1. [25 points] Determine whether each statement is true or false, and explain why.
   
   (a) A homogeneous linear system of equations in three variables and three equations has a unique solution.
   (b) If $T : V \rightarrow W$ is a linear transformation with $\dim(V) = \dim(W)$, then $T$ is an isomorphism.
   (c) The vector spaces $M_{2\times2}(\mathbb{R})$ and $\mathbb{R}_3[x]$ are isomorphic.
   (d) If $A$ is a square matrix whose eigenvalues are 1, 2, and 4, then $A$ is invertible.
   (e) If $T : \mathbb{R}^5 \rightarrow \mathbb{R}_2[x]$ is a linear transformation and $\text{nullity}(T) = 2$, then $T$ is surjective.

2. [30 points] Determine if the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ is diagonalizable. If $A$ is diagonalizable, then find a diagonal form and a diagonalizing matrix of $A$. If $A$ is not diagonalizable, explain why not.
3. [25 points] Let $V = \{(a + 2b + 1) + (a - 1)x + (b + 1)x^2 \mid a, b \in \mathbb{R}\}$. Show that $V$ is a subspace of $\mathbb{R}_2[x]$ and compute its dimension.

4. [20 points] Define $T : \mathbb{R}_2[x] \rightarrow M_{2 \times 3}(\mathbb{R})$ by $T(a + bx + cx^2) = \begin{pmatrix} a + c & c + b & 0 \\ 2a + b + 3c & 0 & -(c + b) \end{pmatrix}$. Show that $T$ is a linear transformation, then find a basis for the range of $T$. 

Rob Carman 2 University of California, Santa Cruz
5. [20 points] Suppose $T$ is an operator on $\mathbb{R}_2[x]$ whose representing matrix with respect to the basis $B = (1 - x + x^2, 1 + x, 1 + x + x^2)$ is $M_T(B, B) = \begin{pmatrix} 7 & 2 & 3 \\ 2 & 3 & -6 \\ -8 & 1 & 2 \end{pmatrix}$. Find $T(2 - 2x + x^2)$.

6. [15 points] The sequence $(1 - x + 2x^3, 2 + x^2)$ is a linearly independent sequence in $\mathbb{R}_3[x]$. Expand it to a basis of $\mathbb{R}_3[x]$. 
7. [25 points] Suppose $A, B$, and $C$ are all $3\times 3$ matrices. Furthermore, suppose $\det(A) = -2$, $\det(B^2) = 9$, and 
\[
\begin{pmatrix}
1 \\
0 \\
0 
\end{pmatrix} \in \text{Null}(C). \text{ Compute the following determinants: } \det(3A), \det(BAB), \det(CA), \det(A^3), \det(A^{-1}).
\]

8. [20 points] Determine whether or not the linear transformation $T : M_{2\times 2}(\mathbb{R}) \rightarrow \mathbb{R}^4$ defined by $T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + 2c \\ c - 2a \\ b - d \\ 3d - 3b \end{pmatrix}$ is an isomorphism. Explain why or why not.
9. [20 points] Find a basis for the kernel of the linear transformation $S : \mathbb{R}^4 \rightarrow \mathbb{R}_2[x]$ defined by $S \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = a - b + (b - c)x + (d - c)x^2$. Then determine whether or not $S$ is surjective.