denote the resulting lattices in A, A', and Z. Fix a non-trivial smooth (and thus unitary) character ψ of Z, such that $\ker(\psi) = C$. We study the smooth representations of H of central character ψ.

We begin with the representation theory of the abelian group AZ. The pairing $A \times A' \to \mathbb{C}^\times$ given by $(a, a') \mapsto \psi([a, a'])$ allows us to identify A' with the unitary dual $\hat{A}$ of A. Since $AZ \cong A \times Z$, the set of characters of AZ which restrict to ψ on Z can be identified naturally with $\hat{A} = A'$. For any element $a' \in A'$, let $\tilde{\psi}_{a'}$ denote the character of AZ whose restriction to Z is ψ, obtained by this identification. Thus $\tilde{\psi}_1$ is the character of $AZ \cong A \times Z$ given by $1 \otimes \psi$.

If $\tilde{\psi}$ is any character of AZ extending ψ on Z, and V is any smooth representation of AZ, then we let $V(AZ, \tilde{\psi})$ denote the linear span of the vectors $pv - \tilde{\psi}(p)v$ for all $p \in AZ$, $v \in V$. Let $V_{AZ,\tilde{\psi}} = V/V(AZ, \tilde{\psi})$ denote the quotient space, which is the maximal quotient of V on which AZ acts via the character $\tilde{\psi}$. The following result is a consequence of sections 5.10 through 5.17 of Bernstein and Zelevinsky [3]:

Proposition 7 *Let V be a smooth representation of AZ, on which Z acts via the character ψ. There exists a smooth sheaf (in the language of [3] an l-sheaf) $\mathcal{F}$ on A' whose stalk $\mathcal{F}_{\dashv'}$ over $a' \in A'$ is isomorphic to the space of co-invariants $V_{AZ,\tilde{\psi}_{a'}}$. Moreover, as a representation of AZ, V is isomorphic to the space of smooth sections of $\mathcal{F}$ with compact support.*

Fix a representation $\tilde{\psi}$ extending ψ to AZ. We study a representation of H obtained by smooth induction,

$$(\rho_{\tilde{\psi}}, W_{\tilde{\psi}}(A)) = \mathrm{I}nd_{AZ}^{H}(\tilde{\psi}).$$

In other words, $W_{\tilde{\psi}}(A)$ is the space of functions $w \colon H \to \mathbb{C}$ satisfying the following conditions:

- There exists a compact open subgroup $K \subset H$ such that $w(hk) = w(h)$ for all

$h \in H$, $k \in K$.

- $w(azh) = \tilde{\psi}(az)w(h)$ for all $a \in A$, $z \in Z$.

In fact, the smoothness (the first of the above conditions) implies that functions $w \in W_{\tilde{\psi}}(A)$ are compactly supported modulo AZ. The action $\rho_{\tilde{\psi}}$ of H on $W_{\tilde{\psi}}(A)$ is by right-translation. Then $(\rho_{\tilde{\psi}}, W_{\tilde{\psi}}(A))$ is a smooth representation of H. We have the following application of Frobenius reciprocity:

$$\mathrm{H}om_H(V, W_{\tilde{\psi}}(A)) = \mathrm{H}om_{AZ}(V_{AZ,\tilde{\psi}}, \mathbb{C}_{\tilde{\psi}}).$$

It is easy to see that the representation $W_{\tilde{\psi}}(A)$ is independent of the chosen extension $\tilde{\psi}$ of ψ, up to isomorphism. Replacing the function $w(h)$ by $w(a'h)$ for various $a' \in A'$ yields an isomorphism from $W_{\tilde{\psi}_1}(A)$ to $W_{\tilde{\psi}_{a'}}(A)$. Therefore, it makes sense to write $(\rho_\psi, W_\psi(A))$ for the representation, without specifying the extension $\tilde{\psi}$ used. We can prove:

Proposition 8 *The representation $(\rho_\psi, W_\psi(A))$ is the unique smooth irreducible representation of H with central character ψ.*

Proof: Suppose that V is a smooth irreducible representation of H with central character ψ. Then there exists a smooth character $\tilde{\psi}$ of AZ extending ψ on Z, such that $V_{AZ,\tilde{\psi}} \neq 0$ by Proposition 2.1. The action of A' on such characters $\tilde{\psi}$ is transitive, so that $V_{AZ,\tilde{\psi}} \neq 0$ for all $\tilde{\psi}$. It follows now from Frobenius reciprocity that V is a submodule of $W_\psi(A)$.

For the character $\tilde{\psi}_{a'}$ extending ψ, the space $W_\psi(A)(AZ, \tilde{\psi}_{a'})$ can be naturally identified with $C_c^\infty(A' \backslash \{a'\})$. Hence the co-invariants $W_\psi(A)_{AZ,\tilde{\psi}_{a'}}$ form a one-dimensional vector space. Since $V_{AZ,\tilde{\psi}} \neq 0$ for all $\tilde{\psi}$, we see that $V_{AZ,\tilde{\psi}} = W_\psi(A)_{AZ,\tilde{\psi}}$. If

$V' = W_\psi(A)/V$, then by exactness of taking co-invariants, we see that $V'_{AZ,\tilde{\psi}} = 0$ for all $\tilde{\psi}$. By Proposition 2.1, this implies that $V' = 0$, and we are done.

□

We introduce the lattice model of the representation W_ψ in order to prove that it is injective. Define the subgroup K_0 of H by

$$K_0 = LL'Z.$$

Then ψ extends to a character ψ_0 of K_0, letting $\psi_0(L) = \psi_0(L') = 1$, since ψ is trivial on $[L, L'] \subset C$. We define the representation of H,

$$W_\psi(K_0) = \mathbb{C}[H] \otimes_{\mathbb{C}[K_0]} \mathbb{C}_{\psi_0}.$$

By $\mathbb{C}[H]$ and $\mathbb{C}[K_0]$, we mean the naïve group rings over $\mathbb{C}$. In the above tensor product, we view $\mathbb{C}[H]$ as a left $\mathbb{C}[K_0]$-module, and view the entire space $W_\psi(K_0)$ as a H-module via the right multiplication action of H on $\mathbb{C}[H]$. Thus $W_\psi(K_0)$ is spanned by elements of the form αh for $\alpha \in \mathbb{C}, h \in H$, subject to the identifications $k_0 h = \psi_0(k_0)h$ for all $k_0 \in K_0$, and subject to the H-action: $h'[\alpha h] = \alpha h h'$. Then for any smooth H-module, we have the usual form of Frobenius reciprocity:

$$\mathrm{H}om_H(W_\psi(K_0), V) = \mathrm{H}om_{K_0}(\mathbb{C}_{\psi_0}, V).$$

We can now show the following:

Proposition 9 *The representation $W_\psi(K_0)$ is isomorphic to $W_\psi(A)$.*

Proof: First, we note that the (K_0, ψ_0)-co-invariants of $W_\psi(K_0)$ are one-dimensional.

The kernel of the projection $W_\psi(K_0) \to W_\psi(K_0)_{K_0,\psi_0}$ is by definition the linear span:

$$\begin{aligned} \mathrm{S}pan\{hk_0 - \psi(k_0)h\} &= \mathrm{S}pan\{hk_0h^{-1}h - \psi(k_0)h\} \\ &= \mathrm{S}pan\{\psi(hk_0h^{-1})h - \psi(k_0)h\}. \end{aligned}$$

The projection annihilates h unless $\psi(hk_0h^{-1}) = \psi(k_0)$ for all $k \in K_0$, in which case $h \in K_0$. Therefore the image of the projection is precisely $\mathbb{C}[K_0] \otimes_{\mathbb{C}[K_0]} \mathbb{C}_{\psi_0} = \mathbb{C}_{\psi_0}$, and is one-dimensional.

The set of characters of K_0 that extend ψ on Z can be identified with the discrete quotient H/K_0. If $\bar{h} \in H/K_0$, then we write $\psi_{\bar{h}}$ for the corresponding character of K_0. Moreover, if V is a smooth representation of K_0, on which Z acts by ψ, then we may decompose V as a direct sum:

$$V = \bigoplus_{\bar{h} \in H/K_0} V_{K_0,\psi_{\bar{h}}}.$$

Since H acts transitively on H/K_0, we see that if V is a smooth representation of H, then the dimension of $V_{K_0,\psi_{\bar{h}}}$ is independent of $\bar{h}$. Hence, as in the proof of Proposition 2.2, we see that $W_\psi(K_0)$ is irreducible. By uniqueness, it is isomorphic to $W_\psi(A)$.

$\square$

Hereafter, we write W_ψ for the unique smooth irreducible representation of H of central character ψ, when we only care about the isomorphism class.

Corollary 1 *The category $Rep(H,\psi)$ of smooth representations of H with central character ψ is equivalent to the category of (complex) vector spaces. Specifically, the functor $\mathrm{H}om_H(W_\psi, \cdot)$ is exact on $Rep(H,\psi)$, and any object W of $Rep(H,\psi)$ satisfies*

$$W \cong \mathrm{H}om_H(W_\psi, W) \otimes_{\mathbb{C}} W_\psi.$$

Proof: The functor $\mathrm{H}om_H(W_\psi, \cdot)$ is isomorphic to the functor $\mathrm{H}om_{K_0}(\mathbb{C}_{\psi_0}, \cdot)$ by Frobenius reciprocity. This functor is exact on the category of representations of central character ψ since K_0 is compact modulo its center. Thus $\mathrm{H}om_H(W_\psi, \cdot)$ is an exact functor from $Rep(H, \psi)$ to the category of vector spaces.

Finally, consider the canonical map, defined on a spanning subset as follows:

$$\begin{array}{rcl} \mathrm{H}om(W_\psi, W) \otimes_{\mathbb{C}} W_\psi & \to & W \\ \sigma \otimes w & \mapsto & \sigma(w). \end{array}$$

Upon taking the (K_0, ψ_0)-co-invariants, and applying Frobenius reciprocity, the above map yields a morphism of K_0 representations:

$$\mathrm{H}om(\mathbb{C}_{\psi_0}, W_{K_0,\psi_0}) \otimes_{\mathbb{C}} \mathbb{C}_{\psi_0} \to W_{K_0,\psi_0}.$$

This is trivially an isomorphism. The same holds, replacing ψ_0 by any other character of K_0 extending ψ. Therefore the original map is an isomorphism, by decomposing the source and target into its K_0-isotypic quotients.

□

The representation $W_\psi(A)$ of H has a natural unitary structure, which is most easily constructed with the Schrödinger realization. Namely, a function $w \in W_\psi(A)$ is determined by its restriction to $A' \subset H$, which is a smooth compactly-supported function. Given two elements $w, w' \in W_\psi(A)$, we use this restriction to define an inner product (up to a choice of Haar measure):

$$\langle w, w' \rangle = \int_{A'} w(a') \overline{w'(a')} da'.$$

It is not difficult to show that the action of H respects this inner product, so $W_\psi(A)$ is unitarizable.

Let G be a group as in the first section, with Heisenberg parabolic $P = LH$, and $G_1 = [L, L]$. Let $G^J = G_1 H$. We quote the following result from Kazhdan and Savin [13]:

Proposition 10 *Suppose that (G, P, Q) is a triple as classified in the first section, so that G is simply-laced, $G \neq SL_2$. The representation $(\rho_\psi, W_\psi(A))$ of H extends uniquely to a smooth representation $(\tilde{\rho}_\psi, W_\psi)$ of G^J, which again respects the inner product above.*

This should be contrasted with the non-simply-laced case; for instance, when G is a symplectic group, the representation W_ψ extends only after passing to a metaplectic cover of G_1. It is difficult to explicitly write down the extension $\tilde{\rho}_\psi$, however it is possible to describe the action of $\tilde{\rho}_\psi(q_1)$ on $W_\psi(A)$ when $q_1 \in G_1$ stabilizes the polarization $H = AA'Z$. In this case, there is a natural adjoint action of q_1 on the vector space $W_\psi(A)$ given by:

$$[Ad(q_1)w](h) = w(q_1 h q_1^{-1}).$$

Proposition 11 *For any $q_1 \in G_1$ that stabilizes the polarization $AA'Z$, the endomorphism $Ad(q_1) \circ \tilde{\rho}_\psi(q_1)$ of $W_\psi(A)$ is equal to the scalar $\delta_1(q_1)^{1/2}$, where δ_1 is the multiplier for the action of q_1 on the Haar measure of A (or equivalently of A').*

Proof: First, we can see that $Ad(q_1) \circ \tilde{\rho}_\psi(q_1)$ commutes with the action of H. For we have:

$$[Ad(q_1) \circ \tilde{\rho}_\psi(q_1) \circ \tilde{\rho}_\psi(h) w](x) = [Ad(q_1) \circ \tilde{\rho}_\psi(q_1 h q_1^{-1}) \circ \tilde{\rho}_\psi(q_1) w](x)$$

$$
\begin{aligned}
&= [\tilde{\rho}_\psi(q_1)w](q_1 x q_1^{-1} q_1 h q_1^{-1}) \\
&= [\tilde{\rho}_\psi(q_1)w](q_1 x h q_1^{-1}) \\
&= [\tilde{\rho}_\psi(h) \circ Ad(q_1) \circ \tilde{\rho}_\psi(q_1)w](x).
\end{aligned}
$$

Therefore $Ad(q_1) \circ \tilde{\rho}_\psi(q_1)$ is an endomorphism of $W_\psi(A)$ that commutes with the action of H. By irreducibility, this implies that $Ad(q_1) \circ \tilde{\rho}_\psi(q_1)$ is a scalar. To determine the scalar, it suffices to look at the action of $Ad(q_1) \circ \tilde{\rho}_\psi(q_1)$ on the norm $\langle w, w \rangle$ of a non-zero $w \in W_\psi(A)$. Since $\tilde{\rho}_\psi(q_1)$ preserves the unitary structure, it therefore suffices to study the action of $Ad(q_1)$ on the norm. It is easy to check that $\langle Ad(q_1)w, Ad(q_1)w \rangle = \delta_1(q_1)\langle w, w \rangle$. Hence the action of $Ad(q_1) \circ \tilde{\rho}_\psi(q_1)$ is via a scalar $\eta(q_1)\delta_1(q_1)^{1/2}$, where η is a unitary character of Q_1. To determine this character η, we use a trick suggested by G. Savin.

There exists a simple root γ and an associated map ϕ_γ from SL_2 into G_1 defined over k, such that $M_1 = [M_1 M_1]T_\gamma$, where T_γ is the image of the usual maximal torus of SL_2 in G_1. Thus the character η of $Q_1 = M_1 N_1$ is determined by its values on T_γ. Let S_γ denote the image of SL_2 under ϕ_γ. Then, following the techniques in the proof of Theorem 2 of [13], there exists a different polarization, $H = A_\gamma A'_\gamma Z$, which is preserved by S_γ. Corresponding to this polarization, there is another representation $W(A_\gamma)$ of $G_1 H$, and an intertwining operator between W and $W(A_\gamma)$ given by a (partial) Fourier transform. From this realization, we see that S_γ acts via an adjoint action on functions in $W(A_\gamma)$ modulo a character again. However, S_γ is simple, so this character is trivial, and moreover, restricting this character to T_γ yields the character η, so η is trivial.

□

Chapter 4

The Fourier-Jacobi Map

Let G be a simple, simply-laced, simply-connected split group over k, with parabolics $P = LH, Q = MN$ as classified in the first section. Let $Sm(G)$ (respectively $Sm(G_1)$) denote the category of smooth representations of G (resp., G_1) on complex vector spaces. Suppose that $(\pi, V) \in Sm(G)$. Define $G^J = G_1 H$. Since G^J centralizes Z, G^J acts naturally on the space of co-invariants $V_{Z,\psi}$. Denote this representation by $\pi_{Z,\psi}$. Consider now the vector space

$$V_1 = \mathrm{H}om_H(W_\psi, V_{Z,\psi}).$$

Then G_1 acts on elements σ of V_1 by setting

$$[g\sigma](w) = \pi_{Z,\psi}(g)\sigma(g^{-1}w).$$

Note that we use the unique extension of (ρ_ψ, W_ψ) to a representation of G^J. This yields a representation (π_1, V_1) of G_1. We have constructed a map, which we call the

Fourier-Jacobi map:

$$\begin{aligned} FJ\colon Sm(G) &\to Sm(G_1) \\ (\pi, V) &\mapsto (\pi_1, V_1) \end{aligned}$$

Remark 1 *In the study of automorphic forms on symplectic groups, numerous authors have studied the "Fourier-Jacobi" expansion, for instance Eichler and Zagier in [6]. The representation theory underlying this expansion has been studied more recently, and one may consult Berndt and Schmidt in [2]. Our functor FJ is the representation theoretic analogue of the Fourier-Jacobi expansion, where we work with simply-laced groups rather than the symplectic group.*

It is not difficult to prove the following:

Proposition 12 *FJ is an exact functor from $Sm(G)$ to $Sm(G_1)$. It is independent of the choice of non-trivial character ψ.*

Proof: FJ is an exact functor because it the composition of two exact functors. The exactness of taking co-invariants is well-known (the theory of Jacquet modules). The exactness of the functor $\mathrm{H}om(W_\psi, \cdot)$ is discussed in the previous section.

We may identify the characters of Z with elements of the opposite subgroup $\bar{Z} \in G$. Since we assume $G \neq SL_2$, the Levi component L of P acts transitively on the non-trivial elements of $\bar{Z}$ (this is a consequence of the existence of a single bond to the highest root in the extended Dynkin diagram). Hence if ψ_1, ψ_2 are non-trivial characters of Z, there exists an element $l \in L$ satisfying $\psi_1(z) = \psi_2(lzl^{-1})$. Now if $v = \pi(z)w - \psi_1(z) \cdot w \in V(Z, \psi_1)$, we have

$$\pi(l)v = \pi(l)\pi(z)w - \pi(l)\psi_1(z) \cdot w,$$

$$= \pi(lzl^{-1})(\pi(l)w) - \psi_1(z)(\pi(l)w),$$
$$= \pi(lzl^{-1})(\pi(l)w) - \psi_2(lzl^{-1})(\pi(t)w).$$

Hence $\pi(l)$ maps $V(Z,\psi_1)$ isomorphically onto $V(Z,\psi_2)$. Therefore $\pi(l)$ yields a vector space isomorphism from V_{Z,ψ_1} to V_{Z,ψ_2}. Similarly, l yields a vector space isomorphism from W_{ψ_1} to W_{ψ_2} in such a way that we have an isomorphism

$$\mathrm{H}om_H(W_{\psi_1}, V_{Z,\psi_1}) \simeq \mathrm{H}om_H(W_{\psi_2}, V_{Z,\psi_2}).$$

□

The precise relation between (π_1, V_1) and the representation $(\pi_{Z,\psi}, V_{Z,\psi})$ is given by the following:

Proposition 13 *The representation* $(\pi_{Z,\psi}, V_{Z,\psi})$ *decomposes as a tensor product:*

$$(\pi_{Z,\psi}, V_{Z,\psi}) = (\pi_1, V_1) \otimes (\rho_\psi, W_\psi).$$

Proof: Consider the canonical map $V_1 \otimes W_\psi \to V_{Z,\psi}$ given by $(\sigma, w) \mapsto \sigma(w)$. This is clearly a G^J-equivariant map, with the natural action of G^J on $V_1 \otimes W_\psi$ via $g(\sigma \otimes w) = \pi_1(g)\sigma \otimes \rho_\psi(g)w$ for all $g \in G^J$. Note that we are letting G^J act on V_1 by having H act trivially. This canonical map is an isomorphism by Corollary 1.

□

Chapter 5

Degenerate Principal Series

In this section, we apply the Fourier-Jacobi map to degenerate principal series representations of the group G, induced from the parabolic Q. If (G, P, Q) is not terminal (i.e., Q does not contain P), then recall that $Q_1 = M_1 N_1 = G_1 \cap Q$ is a maximal parabolic subgroup of G_1 with abelian unipotent radical. We let W, W_L, W_M denote the Weyl groups of G, L, and M. We choose a representative w_β in G for the root reflection in the highest root β.

5.1 Character identities

Let T denote the maximal torus of G as before, $T_1 = T \cap G_1$, and let

$$X(T) = \mathrm{H}om(T, k^\times)$$

be the weight space. If $\nu \in X(T)$ is W_M-invariant then ν yields a character $\chi_\nu \colon M \to \mathbb{R}^\times$ satisfying

$$\chi_\nu(t) = |\nu(t)|.$$

The map $\nu \mapsto \chi_\nu$ extends further to a homomorphism

$$(X(T) \otimes_{\mathbb{Z}} \mathbb{R})^{W_M} \to \mathrm{H}om(M, \mathbb{R}^\times),$$

by setting $\chi_{s\nu}(m) = \chi_\nu(m)^s$ for all $s \in \mathbb{R}$, $\nu \in X(T)^{W_M}$. The same facts hold for M_1 instead of M.

Let ρ (respectively ρ_1) denote half of the sum of the roots occurring in N (resp., N_1). We recall that the structure constant D of N is given by $D = \langle \rho, \tau \rangle$, where τ is the simple root associated to Q, and $D_1 = \langle \rho_1, \tau \rangle$. It is easy to see that

$$\rho \in (X(T) \otimes_{\mathbb{Z}} \mathbb{R})^{W_M},$$

and similarly for ρ_1. Let $\nu = \langle \rho, \tau \rangle^{-1} \rho$ and $\nu_1 = \langle \rho_1, \tau \rangle^{-1} \rho_1$. Let $\chi = \chi_\nu$ and $\chi_1 = \chi_{\nu_1}$ be the associated characters of M and M_1. If $t \in T_1$, then it is not hard to show that

$$\chi(t_1) = \chi_1(t_1).$$

Hence χ agrees with χ_1 on M_1.

Let $\Delta \colon Q \to \mathbb{R}^\times$ denote the modular character corresponding to the adjoint action of Q on N. Similarly define $\Delta_1 \colon Q_1 \to \mathbb{R}^\times$ for the adjoint action of Q_1 on N_1. Q_1 also acts by conjugation on the polarizing subgroup AZ. Let $\delta_1 \colon Q_1 \to \mathbb{R}^\times$ denote the corresponding modular character. We will use the following identities among characters:

$$\begin{aligned} \Delta(q_1) &= \Delta_1(q_1)\delta_1(q_1), \\ \chi(q_1) &= \chi_1(q_1), \\ \Delta(q)^{\frac{1}{2D}} &= \chi(q), \end{aligned}$$

$$\Delta_1(q)^{\frac{1}{2D_1}} = \chi_1(q_1).$$

5.2 Bruhat filtration on principal series

We now define the degenerate principal series:

Definition 4 *Let $I(s)$ denote the space of all functions $f\colon G \to \mathbb{C}$ satisfying the following conditions:*

- *There exists a compact open subgroup $K \subset G$ such that $f(gk) = f(g)$ for all $g \in G, k \in K$.*
- *For all $m \in M$, $n \in N$, $g \in G$, we have $f(mng) = \chi(m)^s \Delta(m)^{1/2} f(g)$.*

Then the action of G on $I(s)$ by right-translations is a smooth representation of G. We define the smooth representation $I_1(s)$ of G_1 identically, inserting the subscript 1 wherever possible.

We will study the action of the Fourier-Jacobi map on these degenerate principal series in a few steps. Recall the Bruhat decomposition $G = \bigsqcup_w QwP$ as w ranges over a set of representatives for the double cosets $W_M \backslash W / W_L$. Motivated by this decomposition, and the work of Bernstein and Zelevinsky [3], we define a subspace of the degenerate principal series for any $w \in W$:

$$I^w(s) = \{ \quad f \in I(s) | Supp(f) \subset QwP, \quad f \text{ compactly supported modulo } Q\}.$$

By a dimension argument, we see that $I^w(s) = 0$ unless $w = w_\beta$, in which case $I^{w_\beta}(s)$ consists of those functions supported in the big cell $Qw_\beta P$, and compactly supported modulo Q. We require the following structural lemma:

Lemma 1 *The subgroup wZw^{-1} is not contained in Q if and only if $W_M w W_L = W_M w_\beta W_L$.*

Proof: Since $w_\beta Z w_\beta$ corresponds to the root space $\mathfrak{g}_{-\beta}$, $w_\beta Z w_\beta^{-1}$ is not contained in Q. We are left to check that no other double coset in $W_M \backslash W / W_L$ sends Z outside of Q. Recall that the Weyl group W_L of L is precisely the subgroup of the Weyl group that stabilizes Z. Hence a single coset wW_L is uniquely determined in W/W_L by wZw^{-1}, or equivalently by the root $w(\beta)$. Moreover, the Weyl group W_M of M acts transitively on the root spaces in the unipotent radical N and its opposite N^-. From this we see that if wZw^{-1} does not lie in Q, then there exists $w' \in W_M$ such that $w'w(\beta) = w_\beta(\beta)$. Hence w is in the same double-coset in $W_M \backslash W / W_L$ as w_β. This verifies uniqueness.

□

From this lemma, we deduce the following useful result:

Proposition 14 *The co-invariants of $I(s)$ are determined by those on the big cell,*

$$I(s)_{Z,\psi} = I^{w_\beta}(s)_{Z,\psi}.$$

Proof: Recall that $I(s)(Z, \psi)$ is generated by functions of the form $f'(g) = f(gz) - \psi(z)f(g)$ for functions $f \in I(s)$. Therefore, if $f \in I(s)$, then in the quotient $I(s)_{Z,\psi}$, we have

$$f(g) \equiv f(g) - \frac{1}{1 - \psi(z)} \left[f(gz) - \psi(z) f(g) \right].$$

Now if $wZw^{-1} \subset Q$, then $f(gz) = f(g)$ for all $f \in I(s)$, $z \in Z$, and $g \in QwP$. By our last lemma, this implies that $f(gz) = f(g)$ whenever g is not contained in the big cell $Qw_\beta P$. From this we see that the right hand side of the above equation is supported on the big cell when $z \notin \ker(\psi)$. It is also compactly supported modulo Q, since f itself is. Therefore every function $f \in I(s)$ is congruent to a function in $I^{w_\beta}(s)$ modulo $I(s)(Z,\psi)$. This proves our proposition.

□

5.3 The effect of Fourier-Jacobi

Theorem 4 *The Fourier-Jacobi map sends $I(s)$ to $I_1(s)$. In other words, there is a natural intertwining operator that identifies the representations $(I(s))_1$ and $I_1(s)$.*

Proof: By previous results, it suffices to prove that $I^{w_\beta}(s)_{Z,\psi} \simeq I_1(s) \otimes W_\psi$ as representations of G^J. We introduce a new representation of G^J by defining

$$Y = \{F \in C^\infty(G^J) | \quad F(q_1 j) = \chi(q_1)^s \Delta(q_1)^{1/2} F(j), \quad F(azj) = \psi(z)F(j), \quad F(j) \text{ is compactly supported modulo } Q_1 AZ\}.$$

We prove our theorem now in two steps: first we show that $I^{w_\beta}(s)_{Z,\psi} \simeq Y$, by writing an explicit intertwining operator.

Begin with a function $f \in I^{w_\beta}(s)$. Then for fixed $j \in G^J$, $f(w_\beta jz)$ is compactly supported in the variable $z \in Z$. Therefore we may define a function F on G^J by setting

$$F(j) = \int_Z f(w_\beta jz)\overline{\psi(z)}dz.$$

This yields a G^J-intertwining homomorphism from $I^{w_\beta}(s)$ to Y. By standard techniques in the theory of Jacquet's functors, we see that the kernel of $f \mapsto F$ is precisely the space $I^{w_\beta}(s)(Z,\psi)$. To see that the image is all of Y, note that one can recover f from F by setting

$$f(q_1 w_\beta j) = \chi(q_1)^s \Delta(q_1)^{1/2} F(j),$$

for all $q_1 \in Q_1, j \in G^J$. Note here that $Q w_\beta G^J = Q w_\beta P$, so this is sufficient. Therefore $I^{w_\beta}(s)_{Z,\psi}$ is isomorphic to Y.

Second, we show that $I_1(s) \otimes W_\psi$ is also isomorphic to Y. Suppose that $f_1 \in I_1(s)$ and $w \in W_\psi$, and define

$$F(g_1 h) = f_1(g_1)[\tilde{\rho}(g_1) w](g_1 h g_1^{-1}),$$

for all $g_1 \in G_1$ and $h \in H$. Then F is a function on G^J, and $f_1 \otimes w \mapsto F$ is a G^J-intertwining homomorphism from $I_1(s) \otimes W_\psi$ to Y. The fact that $F \in Y$ follows from the character identities of the previous section, together with Proposition 11. It is not hard to check that $f_1 \otimes w \mapsto F$ is an injective G^J-intertwining homomorphism. Surjectivity follows from the identification of $C^\infty(G_1 \times H)$ with $C^\infty(G_1) \otimes C^\infty(H)$. Thus $I_1(s) \otimes W_\psi$ is isomorphic to Y also, and we are done.

□

Chapter 6

Reducibility Points and Composition Series

6.1 Lengths of degenerate principal series

In this section, we consider the reducibility points of the degenerate principal series $I(s)$. The following lemma applies the Fourier-Jacobi map to study the length of $I(s)$ at a reducibility point:

Lemma 2 *Suppose that the length of $I(s)$ is greater than the length of $I_1(s)$. Then $s = \pm D$, the trivial representation occurs as a Jordan Hölder component of $I(s)$, and the length of $I(s)$ is exactly one more than the length of $I_1(s)$.*

Proof: Let L, L_1 denote the lengths of $I(s)$ and $I_1(s)$ respectively. If $L > L_1$, then the Fourier-Jacobi map must kill $L - L_1$ components of $I(s)$. By Corollary 3 (which does not depend on this Lemma), the trivial representation of G is the only one killed by the Fourier-Jacobi map. Hence the trivial representation occurs with multiplicity $L - L_1$ in $I(s)$. Thus the U-co-invariants $I(s)_{U,1}$ contain the trivial representation of T as a subquotient. But Bernstein and Zelevinsky [3] precisely describe the U-co-invariants, to show that $I(s)_{U,1}$ is "glued" from the components

$$F_w = (\chi^s \circ w)\Delta^{1/2},$$

where w ranges over the set $W^{M,T} = W_M \backslash W$. Recall that $\Delta^{1/2} = \chi^D$ from Chapter 5.1. In particular, we see that the trivial representation occurs only when $s = \pm D$, and only once at these values of s. Hence $s = \pm D$, and $L - L_1 = 1$.

□

This lemma implies the following bound on the length of $I(s)$:

Corollary 2 *The length of $I(s)$ is less than or equal to 2.*

Proof: We prove the statement of the corollary, and also that reducibility points occur for $|s| \leq D$ inductively on the rank of N. As a base step, we suppose first that (G, P, Q) is a terminal triple, or equivalently that the rank of N is equal to 1. Then the Fourier-Jacobi map sends every $I(s)$ to a character of G_1. Hence the only reducibility points of $I(s)$ are when $s = \pm D$, and the length at these reducibility points is equal to 2 by the previous lemma. Now we inductively assume that our corollary is true for $I_1(s)$. Then all reducibility points of $I(s)$ occur for values of s where $I_1(s)$ is reducible, or when $s = \pm D$. Clearly $D_1 < D$, so our previous lemma implies that the length at these reducibility points is less than or equal to 2.

□

6.2 Reducibility Points via L-functions

The Fourier-Jacobi map provides an upper bound on the length of degenerate principal series: $I(s)$ has at most 2 Jordan-Hölder components. However, we would like to show that $I(s)$ is reducible whenever allowed by the Fourier-Jacobi map. In other words, we would like to know that when $I_1(s)$ is reducible, so is $I(s)$. In this section, we use the results of Shahidi and Muić [15] to completely relate the reducibility points of the degenerate principal series to the vanishing of local L-functions for the Steinberg representation. G. Savin has remarked that it is possible to locate the reducibility points of $I(s)$ in a simpler, or at least more self-contained, fashion by using intertwining operators; this has been accomplished for GL_{2n} in the work of Kudla and Sweet [14]. However, the deep results of Shahidi and Muić yield an exceptional and efficient tool to study reducibility points.

The principal result we use is an immediate consequence of Proposition 3.3 and its subsequent remarks in Shahidi and Muić [15]:

Proposition 15 *Let G, Q, M, N satisfy our usual hypotheses: G is split, simple, simply-laced, and Q a maximal parabolic with abelian unipotent radical N and Levi M. Let M^L denote the dual Levi factor to M in the Langlands dual G^L, with its action on the unipotent radical N^L of the dual parabolic. Call the differentiated action $r\colon M^L \to Aut(\mathfrak{n}^L)$, so that r is irreducible (since N^L is abelian). Let St_M denote the Steinberg representation of M. Then for $s > 0$, $I(s)$ is reducible (and so is $I(-s)$) if and only if*

$$L(1-s, St_M, r)^{-1} = 0.$$

From the theory of intertwining operators, we know that when $s > 0$, $I(s)$ has a unique spherical quotient; symmetrically, when $s < 0$, $I(s)$ has a unique spherical subrepresentation.

To compute these local L-functions, we work on the side of representations of the Weil group W_k of k (one may consult the article of Tate [18] for details about Weil groups). Let $\alpha\colon W_k \to SL_2$ be the homomorphism

$$\alpha(w) = \begin{pmatrix} \|w\|^{1/2} & 0 \\ 0 & \|w\|^{-1/2} \end{pmatrix}.$$

Let $\phi\colon SL_2 \to M^L$ denote the principal homomorphism as in the paper of Gross [9]. Then $\sigma = \phi \circ \alpha\colon W_k \to M^L$ is the Langlands parameter of the Steinberg representation St_M. More explicitly, we may compute in the split case

$$\sigma(w) = \rho_M(\|w\|),$$

where ρ_M denotes half the sum of the positive roots of M, viewed as coroots of M^L. Now let

$$R = r \circ \phi\colon SL_2 \longrightarrow GL(\mathfrak{n}^L)$$

denote the adjoint action of the principal SL_2 on the Lie algebra $\mathfrak{n}^L$ of N^L. The representation $R \circ \alpha$ of the Weil group comes with the structure of a "Weil-Deligne" representation, i.e., it comes with a unipotent element u of $GL(\mathfrak{n}^L)$. This element is simply

$$u = R\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let $\mathfrak{n}_u^L = \{X \in \mathfrak{n} | Ad(u)X = 0\}$. Since

$$\alpha(Fr) = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix},$$

for a geometric Frobenius element Fr the local L-function $L(s, St_M, r)$ may be explicitly written as

$$L(s, St_M, r) = \det \left(1 - R \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} q^{-s} |\mathfrak{n}_u^L \right)^{-1}.$$

This allows us to compute reducibility points via the following proposition:

Proposition 16 *Suppose that the representation R of the principal SL_2 on $\mathfrak{n}^L$ breaks up into irreducible representations as*

$$R = \bigoplus_i Sym^{t_i},$$

for some set of non-negative integers t_i (here of course Sym^t denotes the t-th symmetric power of the standard representation). Then the local L-function may be computed;

$$L(s, St_M, r) = \prod_i \left(1 - q^{-s-\frac{t_i}{2}}\right)^{-1}.$$

Hence the reducibility points of the degenerate principal series $I(s)$ occur precisely at the points

$$s = \pm(1 + \frac{t_i}{2}).$$

Proof: Let (Sym^t, S^t) denote the t-th symmetric power representation of SL_2. Then it is not hard to check that the space

$$S_u^t = \{X \in S^t | Sym^t \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} X = X\}$$

is one-dimensional. Moreover, one may check directly that this subspace is an eigenspace for the action of $Sym^t \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}$, with eigenvalue equal to $q^{-t/2}$. Our proposition now follows directly from decomposing $\mathfrak{n}^L$ into irreducible representations of the form S^{t_i}, and noting that $\mathfrak{n}_u^L = \bigoplus_i S_u^{t_i}$. The reducibility points can be computed from the previous proposition.

□

We tabulate for future reference the groups and representations considered above in many cases:

G	M	r	R
SL_n	$S(GL_i \times GL_j)$	$Stand_i \otimes Stand_j$	$Sym^{i-1} \otimes Sym^{j-1}$
$Spin_{2n}$	$CSpin_{2n-2}$	$Stand_{2n-2}$	$Sym^{2n-4} \oplus Sym^0$
$Spin_{2n}$	$\widetilde{GL}_n$	$\bigwedge^2(Stand_n)$	$\bigwedge^2(Sym^{n-1})$
E_6	$CSpin_{10}$	$\frac{1}{2}$-spin representation	$Sym^{10} \oplus Sym^4$
E_7	CE_6	27-dimensional representation	$Sym^{16} \oplus Sym^8 \oplus Sym^0$

We can use this table together with our previous proposition to tabulate the reducibility points of $I(s)$ for various triples (G, P, Q). This is carried out in the appendix. For now, we just observe (checking case-by-case) that every reducibility point of $I_1(s)$ is also a reducibility point of $I(s)$.

Chapter 7

The Rank of Representations

7.1 N-rank

Fix a representation $(\pi, V) \in Sm(G)$, and consider a character ϕ of N. Suppose N has rank r. The Levi component M of Q acts on such characters in the obvious manner; if $m \in M$, then we set $[m\phi](n) = \phi(mnm^{-1})$. From this action, we have vector space isomorphisms

$$V(N,\phi) \to V(N, m\phi), v \mapsto \pi(m)v$$

for any $m \in M$. Therefore, we see that the N, ϕ co-invariants of V essentially depend only on the M-orbit of ϕ; explicitly, we have

$$V_{N,\phi} = 0 \Longleftrightarrow V_{N,m\phi} = 0.$$

The M-orbits of characters of N correspond to the M-orbits of the opposite nilradical $\bar{N}$. Richardson et al. [17] show that over an algebraically closed field, there is exactly one orbit for each integer $s \leq r$. The integer s associated to an orbit is called the rank of the orbit. The rank of a character of N is defined to be the rank of its $M(\bar{k})$-orbit. In our case, we work with simply connected groups and over the base field k. We may describe the M orbits on $\bar{N}$ with the following:

Proposition 17 *Though the orbit of $M(\bar{k})$ on $\bar{N}(\bar{k})$ of rank r may split into multiple orbits of M on N, there is exactly one orbit of M on N of each rank $s < r$. Equivalently, there is exactly one M-conjugacy class of characters of N of rank less than r.*

Proof: In Proposition 2.13 of [17], it is shown that every element $\bar{n} \in \bar{N}$ is conjugate to an element of the form $\prod_{i=0}^{s-1} u_{-\beta_i}$ for some $s \leq r$, where $u_{-\beta_i}$ are contained in root subgroups associated to an orthogonal sequence $(-\beta_0, \ldots, -\beta_{s-1})$ of roots in $\bar{N}$. This particular statement and its proof in [17] carry over to our case without changes. Moreover, by conjugation by elements of the Weyl group of M, every element $\bar{n} \in \bar{N}$ is conjugate to a product $\prod_{i=0}^{s-1} u_{-\beta_i}$ for the canonical sequence of roots β_i. Thus every character of N is M-conjugate to a character of the form $\prod_{i=0}^{s-1} \psi_i$ for non-trivial characters ψ_i of the root subgroup of Z_i of β_i. Here, we recall the decomposition of N:

$$N = \prod_{i=0}^{r-1} A_i \times Z_i,$$

and consider $\prod_{i=0}^{s-1} \psi_i$ as a character of N in the obvious way. Now, we apply induction on the rank of N. When N has rank 1, the assertion is trivial. Now suppose that $\phi = \prod_{i=0}^{s-1} \psi_i$ is a character of rank s of N, and $0 < s < r$. Then $\phi_1 = \prod_{i=1}^{s-1} \psi_i$ is a character of N_1 of rank $s-1 < r-1$. Hence ϕ_1 is conjugate via $M_1 = M \cap G_1$ to any given character of N_1 of rank $s-1$ by our inductive hypothesis. Since G_1 centralizes Z, conjugation of ϕ by M_1 doesn't change ψ_0. Since G is simply-laced and $r > 1$, so $G \neq SL_2$, T acts transitively on the characters of Z. Therefore, conjugation by TM_1 is transitive on characters of the form $\psi_0 \cdot \prod_{i=1}^{s-1} \psi_i$ and we are done.

□

The following definition is a direct generalization of the work of Howe in [12].

Definition 5 *A smooth representation* (π, V) *of* G *has* N*-rank* s *if* $s < r$ *and* s *is the maximal integer such that for some (and therefore every) character* ϕ_s *of rank* s*,*

$$V_{N,\phi_s} \neq 0.$$

We follow the convention that if $V = 0$*, then the* N*-rank of* (π, V) *is* -1*. If there exists a character* ϕ_r *of rank* r *such that* $V_{N,\phi_r} \neq 0$*, then we say that the* N*-rank of* (π, V) *is* r*.*

It is not too surprising from this definition that there is a close relationship between N-rank of a smooth representation and the Fourier-Jacobi map. A useful result is the following:

Proposition 18 *Let* (π, V) *be a smooth representation of* G*. Let* (π_1, V_1) *be the smooth representation of* G_1 *obtained via the Fourier-Jacobi map. Then if* s *denotes the* N_1*-rank of* (π_1, V_1)*, then the* N*-rank of* (π, V) *is* $s+1$*. In other words, the Fourier-Jacobi map decreases* N*-rank by 1.*

Proof: Fix non-trivial characters ψ_i of Z_i for $0 \leq i \leq r-1$. We know from before that $V_{Z,\psi_0} = V_1 \otimes W_{\psi_0}$. If V_1 has N_1-rank s, we know that

$$\begin{aligned} (V_1)_{\prod_{i=1}^t Z_i, \prod_{i=1}^t \psi_i} &= 0 \text{ if } t > s, \\ (V_1)_{\prod_{i=1}^t Z_i, \prod_{i=1}^t \psi_i} &\neq 0 \text{ if } t = s \end{aligned}$$

Even if $t = 0$, the above expression makes sense, and just reads $V_1 = 0$. Here we note that $(Z_1, \ldots, Z_t)$ is a sequence of orthogonal root subgroups of length t in $G_1 \cap N = N_1$. Now the action of Z_i on W_ψ is trivial for all $i \geq 1$. Hence the (Z_i, ψ_i) co-invariants of W_ψ are trivial for all $i \geq 1$. We can deduce

$$\begin{aligned} V_{\prod_{i=0}^t Z_i, \prod_{i=0}^t \psi_i} &= (V_{Z,\psi})_{\prod_{i=1}^t Z_i, \prod_{i=1}^t \psi_i} \\ &= (V_1 \otimes W_\psi)_{\prod_{i=1}^t Z_i, \prod_{i=1}^t \psi_i} \end{aligned}$$

If $t > s$ then this last quantity is equal to zero, and if $t = s$ then this last quantity is non-zero. Therefore V has N-rank $s + 1$.

□

We may now completely characterize smooth representations of N-rank 0 as follows:

Corollary 3 *Given a smooth irreducible representation (π, V) of a split group G, the following conditions are equivalent:*

1 (π, V) has N-rank equal to 0.

2 The Fourier-Jacobi map kills (π, V), i.e., $V_1 = 0$.

3 The representation (π, V) is one-dimensional.

Proof: The equivalence of (1) and (2) follows directly from our previous proposition. So suppose that (π, V) is one-dimensional. Then the subgroup Z of G must act on V by some character ψ_0. Choose some $\psi \neq \psi_0$, so that

$$V(Z, \psi) = Span\{\pi(z)v - \psi(z) \cdot v | \text{ for all } z \in Z, v \in V\},$$

$$= Span\{\psi_0(z) \cdot v - \psi(z) \cdot v | \text{ for all } z \in Z, v \in V\},$$
$$= V.$$

We have $V_{Z,\psi} = V/V(Z,\psi) = 0$. Therefore the Fourier-Jacobi map kills (π, V).

Conversely, suppose that (π, V) is smooth and irreducible, and killed by the Fourier-Jacobi map. Then Z acts trivially on the vector space V. Now the Weyl group W of G acts transitively on the root subgroups, since we assume G is simply-laced. Hence if Z acts trivially on V, the representation (π, V) is irreducible even upon restriction to a maximal torus T. Irreducible smooth representations of T are all one-dimensional since T is abelian, so we are done.

□

7.2 Local Character Expansion

Traditionally, the size of a representation of a p-adic group has been studied via the local character expansion. If (π, V) is a smooth admissible representation of G, then its character Θ_π is an invariant distribution on G. Let $\mathcal{O}$ denote the (finite) set of nilpotent orbits in the Lie algebra $\mathfrak{g}$ of G. There exist canonical functions $\hat{\mu}_{\mathfrak{o}}$ on the regular set for each $\mathfrak{o} \in \mathcal{O}$ (independent of (π, V)), and constants $c_{\mathfrak{o}}$ for each $\mathfrak{o} \in \mathcal{O}$ (depending on (π, V)), such that there exists an open neighborhood Ξ of 0 in $\mathfrak{g}$ on which

$$\Theta_\pi(\exp X) = \sum_{\mathfrak{o} \in \mathcal{O}} c_{\mathfrak{o}} \hat{\mu}_{\mathfrak{o}}(X),$$

for all $X \in \Xi$.

The set $\mathcal{O}$ carries a natural partial ordering by setting $\mathfrak{o} \leq \mathfrak{o}'$ if $\mathfrak{o}$ is contained in the closure of $\mathfrak{o}'$. One way of measuring the size of the representation (π, V), is by looking at the support of the constants $c_{\mathfrak{o}}$, called the wave front set:

$$WF(\pi, V) = \{\mathfrak{o} \in \mathcal{O} | c_{\mathfrak{o}} \neq 0\}.$$

For some representations (π, V), the support of the local character expansion is automatically small. For instance, suppose that $Q = MN$ is a maximal parabolic subgroup, and N is abelian as before. Let $\mathfrak{o}_N$ denote the stable Richardson orbit, i.e., the intersection of the Richardson orbit over $\bar{k}$ with $\mathfrak{g}(k)$, and suppose that π is a representation induced from a character of M. Then it follows that

$$WF(\pi, V) \subset \{\mathfrak{o} \in \mathcal{O} | \mathfrak{o} \subset \overline{\mathfrak{o}_N}\}.$$

In other words the wave front set of (π, V) is contained in the closure of the stable Richardson orbit.

In this case, there is a close connection between N-rank and the local character expansion. Let $\mathcal{O}_N \subset \mathcal{O}$ denote the partially ordered subset of $\mathcal{O}$, consisting of nilpotent orbits contained in the closure of the stable Richardson orbit $\mathfrak{o}_N$. Then $\mathcal{O}_N$ is totally ordered, until the Richardson orbit; we can write

$$\mathcal{O}_N = \{0, \mathfrak{o}_1, \ldots, \mathfrak{o}_{r-1}\} \cup \{\mathfrak{o} \subset \mathfrak{o}_N\}$$

where $\mathfrak{o}_i < \mathfrak{o}_j$ precisely when $i < j$. The constant r is the aforementioned rank of N. The above analysis of $\mathcal{O}_N$ follows from Proposition 17. The following result will be useful:

Proposition 19 *Suppose that (π, V) is a smooth representation of finite length, and $WF(\pi, V)$ is contained in $\mathcal{O}_N$. Then $WF(\pi_i, V_i)$ is contained in $\mathcal{O}_N$ for every Jordan-*

Hölder component (π_i, V_i) *of* (π, V).

Proof: Suppose that, to the contrary, there exists a nilpotent orbit $\mathfrak{o} \notin \mathcal{O}_N$, such that $\mathfrak{o}$ is contained in the wave front set of some component (π_i, V_i). By the finiteness of $\mathcal{O}$, we may assume that $\mathfrak{o}$ is maximal with this property. Let $c^i_{\mathfrak{o}}$ denote the coefficient of $\hat{\mu}_{\mathfrak{o}}$ in the character expansion of (π_i, V_i). Then for any i, either $c^i_{\mathfrak{o}}$ is equal to 0, or $c^i_{\mathfrak{o}}$ is positive, since by maximality and results of Mœglin-Waldspurger [8], it is the dimension of a certain space of degenerate Whittaker models. At least one $c^i_{\mathfrak{o}}$ must be non-zero, by definition of $\mathfrak{o}$, therefore

$$c_{\mathfrak{o}} = \sum_i c^i_{\mathfrak{o}} > 0.$$

This is a contradiction, since $WF(\pi, V) \subset \mathcal{O}_N$, and we are done.

□

For representations whose wave front set is contained in $\mathcal{O}_N$, we can connect N-rank with the local character expansion of representations:

Proposition 20 *Suppose that* (π, V) *is an admissible representation, such that* $WF(\pi, V) \subset \mathcal{O}_N$. *Then* (π, V) *has* N*-rank equal to* s *with* $0 \leq s < r$ *if and only if* $WF(\pi, V) \subset \{0, \mathfrak{o}_1, \ldots, \mathfrak{o}_s\}$.

Proof: By a result of Mœglin and Waldspurger [8], we see that the maximal element of the wave front set of (π, V) is $\mathfrak{o}_s$ if and only if $V_{n,\phi_t} = 0$ for every character ϕ_t of rank $t > s$ (in their language, this is the non-existence of a degenerate Whittaker model). Our proposition follows directly from this.

□

7.3 Small representations at reducibility points

The trivial representation occurs as a subrepresentation of $I(s)$ at $s = -D$. Since $s = -D$ is the smallest value of s for which $I(s)$ is reducible, we call $-D$ the first reducibility point. The next smallest value of s where $I(s)$ is reducible is $-D_1$, and so $-D_1$ is called the second reducibility point, and so on. We know from before that $I(s)$ has length two at a reducibility point; the following result describes an irreducible subrepresentation:

Proposition 21 *If $s < 0$ is the n^{th} reducibility point of $I(s)$, then the unique irreducible spherical subrepresentation V_n of $I(s)$ has N-rank $n-1$.*

Proof: At the n^{th} reducibility point, there is a subrepresentation V_n of $I(s)$, such that the Fourier-Jacobi map applied $n-1$ times sends V to the trivial representation. The subrepresentation V_n is spherical, since $s < 0$ (when $s > 0$, $I(s)$ has spherical quotient instead). Since the Fourier-Jacobi map decreases N-rank by 1 each time, and the trivial representation has N-rank 0, the proposition follows.

□

From the results of the previous section, we may re-interpret the above result in terms of the wave front set.

Corollary 4 *Let V_n denote the unique spherical subrepresentation of $I(s)$ at the n^{th} reducibility point. Then the leading term of $WF(V_n)$ is $\mathfrak{o}_{n-1}$. In particular V_2, which appears as the unique spherical subrepresentation of $I(-D_1)$, is a minimal representation in the sense that the leading term of the wave front set is $\mathfrak{o}_1$.is $\mathfrak{o}_1$.*

Appendix A

Fourier-Jacobi Towers

We begin with a triple (G, P, Q), and plot the reducibility points of $I(s)$ as s varies. We then plot the reducibility points of $I_1(s)$, for (G_1, P_1, Q_1) and so on until we arrive at a terminal triple $(G_{r-1}, P_{r-1}, Q_{r-1})$. These "Fourier-Jacobi towers" are included in the appendix. We list the groups G, M by their Cartan type for simplicity.

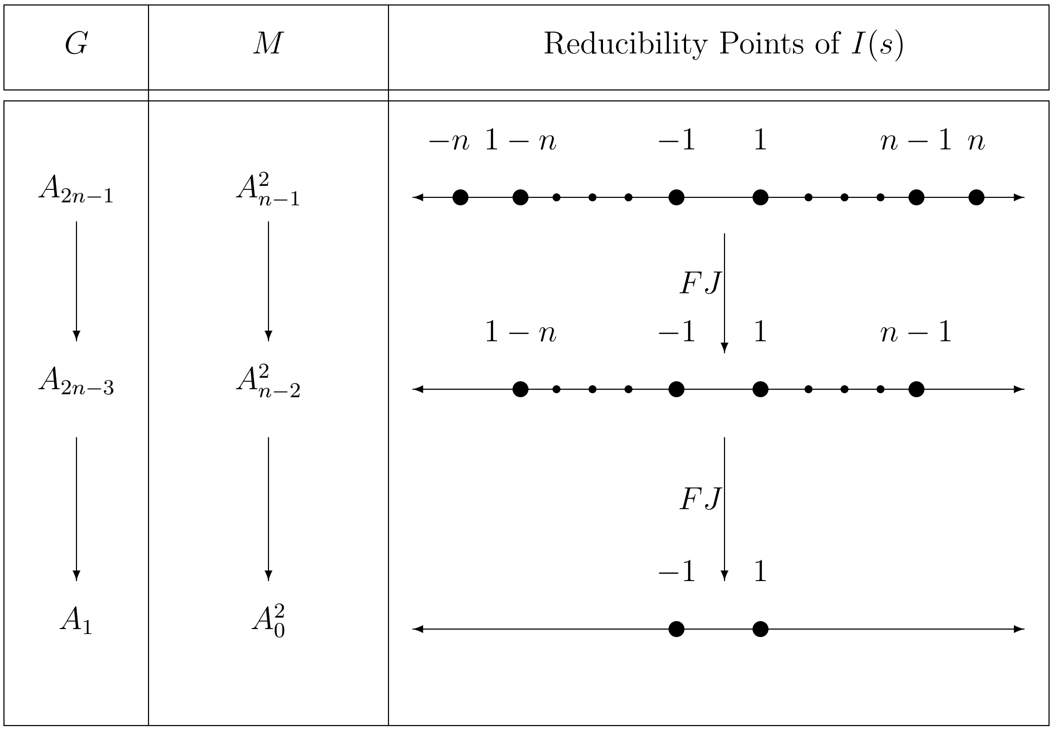


Figure A.1: Fourier-Jacobi Tower for $G = SL_{2n}$, $M = S(GL_n^2)$

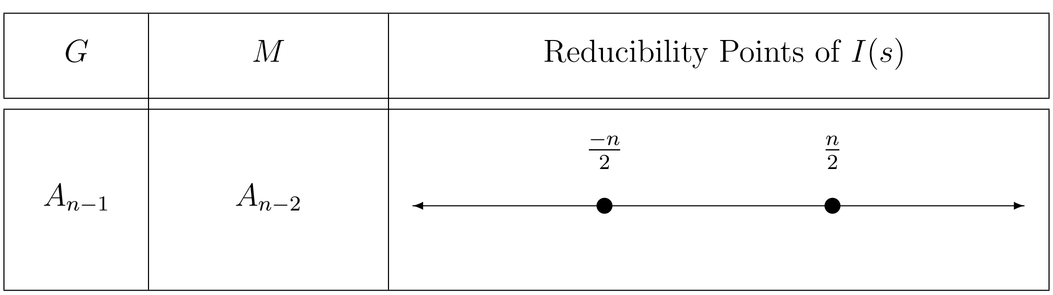


Figure A.2: Fourier-Jacobi Tower for $G = SL_n$, $M = S(GL_1 \times GL_{n-1})$

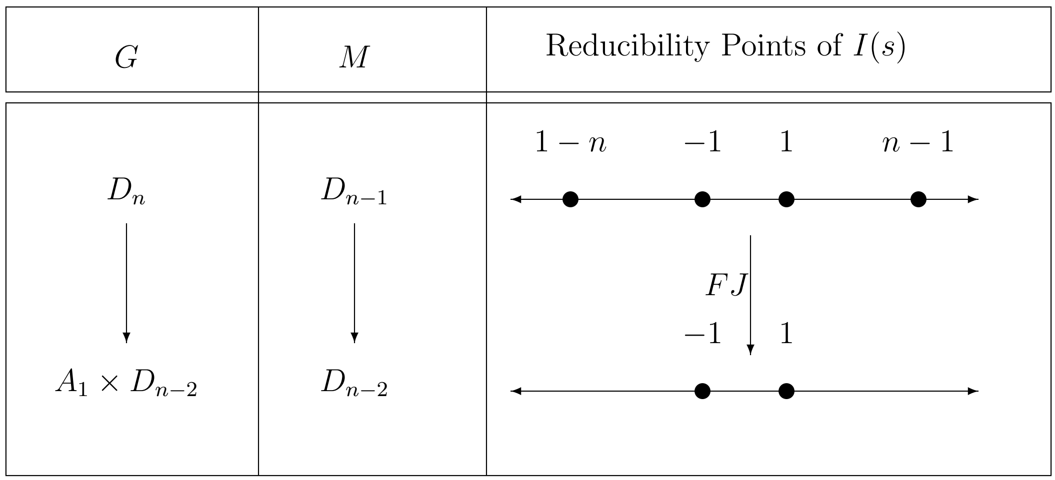


Figure A.3: Fourier-Jacobi Tower for $G = Spin_{2n}$, $M = CSpin_{2n-2}$

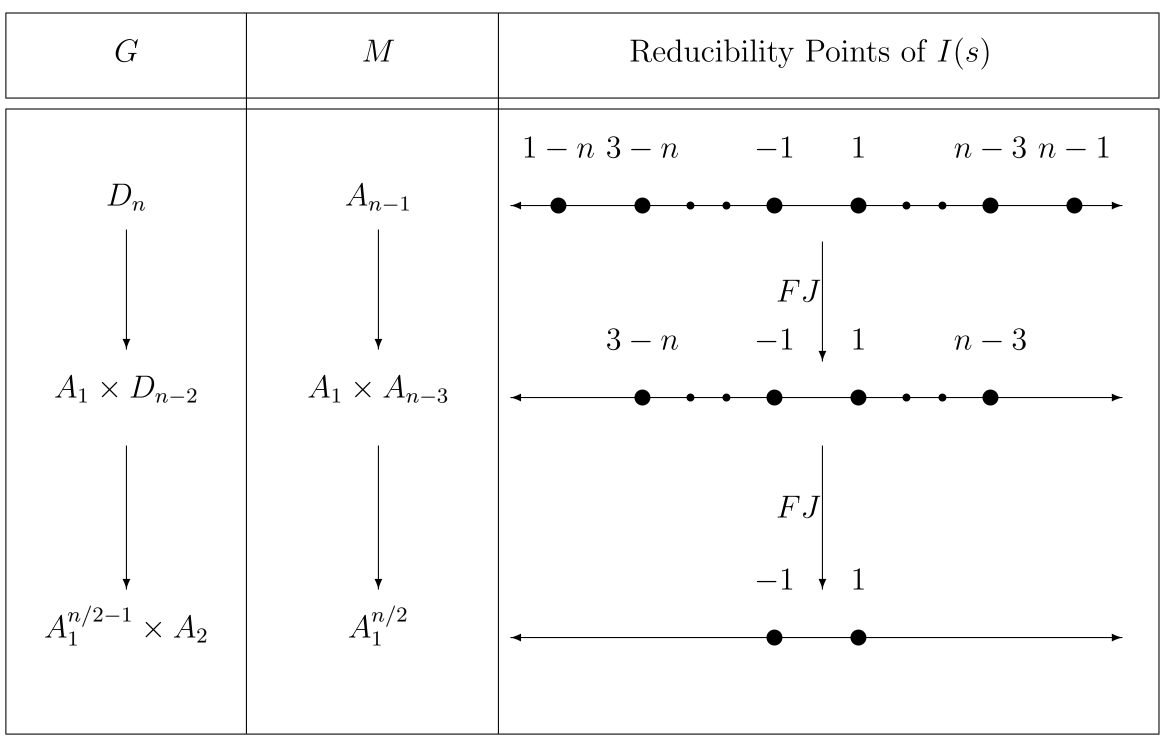


Figure A.4: Fourier-Jacobi Tower for $G = Spin_{2n}$, $M = \widetilde{GL}_n$

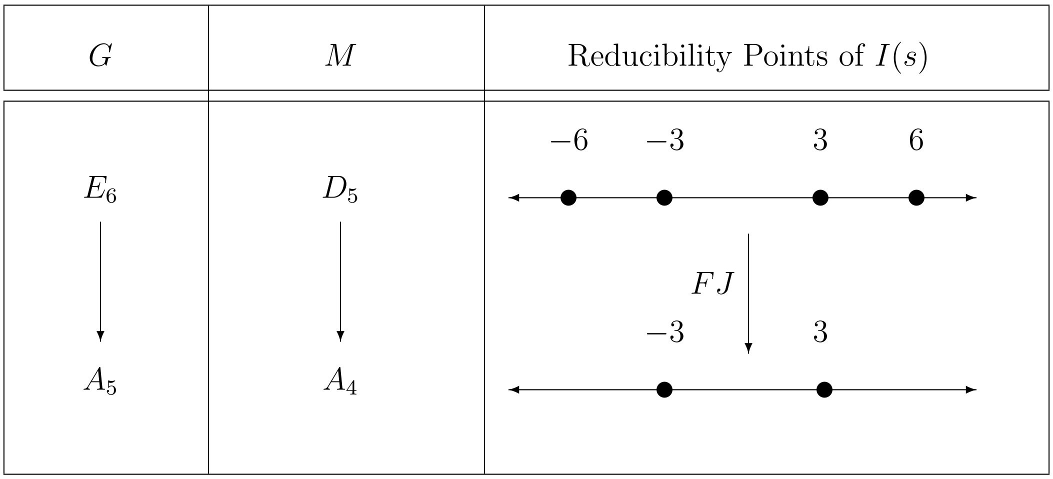


Figure A.5: Fourier-Jacobi Tower for $G = E_6$, $M = CSpin_{10}$

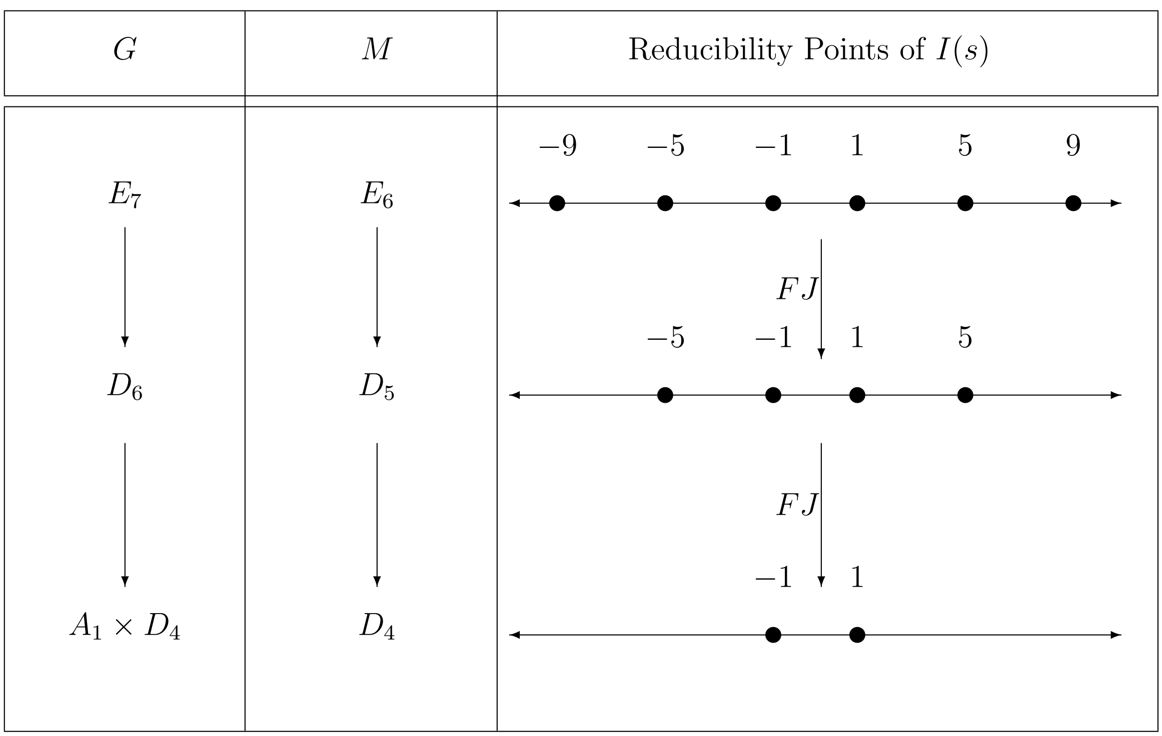


Figure A.6: Fourier-Jacobi Tower for $G = E_7$, $M = CE_6$

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