Worksheet 10.1.
Sequences

1. Calculate the first four terms of the sequence $b_n = \cos \pi n$, starting with $n = 1$.

2. Calculate the first four terms of the sequence $b_n = 2 + (-1)^n$, starting with $n = 1$.

3. Use Theorem 1 to determine the limit of the sequence $b_n = \frac{3n + 1}{2n + 4}$ or state that the sequence diverges.

4. Use Theorem 1 to determine the limit of the sequence $c_n = 4(2^n)$ or state that the sequence diverges.
5. Determine the limit of the sequence \( y_n = \frac{e^n}{2^n} \) or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).

6. Determine the limit of the sequence \( a_n = \frac{\sqrt{n}}{\sqrt{n} + 4} \) or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).

7. Determine the limit of the sequence \( b_n = e^{n^2-n} \) or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).

8. Determine the limit of the sequence \( b_n = \frac{3 - 4^n}{2 + 7 \cdot 4^n} \) or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).

9. Show that \( a_n = \frac{3n^2}{n^2 + 2} \) is strictly increasing. Find an upper bound.
Solutions to Worksheet 10.1

1. Calculate the first four terms of the sequence $b_n = \cos \pi n$, starting with $n = 1$.
   Setting $n = 1, 2, 3, 4$ into the formula for $b_n$ gives:
   
   $b_1 = \cos \pi \cdot 1 = \cos \pi = -1$
   $b_2 = \cos \pi \cdot 2 = \cos 2\pi = 1$
   $b_3 = \cos \pi \cdot 3 = \cos 3\pi = -1$
   $b_4 = \cos \pi \cdot 4 = \cos 4\pi = 1$

2. Calculate the first four terms of the sequence $b_n = 2 + (-1)^n$, starting with $n = 1$.
   The first four terms of $\{b_n\}$ are obtained by setting $n = 1, 2, 3, 4$ in the formula for $b_n$:
   
   $b_1 = 2 + (-1)^1 = 2 - 1 = 1$
   $b_2 = 2 + (-1)^2 = 2 + 1 = 3$
   $b_3 = 2 + (-1)^3 = 2 - 1 = 1$
   $b_4 = 2 + (-1)^4 = 2 + 1 = 3$

3. Use Theorem 1 to determine the limit of the sequence $b_n = \frac{3n + 1}{2n + 4}$ or state that the sequence diverges.
   Using Theorem 1 and the asymptotic behavior of rational functions we get:
   
   $$\lim_{n \to \infty} \frac{3n + 1}{2n + 4} = \lim_{x \to \infty} \frac{3x + 1}{2x + 4} = \frac{3}{2}$$

4. Use Theorem 1 to determine the limit of the sequence $c_n = 4(2^n)$ or state that the sequence diverges.
   By Theorem 1,
   
   $$\lim_{n \to \infty} 4 \cdot 2^n = \lim_{x \to \infty} 4 \cdot 2^x = \infty.$$ 
   Thus, the sequence $4 \cdot (2^n)$ diverges.

5. Determine the limit of the sequence $y_n = \frac{e^n}{2^n}$ or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).
   
   $$\frac{e^n}{2^n} = \left( \frac{e}{2} \right)^n$$ and $\frac{e}{2} > 1$. By the Limit of Geometric Sequences, proved in Example 6, we conclude that $\lim_{n \to \infty} \left( \frac{e}{2} \right)^n = \infty$. Thus, the given sequence diverges to $\infty$.

6. Determine the limit of the sequence $a_n = \frac{\sqrt{n}}{\sqrt{n} + 4}$ or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).
We compute the limit using Theorem 1:
\[
\lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{n} + 4} = \lim_{x \to \infty} \frac{\sqrt{x}}{\sqrt{x} + 4} = \lim_{x \to \infty} \frac{\sqrt{x}}{1 + \frac{4}{\sqrt{x}}} = \lim_{x \to \infty} \frac{1}{1 + 0} = 1
\]

7. Determine the limit of the sequence \( b_n = e^{n^2 - n} \) or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).

We have \( \lim_{x \to \infty} (x^2 - x) = \lim_{x \to \infty} x^2 \left(1 - \frac{1}{x}\right) = \infty \). Since the exponential function \( e^x \) is continuous, it follows from Theorem 4 that \( \lim_{x \to \infty} e^{x^2 - x} = \infty \). Thus, by Theorem 1, we conclude \( \lim_{n \to \infty} e^{n^2 - n} = \infty \); that is, the sequence \( e^{n^2 - n} \) diverges.

8. Determine the limit of the sequence \( b_n = \frac{3 - 4^n}{2 + 7 \cdot 4^n} \) or show that the sequence diverges (justifying each step using the appropriate Limit Laws or Theorems).

Dividing the numerator and denominator by \( 4^n \) yields:
\[
a_n = \frac{3 - 4^n}{2 + 7 \cdot 4^n} = \frac{3}{4^n} - \frac{4^n}{4^n} = \frac{3}{4^n} - \frac{1}{2 + 7}
\]

We now compute the limit using Limit Laws for Sequences and the limit of the geometric sequence (see Example 6) \( \lim_{n \to \infty} \frac{1}{4^n} = \lim_{n \to \infty} \left(\frac{1}{4}\right)^n = 0 \). We obtain:
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\frac{3}{4^n} - 1}{\frac{3}{4^n} + 7} = \lim_{n \to \infty} \frac{\left(\frac{3}{4^n} - 1\right)}{\left(\frac{3}{4^n} + 7\right)} = \frac{3 \lim_{n \to \infty} \frac{1}{4^n} - \lim_{n \to \infty} 1}{2 \lim_{n \to \infty} \frac{1}{4^n} - \lim_{n \to \infty} 7} = \frac{3 \cdot 0 - 1}{2 \cdot 0 + 7} = -\frac{1}{7}
\]

9. Show that \( a_n = \frac{3n^2}{n^2 + 2} \) is strictly increasing. Find an upper bound.

We consider the function \( f(x) = \frac{3x^2}{x^2 + 2} \). Differentiating \( f \) yields:
\[
f'(x) = \frac{6x(x^2 + 2) - 3x^2 \cdot 2x}{(x^2 + 2)^2} = \frac{12x}{(x^2 + 2)^2}
\]

\( f'(x) > 0 \) for \( x > 0 \), hence \( f \) is strictly increasing on this interval. It follows that \( a_n = f(n) \) is also strictly increasing. Next, we show that \( M = 3 \) is an upper bound for \( a_n \) by writing
\[
a_n = \frac{3n^2}{n^2 + 2} \leq \frac{3n^2 + 6}{n^2 + 2} = \frac{3(n^2 + 2)}{n^2 + 2} = 3.
\]
That is, $a_n \leq 3$ for all $n$. 