Worksheet 5.1.
Approximating and Computing Area

1. Calculate the area of the shaded rectangles in the figure. Which approximation to the area under the curve is this?

\[ y = \frac{4 - x}{1 + x^2} \]

2. Estimate \( R_6 \) and \( L_6 \) for the function shown in the graph.
3. Evaluate the following sums (see Equations (3)–(5)).

   a. \( \sum_{k=1}^{20} 2k + 1 \)

   b. \( \sum_{\ell=1}^{10} (\ell^3 - 2\ell^2) \)

4. Evaluate \( \lim_{N \to \infty} \sum_{i=1}^{N} \frac{i^2 - i + 1}{N^3} \).

5. Use Equations (3)–(5) to find a formula for \( R_N \) for \( f(x) = 3x^2 - x + 4 \) over the interval \([0, 1]\).

6. Evaluate \( \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \sqrt{1 - \left( \frac{j}{N} \right)^2} \) by interpreting the limit as an area.
Solutions to Worksheet 5.1

1. Calculate the area of the shaded rectangles in the figure below. Which approximation to the area under the curve is this?

Each rectangle has a width of 1 and the height is taken as the value of the function at the midpoint of the interval. Thus, the area of the shaded rectangles is

\[ 1 \left( \frac{26}{29} + \frac{22}{13} + \frac{18}{5} + \frac{14}{5} + \frac{10}{13} + \frac{6}{29} \right) = \frac{18784}{1885} \approx 9.965 \]

Because there are six rectangles and the height of each rectangle is taken as the value of the function at the midpoint of the interval, the shaded rectangles represent the approximation \( M_6 \) to the area under the curve.

2. Estimate \( R_6 \) and \( L_6 \) for the function shown in the graph.
Let $f(x)$ on $[0, 3]$ be given as pictured. For $n = 6$, $\Delta x = (3 - 0)/6 = \frac{1}{2}$, \( \{x_k\}_{k=0}^6 = \left\{ 0, \frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2}, 3 \right\} \). Therefore

\[
L_6 = \frac{1}{2} \sum_{k=0}^{5} f(x_k) = \frac{1}{2} (3 + 2.8 + 2.6 + 2.2 + 1.7 + 1.2) = 6.75
\]

\[
R_6 = \frac{1}{2} \sum_{k=1}^{6} f(x_k) = \frac{1}{2} (2.8 + 2.6 + 2.2 + 1.7 + 1.2 + .75) = 5.625
\]

3. Evaluate the following sums (See Equations (3)–(5)).

a. \( \sum_{k=1}^{20} 2k + 1 \)

\[
\sum_{k=1}^{20} (2k + 1) = 2 \sum_{k=1}^{20} k + \sum_{k=1}^{20} 1 = 2 \left( \frac{20^2}{2} + \frac{20}{2} \right) + 20 = 440.
\]

b. \( \sum_{\ell=1}^{10} (\ell^3 - 2\ell^2) \)
\[
\sum_{\ell=1}^{10} (\ell^3 - 2\ell^2) = \sum_{\ell=1}^{10} \ell^3 - 2 \sum_{\ell=1}^{10} \ell^2 \\
= \left( \frac{10^4}{4} + \frac{10^3}{2} + \frac{10^2}{4} \right) - 2 \left( \frac{10^3}{3} + \frac{10^2}{2} + \frac{10}{6} \right) \\
= 2255.
\]

4. Evaluate \( \lim_{N \to \infty} \sum_{i=1}^{N} \frac{i^2 - i + 1}{N^3} \).

Let \( s_N = \sum_{i=1}^{N} \frac{i^2 - i + 1}{N^3} \).

Then
\[
s_N = \sum_{i=1}^{N} \frac{i^2 - i + 1}{N^3} = \frac{1}{N^3} \left[ \left( \sum_{i=1}^{N} i^2 \right) - \left( \sum_{i=1}^{N} i \right) + \left( \sum_{i=1}^{N} 1 \right) \right] \\
= \frac{1}{N^3} \left[ \left( \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} \right) - \left( \frac{N^2}{2} + \frac{N}{2} \right) + N \right] = \frac{1}{3} + \frac{2}{3N^2}.
\]

Therefore, \( \lim_{N \to \infty} s_N = \frac{1}{3} \).

5. Use Equations (3)–(5) to find a formula for \( R_N \) for \( f(x) = 3x^2 - x + 4 \) over the interval \([0, 1]\).

Let \( f(x) = 3x^2 - x + 4 \) on the interval \([0, 1]\). Then \( \Delta x = \frac{1 - 0}{N} = \frac{1}{N} \) and \( a = 0 \). Hence,
\[
R_N = \Delta x \sum_{j=1}^{N} f(0 + j\Delta x) = \frac{1}{N} \sum_{j=1}^{N} \left( 3j^2 \frac{1}{N^2} - j \frac{1}{N} + 4 \right) \\
= \frac{3}{N^3} \sum_{j=1}^{N} j^2 - \frac{1}{N^2} \sum_{j=1}^{N} j + \frac{4}{N} \sum_{j=1}^{N} 1 \\
= \frac{3}{N^3} \left( \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} \right) - \frac{1}{N^2} \left( \frac{N^2}{2} + \frac{N}{2} \right) + \frac{4}{N} \\
= 1 + \frac{3}{2N} + \frac{1}{2N^2} - \frac{1}{2} - \frac{1}{2N} + 4.
\]
and

$$
\lim_{N \to \infty} R_N = \lim_{N \to \infty} \left( 4.5 + \frac{1}{N} + \frac{1}{2N^2} \right) = 4.5.
$$

6. Evaluate \( \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \sqrt{1 - \left( \frac{j}{N} \right)^2} \) by interpreting the limit as an area.

The limit

$$
\lim_{N \to \infty} R_N = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \sqrt{1 - \left( \frac{j}{N} \right)^2}
$$

represents the area between the graph of \( y = f(x) = \sqrt{1 - x^2} \) and the \( x \)-axis over the interval \([0, 1]\). This is the portion of the circular disk \( x^2 + y^2 \leq 1 \) that lies in the first quadrant. Accordingly, its area is \( \frac{1}{4} \pi (1)^2 = \frac{\pi}{4} \).