Worksheet 10.4.
Absolute and Conditional Convergence

1. Show that \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2 + 1} = \frac{1}{3} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} + \frac{5}{16} - \ldots \) converges conditionally.

2. Determine if the series \( \sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^3 + 1} \) converges absolutely, conditionally, or not at all.

3. Determine if the series \( \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \) converges absolutely, conditionally, or not at all.
Solutions to Worksheet 10.4

1. Show that \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2 + 1} = \frac{1}{3} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} + \frac{5}{16} - \ldots \) converges conditionally.

   We first show that the series converges, using Leibniz Test for Alternating Series.

   The terms \( a_n = \frac{n}{n^2 + 1} \) tend to zero since \( \lim_{n \to \infty} \frac{n}{n^2 + 1} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n^2}} = 0 \). We show that \( a_n \) forms a decreasing sequence, by showing that the function \( f(x) = \frac{x}{x^2 + 1} \) is decreasing on \( x \geq 1 \). We differentiate \( f \):

   \[
   f'(x) = \frac{1 \cdot (x^2 + 1) - x \cdot 2x}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}
   \]

   We see that \( f'(x) < 0 \) for \( x > 1 \), so \( f \) is decreasing for \( x > 1 \). By the continuity at \( x = 1 \), we conclude that \( f \) is decreasing on \( x \geq 1 \). This implies that the sequence \( \{a_n\} \) is decreasing, that is:

   \( a_1 \geq a_2 \geq a_3 \geq \ldots \)

   We now apply Leibniz Test to conclude that the given alternating series converges.

   To prove conditional convergence, we must show that the positive series \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \) diverges. We do so using the Limit Comparison Test, with the divergent harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \). This gives:

   \[
   \lim_{n \to \infty} \frac{\frac{n}{n^2 + 1}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^2}{n^2 + 1} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n^2}} = \frac{1}{1 + 0} = 1 \neq 0
   \]

   Thus, the positive series \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \) also diverges.

   Since the given series converges but diverges absolutely, this series converges conditionally.

2. Determine if the series \( \sum_{n=1}^{\infty} \frac{(-1)^{n}n^4}{n^3 + 1} \) converges absolutely, conditionally, or not at all.

   We compute the limit:

   \[
   \lim_{n \to \infty} \frac{n^4}{n^3 + 1} = \lim_{n \to \infty} \frac{n}{1 + \frac{1}{n^3}} = \infty
   \]

   It follows that the general term \( \frac{(-1)^{n}n^4}{n^3 + 1} \) of the series does not tend to zero, hence this series diverges by the Divergence Test.
3. Determine if the series \( \sum_{n=1}^{\infty} \frac{\sin n}{n^2} \) converges absolutely, conditionally, or not at all.

The positive series is \( \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \). Since \( \frac{|\sin n|}{n^2} \leq \frac{1}{n^2} \) for \( n \geq 1 \), the Comparison Test and the convergence of the \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) imply that the series \( \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \) converges. Hence, the given series is absolutely convergent.