Worksheet 7.5.
The Method of Partial Fractions

1. Use the Method of Partial Fractions to evaluate \( \int \frac{(3x + 5)dx}{x^2 - 4x - 5} \).

2. Use the Method of Partial Fractions to evaluate \( \int \frac{3dx}{(x + 1)(x^2 + x)} \).
3. Use the Method of Partial Fractions to evaluate \( \int \frac{dx}{x(x^2 + 25)} \).

4. Use the Method of Partial Fractions to evaluate \( \int \frac{10dx}{(x - 1)^2(x^2 + 9)} \).

5. Use long division to write \( \frac{x^3 - 1}{x^2 - x} \) as the sum of a polynomial and a proper rational fraction. Then calculate the integral.
Solutions to Worksheet 7.5

1. Use the Method of Partial Fractions to evaluate \[ \int \frac{(3x + 5)\, dx}{x^2 - 4x - 5}. \]

The denominator factors as \( x^2 - 4x - 5 = (x - 5)(x + 1) \), so the partial fraction decomposition has the form

\[ \frac{3x + 5}{x^2 - 4x - 5} = \frac{3x + 5}{(x - 5)(x + 1)} = \frac{A}{x - 5} + \frac{B}{x + 1} \]

Clearing denominators gives

\[ 3x + 5 = A(x + 1) + B(x - 5) \]

Setting \( x = 5 \),

\[ 20 = A(6) + 0 \Rightarrow A = \frac{20}{6} = \frac{10}{3} \]

Setting \( x = -1 \),

\[ 2 = 0 + B(-6) \Rightarrow B = -\frac{1}{3} \]

The result is

\[ \int \frac{(3x + 5)\, dx}{x^2 - 4x - 5} = 10 \int \frac{dx}{x - 5} - \frac{1}{3} \int \frac{dx}{x + 1} \]

\[ = \frac{10}{3} \ln |x - 5| - \frac{1}{3} \ln |x + 1| + C \]

2. Use the Method of Partial Fractions to evaluate \[ \int \frac{3\, dx}{(x + 1)(x^2 + x)}. \]

The partial fraction decomposition has the form

\[ \frac{3}{(x + 1)(x^2 + x)} = \frac{3}{(x + 1)(x)(x + 1)} = \frac{3}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} \]

Clearing denominators gives

\[ 3 = A(x + 1)^2 + Bx(x + 1) + Cx \]

Setting \( x = 0 \) gives

\[ 3 = A(1) + 0 + 0 \Rightarrow A = 3 \]

Setting \( x = -1 \) gives

\[ 3 = 0 + 0 + C(-1) \Rightarrow C = -3 \]
Now plug in $A = 3$ and $C = -3$:

$$3 = 3(x + 1)^2 + Bx(x + 1) - 3x$$

The constant $B$ can be determined by plugging in for $x$ any value other than 0 or $-1$. Plugging in $x = 1$ gives

$$3 = 3(4) + B(1)(2) - 3 \Rightarrow B = -3$$

The result is

$$\frac{3}{(x+1)(x^2+x)} = \frac{3}{x} + \frac{-3}{x+1} + \frac{-3}{(x+1)^2}$$

$$\int \frac{3 \, dx}{(x+1)(x^2+x)} = 3 \int \frac{dx}{x} - 3 \int \frac{dx}{x+1} - 3 \int \frac{dx}{(x+1)^2}$$

$$= 3 \ln |x| - 3 \ln |x+1| + \frac{3}{x+1} + C$$

3. Use the Method of Partial Fractions to evaluate $\int \frac{dx}{x(x^2+25)}$.

The partial fraction decomposition has the form

$$\frac{1}{x(x^2+25)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 25}$$

Clearing denominators,

$$1 = A(x^2 + 25) + (Bx + C)x$$

Setting $x = 0$,

$$1 = A(25) \Rightarrow A = \frac{1}{25}$$

This gives

$$1 = \frac{1}{25}x^2 + 1 + Bx^2 + Cx = \left( B + \frac{1}{25} \right) x^2 + Cx + 1$$

Equating $x^2$-coefficients,

$$0 = B + \frac{1}{25} \Rightarrow B = -\frac{1}{25}$$

Equating $x$-coefficients, $C = 0$. The result is

$$\frac{1}{x(x^2+25)} = \frac{\frac{1}{25}}{x} + \frac{-\frac{1}{25}x}{x^2+25}$$

$$\int \frac{dx}{x(x^2+25)} = \frac{1}{25} \int \frac{dx}{x} - \frac{1}{25} \int \frac{x \, dx}{x^2+25}$$

For the integral on the right, use $u = x^2 + 25$, $du = 2x \, dx$. Then

$$\int \frac{dx}{x(x^2+25)} = \frac{1}{25} \ln |x| - \frac{1}{50} \ln |x^2 + 25| + C$$
4. Use the Method of Partial Fractions to evaluate \( \int \frac{10 \, dx}{(x - 1)^2(x^2 + 9)} \).

The partial fraction decomposition has the form

\[
\frac{10}{(x - 1)^2(x^2 + 9)} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{Cx + D}{x^2 + 9}
\]

Clearing denominators,

\[
10 = A(x - 1)(x^2 + 9) + B(x^2 + 9) + (Cx + D)(x - 1)^2
\]

Setting \( x = 1 \),

\[
10 = 0 + B(10) + 0 \Rightarrow B = 1
\]

Expanding the right-hand side,

\[
10 = (A + C)x^3 + (C + B - 2C - A)x^2 + (9A + C - 2D)x + (9 - 9A + D)
\]

Equating coefficients,

\[
\begin{align*}
A + C &= 0 \\
B - A - 2C + D &= 0 \\
9A + C - 2D &= 0 \\
9 - 9A + D &= 10
\end{align*}
\]

From the first equation, \( C = -A \), and from the fourth equation \( D = 1 + 9A \). Substituting these into the third equation,

\[
9A - A - 2(1 + 9A) = 0 \Rightarrow A = -\frac{1}{5}
\]

From this we get \( B = 1 \), \( C = \frac{1}{5} \), and \( D = -\frac{4}{5} \). The result is

\[
\int \frac{10 \, dx}{(x - 1)^2(x^2 + 9)} = \frac{-1}{5} \int \frac{dx}{x - 1} + \frac{1}{5} \int \frac{dx}{(x - 1)^2} + \frac{1}{5} \int \frac{dx}{x^2 + 9}
\]

\[
= -\frac{1}{5} \ln |x - 1| - \frac{1}{10} \ln |x^2 + 9| - \frac{4}{5} \left( \frac{1}{3} \right) \tan^{-1} \left( \frac{x}{3} \right) + C
\]

\[
= -\frac{1}{5} \ln |x - 1| - \frac{1}{10} \ln |x^2 + 9| - \frac{4}{15} \tan^{-1} \left( \frac{x}{3} \right) + C
\]
5. Use long division to write \( \frac{x^3 - 1}{x^2 - x} \) as the sum of a polynomial and a proper rational fraction. Then calculate the integral.

Long division gives
\[
\frac{x^3 - 1}{x^2 - x} = x + 1 + \frac{x - 1}{x(x - 1)} = x + 1 + \frac{1}{x}
\]

Therefore the integral is
\[
\int \frac{x^3 - 1}{x^2 - x} \, dx = \int (x + 1) \, dx + \int \frac{dx}{x} = \frac{1}{2} x^2 + x + \ln |x| + C
\]