The MIU approximation (chapter 2, 3rd ed.)

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miu_dynamics_3e.tex

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1 Introduction

This note provides more details on the derivation of the linear approximation used in Chapter 2 (3rd ed.) to study dynamics in the basic money-in-the-utility function (MIU) model and on solving linear, rational expectations models.

2 The MIU model

The basic equilibrium conditions of the MIU model of Chapter 2 are given by

$$\lambda_t = u_c(c_t, m_t, 1-n_t) \tag{1}$$

$$y_t = f(k_{t-1}, n_t, z_t) \tag{2}$$

$$u_m(c_t, m_t, 1-n_t) = \lambda_t \left(\frac{i_t}{1+i_t}\right) \tag{3}$$

$$u_l(c_t, m_t, 1-n_t) = \lambda_t f_n(k_{t-1}, n_t, z_t) \tag{4}$$

$$\lambda_t = \beta \mathrm{E}_t R_t \lambda_{t+1}, \tag{5}$$

$$R_t = 1 - \delta + f_k(k_t, n_{t+1}, z_{t+1}) \tag{6}$$

$$R_t = \mathrm{E}_t \frac{1+i_t}{1+\pi_{t+1}}.$$

$$y_t = c_t + x_t, \tag{7}$$

$$x_t = k_t - (1-\delta)k_{t-1}$$

$$m_t = \left(\frac{1+u_t}{1+\pi_t}\right) m_{t-1}, \tag{8}$$

$$z_t = \rho z_{t-1} + e_t, \tag{9}$$

$$u_t = \gamma u_{t-1} + \phi z_{t-1} + \varphi_t, \quad 0 \le \gamma < 1, \tag{10}$$

where all variables are defined in the text.

3 The linear approximation

The next step is to derive first-order linear approximations to the model's equilibrium conditions.

3.1 Functional forms

The utility function:

$$u\left(c_t, m_t, 1-n_t\right) = \frac{\left[aC_t^{1-b} + (1-a)m_t^{1-b}\right]^{\frac{1-\Phi}{1-b}}}{1-\Phi} + \Psi\left[\frac{(1-n_t)^{1-\eta}}{1-\eta}\right].$$

The production function:

$$y_t = e^{z_t} k_{t-1}^{\alpha} n_t^{1-\alpha}$$

3.2 Production function

$$y_t = e^{z_t} k_{t-1}^{\alpha} n_t^{1-\alpha}$$

$$y^{ss}\left(1+\hat{y}_t\right) = \left(1+z_t\right)\left(k^{ss}\right)^a \left(1+\hat{k}_{t-1}\right)^{\alpha} \left(n^{ss}\right)^{1-\alpha}\left(1+\hat{n}_t\right)^{1-\alpha}$$

Since

$$y^{ss} = \left(k^{ss}\right)^a \left(n^{ss}\right)^{1-\alpha},$$

$$\begin{aligned} \left(1+\hat{y}_t\right) &= \left(1+z_t\right)\left(1+\hat{k}_{t-1}\right)^{\alpha}\left(1+\hat{n}_t\right)^{1-\alpha} \\ &\approx 1 + \alpha\hat{k}_{t-1} + (1-\alpha)\hat{n}_t + z_t \end{aligned}$$

$$\hat{y}_t = \alpha\hat{k}_{t-1} + (1-\alpha)\hat{n}_t + z_t \tag{11}$$

3.3 Goods market clearing

$$k_t = (1-\delta)k_{t-1} + y_t - c_t$$

$$\begin{aligned} k^{ss}\left(1+\hat{k}_t\right) &= (1-\delta)k^{ss}\left(1+\hat{k}_{t-1}\right) + y^{ss}\left(1+\hat{y}_t\right) \\ &\quad - c^{ss}\left(1+\hat{c}_t\right) \end{aligned}$$

$$\hat{k}_t = (1-\delta)\hat{k}_{t-1} + \left(\frac{y^{ss}}{k^{ss}}\right)\hat{y}_t - \left(\frac{c^{ss}}{k^{ss}}\right)\hat{c}_t$$

or

$$\left(\frac{y^{ss}}{k^{ss}}\right)\hat{y}_t = \left(\frac{c^{ss}}{k^{ss}}\right)\hat{c}_t + \hat{k}_t - (1-\delta)\hat{k}_{t-1}$$

In addition,

$$x^{ss}(1+\hat{x}_t) = k^{ss}\left(1+\hat{k}_t\right) - (1-\delta)k^{ss}\left(1+\hat{k}_{t-1}\right)$$

which implies

$$\left(\frac{x^{ss}}{k^{ss}}\right)\hat{x}_t = \hat{k}_t - (1-\delta)\hat{k}_{t-1}$$

but $x^{ss}/k^{ss} = \delta$, so

$$\delta\hat{x}_t = \hat{k}_t - (1-\delta)\hat{k}_{t-1}$$

Hence,

$$\left(\frac{y^{ss}}{k^{ss}}\right)\hat{y}_t = \left(\frac{c^{ss}}{k^{ss}}\right)\hat{c}_t + \delta\hat{x}_t \tag{12}$$

and

$$\hat{k}_t = (1-\delta)\hat{k}_{t-1} + \delta\hat{x}_t \tag{13}$$

3.4 Labor hours choice

$$u_l(c_t, m_t, 1-n_t) = \lambda_t f_n(k_{t-1}, n_t, z_t)$$

$$\frac{u_l}{\lambda_t} = \frac{\Psi(1-n_t)^{-\eta}}{\lambda_t} = (1-\alpha)\left(\frac{y_t}{n_t}\right)$$

$$\frac{\Psi L_t^{-\eta}}{\lambda_t} = (1-\alpha)\left(\frac{y_t}{n_t}\right)$$

$$\frac{\Psi l^{ss}(1+\hat{l}_t)^{-\eta}}{\bar{\lambda}(1+\hat{\lambda}_t)} = (1-\alpha)\left(\frac{y^{ss}}{n^{ss}}\right)\left(\frac{1+\hat{y}_t}{1+\hat{n}_t}\right)$$

But

$$\frac{\Psi l^{ss}}{\bar{\lambda}} = (1-\alpha)\left(\frac{y^{ss}}{n^{ss}}\right)$$

so

$$\frac{(1+\hat{l}_t)^{-\eta}}{(1+\hat{\lambda}_t)} = \left(\frac{1+\hat{y}_t}{1+\hat{n}_t}\right)$$

$$\left(1-\eta\hat{l}_t\right)(1-\lambda_t) \approx 1-\eta\hat{l}_t - \lambda_t \approx 1+\hat{y}_t - \hat{n}_t$$

But

$$l_t = 1-n_t$$

$$l^{ss}(1+\hat{l}_t) = 1-n^{ss}(1+\hat{n}_t)$$

$$l^{ss}\hat{l}_t = -n^{ss}\hat{n}_t \Rightarrow \hat{l}_t = -\left(\frac{n^{ss}}{l^{ss}}\right)\hat{n}_t$$

So

$$1-\eta\hat{l}_t - \lambda_t = 1+\eta\left(\frac{n^{ss}}{l^{ss}}\right)\hat{n}_t - \lambda_t \approx 1+\hat{y}_t - \hat{n}_t$$

$$-\eta\hat{l}_t - \lambda_t = \eta\left(\frac{n^{ss}}{l^{ss}}\right)\hat{n}_t - \lambda_t \approx \hat{y}_t - \hat{n}_t$$

$$\left[1+\eta\left(\frac{n^{ss}}{l^{ss}}\right)\right]\hat{n}_t = \hat{y}_t + \lambda_t \tag{14}$$

3.5 Marginal utility of consumption

$$\lambda_t = a\left[ac_t^{1-b} + (1-a)m_t^{1-b}\right]^{\frac{b-\Phi}{1-b}} c_t^{-b}$$

Define

$$H_t = ac_t^{1-b} + (1-a)m_t^{1-b}.$$

Then

$$\lambda_t = aH_t^{\frac{b-\Phi}{1-b}} c_t^{-b}$$

$$\bar{\lambda}(1+\hat{\lambda}_t) = a\left(H^{ss}\right)^{\frac{b-\Phi}{1-b}}\left(c^{ss}\right)^{-b}\left(1+\left(\frac{b-\Phi}{1-b}\right)\hat{h}_t\right)(1-b\hat{c}_t)$$

$$1+\hat{\lambda}_t = \left(1+\left(\frac{b-\Phi}{1-b}\right)\hat{h}_t\right)(1-b\hat{c}_t) \approx 1+\left(\frac{b-\Phi}{1-b}\right)\hat{h}_t - b\hat{c}_t$$

$$\hat{\lambda}_t = \left(\frac{b-\Phi}{1-b}\right)\hat{h}_t - b\hat{c}_t$$

$$(1+\hat{h}_t) = \frac{a\left(c^{ss}\right)^{1-b}}{H^{ss}}(1+\hat{c}_t)^{1-b} + \frac{(1-a)\left(m^{ss}\right)^{1-b}}{\left(H^{ss}\right)}(1+\hat{m}_t)^{1-b}$$

$$(1+\hat{h}_t) = \frac{a\left(c^{ss}\right)^{1-b}}{H^{ss}}\left[1+(1-b)\hat{c}_t\right] + \frac{(1-a)\left(m^{ss}\right)^{1-b}}{H^{ss}}\left[1+(1-b)\hat{m}_t\right]$$

$$\hat{h}_t = \frac{a\left(c^{ss}\right)^{1-b}}{H^{ss}}(1-b)\hat{c}_t + \frac{(1-a)\left(m^{ss}\right)^{1-b}}{H^{ss}}(1-b)\hat{m}_t$$

$$\hat{h}_t = (1-b)\left[\gamma\hat{c}_t + (1-\gamma)\hat{m}_t\right]$$

where

$$\gamma = \frac{a\left(c^{ss}\right)^{1-b}}{H^{ss}}$$

So

$$\begin{aligned}
\hat{\lambda}_t &= \left(\frac{b-\Phi}{1-b}\right)\hat{h}_t - \hat{c}_t \\
&= \left(\frac{b-\Phi}{1-b}\right)(1-b)\left[\gamma\hat{c}_t + (1-\gamma)\hat{m}_t\right] - b\hat{c}_t \\
&= (b-\Phi)\left[\gamma\hat{c}_t + (1-\gamma)\hat{m}_t\right] - b\hat{c}_t
\end{aligned}$$

$$\hat{\lambda}_t = \Omega_1\hat{c}_t + \Omega_2\hat{m}_t \tag{15}$$

where $\Omega_1 = b(\gamma-1) - \gamma\Phi$ and $\Omega_2 = (b-\Phi)(1-\gamma)$. Note that if $b = \Phi$, $\hat{\lambda}_t = -b\hat{c}_t$.

3.6 Marginal product, real return condition

$$R_t = 1 + r_t = 1 - \delta + \alpha \mathrm{E}_t \left(\frac{y_{t+1}}{k_t} \right)$$

$$R_t = 1 + r_t = 1 - \delta + \alpha \left(\frac{y^{ss}}{k^{ss}} \right) \mathrm{E}_t \left(1 + \hat{y}_{t+1} - \hat{k}_t \right)$$

$$R_t = 1 + r_t = 1 - \delta + \alpha \left(\frac{y^{ss}}{k^{ss}} \right) + \alpha \left(\frac{y^{ss}}{k^{ss}} \right) \mathrm{E}_t \left(\hat{y}_{t+1} - \hat{k}_t \right)$$

$$R_t = 1 + r_t = 1 + r^{ss} + \alpha \left(\frac{y^{ss}}{k^{ss}} \right) \mathrm{E}_t \left(\hat{y}_{t+1} - \hat{k}_t \right)$$

To a first order,

$$\hat{r}_t = r_t - r^{ss} = \alpha \left(\frac{y^{ss}}{k^{ss}} \right) \mathrm{E}_t \left(\hat{y}_{t+1} - \hat{k}_t \right) \tag{16}$$

3.7 Money holdings

$$\frac{u_m(c_t, m_t, 1 - n_t)}{u_c(c_t, m_t, 1 - n_t)} = \left(\frac{i_t}{1 + i_t} \right)$$

$$\begin{aligned}
\frac{u_m(c_t, m_t, 1 - n_t)}{u_c(c_t, m_t, 1 - n_t)} &= \frac{(1-a) m_t^{-b}}{a c_t^{-b}} \approx \frac{(1-a) m^{ss}}{a c^{ss}} \left(1 - b\hat{m}_t + b\hat{c}_t \right) \\
&= \left(\frac{i^{ss}}{1 + i^{ss}} \right) \left(1 - b\hat{m}_t + b\hat{c}_t \right).
\end{aligned}$$

Therefore,

$$-b\hat{m}_t + b\hat{c}_t \approx \left(\frac{1 + i^{ss}}{i^{ss}} \right) \left(\frac{i_t}{1 + i_t} \right) - 1.$$

But

$$\left(\frac{1 + i^{ss}}{i^{ss}} \right) \left(\frac{i_t}{1 + i_t} \right) - 1 = \frac{i_t (1 + i^{ss})}{i^{ss} (1 + i_t)} - 1,$$

so ignoring second order terms,

$$\frac{i_t (1 + i^{ss})}{i^{ss} (1 + i_t)} - 1 \approx \left(\frac{i_t - i^{ss}}{i^{ss}} \right) = \left(\frac{1}{i^{ss}} \right) \hat{\imath}_t$$

$$-b \left(\hat{m}_t - \hat{c}_t \right) = \left(\frac{1}{i^{ss}} \right) \hat{\imath}_t$$

or

$$\hat{m}_t = \hat{c}_t - \left(\frac{1}{b} \right) \left(\frac{1}{i^{ss}} \right) \hat{\imath}_t \tag{17}$$

3.8 Euler condition

$$\lambda_t = \beta \mathrm{E}_t R_t \lambda_{t+1}$$

$$\bar{\lambda}\left(1+\hat{\lambda}_t\right) = \beta\bar{\lambda}(1+r_t)\mathrm{E}_t\left(1+\hat{\lambda}_{t+1}\right)$$

$$\left(1+\hat{\lambda}_t\right) = \beta(1+r_t)\mathrm{E}_t\left(1+\hat{\lambda}_{t+1}\right)$$

$$1+\hat{\lambda}_t = \beta\left(1+r_t+\mathrm{E}_t\hat{\lambda}_{t+1}\right) = \left(\frac{1+r_t+\mathrm{E}_t\hat{\lambda}_{t+1}}{1+r^{ss}}\right)$$

$$1+\hat{\lambda}_t \approx 1+r_t-r^{ss}+\mathrm{E}_t\hat{\lambda}_{t+1}$$

To first order:

$$\hat{\lambda}_t = (r_t - r^{ss}) + \mathrm{E}_t\hat{\lambda}_{t+1} \tag{18}$$

3.9 Fisher equation

$$R_t = \mathrm{E}_t\left(\frac{1+i_t}{1+\pi_{t+1}}\right)$$

$$R_t = 1+r_t \approx (1+i_t-\mathrm{E}_t\pi_{t+1})$$

$$r_t - r^{ss} \approx i_t - r^{ss} - \mathrm{E}_t\pi_{t+1}$$

around a zero-steady-state rate of inflation. So

$$\hat{r}_t \approx \hat{\imath}_t - \mathrm{E}_t\pi_{t+1} \tag{19}$$

3.10 Real money growth

$$m_t = \left(\frac{1+u_t}{1+\pi_t}\right)m_{t-1}$$

$$\hat{m}_t = u_t - \pi_t + \hat{m}_{t-1} \tag{20}$$

3.11 Collecting all equations

Unknowns: $\hat{y}_t$, $\hat{k}_t$, $\hat{n}_t$, $\hat{x}_t$, $\hat{c}_t$, $\hat{\lambda}_t$, $\hat{r}_t$, $\hat{\imath}_t$, π_t, $\hat{m}_t$ – 10 variables.

Ten equations, (11) - (20) plus the specification of the processes governing the exogenous productivity and money growth disturbances.

$$\hat{y}_t = z_t + \alpha\hat{k}_{t-1} + (1-\alpha)\hat{n}_t$$

$$\left(\frac{y^{ss}}{k^{ss}}\right)\hat{y}_t = \left(\frac{c^{ss}}{k^{ss}}\right)\hat{c}_t + \delta\hat{x}_t$$

$$\hat{k}_t = (1-\delta)\hat{k}_{t-1} + \delta\hat{x}_t$$

$$\left[1+\eta\left(\frac{n^{ss}}{l^{ss}}\right)\right]\hat{n}_t = \hat{y}_t + \lambda_t$$

$$\hat{m}_t = \hat{c}_t - \left(\frac{1}{b}\right)\left(\frac{1}{i}\right)\hat{\imath}_t$$

$$\hat{\lambda}_t = \Omega_1 \hat{c}_t + \Omega_2 \hat{m}_t$$

$$\hat{r}_t = \alpha\left(\frac{y^{ss}}{k^{ss}}\right)\mathrm{E}_t\left(\hat{y}_{t+1} - \hat{k}_t\right)$$

$$\hat{\lambda}_t = \hat{r}_t + \mathrm{E}_t\hat{\lambda}_{t+1}$$

$$\hat{\imath}_t = \hat{r}_t + \mathrm{E}_t\pi_{t+1}$$

$$\hat{m}_t = u_t - \pi_t + \hat{m}_{t-1}$$

$$z_t = \rho_z z_{t-1} + e_t$$

$$u_t = \rho_u u_{t-1} + \phi z_{t-1} + \varphi_t.$$

4 Solving Linear Rational Expectations Models with Forward-Looking Variables

This sections provides a brief overview of the approach used to solve linear rational expectations models numerically. The basic reference is Blanchard and Kahn (1980). This discussion follows Uhlig (1999), to which the reader is referred for more details. General discussions can be found in Farmer (1993, chapter 3) or the user's guide in Hoover, Hartley, and Salyer (1998). See also Turnovsky (1995), Wickens (2008, Appendex 15.8), and Cochrane (2007). Further details can be found in Uhlig (1999), who also provides the software tools to solve linear rational expectations models in Matlab. Standard solution methods require that the model be written in state-space form. Dynare is a popular matlab-based program for solving models that allows the models to be written in a more natural form. Dynare is also popular for and estimating rational expectations models and for obtaining second-order approximations to non-linear models. The focus in these notes is on first-order linear approximations to non-linear structural equations.

Remark 1 *Work by King, et al and more recently Christiano?*

Let X_t denote variables predetermined at time t, while x_t denote non-predetermined variables; x_t are forward-looking variables, also called jump variables. By predetermined, we mean that X_t is known at time t and not jointly determined with x_t, while x_t are endogenously determined at time t. Let n_1 denote the number of predetermined variables and n_2 the number of forward-looking variables. We assume the model can be written in the state-spate form given by

$$A_1\left[\begin{array}{c} X_{t+1} \\ \mathrm{E}_t x_{t+1} \end{array}\right] = A_2\left[\begin{array}{c} X_t \\ x_t \end{array}\right] + \left[\begin{array}{c} \psi_{t+1} \\ 0 \end{array}\right],$$

where A_1 is non-singular. (See King, XXX (XXXX) for a treatment of the case in which A_1 is singular.) Pre-multiplying by A_1^{-1} yields

$$E_t Z_{t+1} \equiv \begin{bmatrix} X_{t+1} \\ \mathrm{E}_t x_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ x_t \end{bmatrix} + \begin{bmatrix} \psi_{t+1} \\ 0 \end{bmatrix} = AZ_t + \varepsilon_{t+1}, \tag{21}$$

where $A = A_1{}^{-1} A_2$,

$$\mathrm{E}_t Z_{t+1} \equiv \begin{bmatrix} X_{t+1} \\ \mathrm{E}_t x_{t+1} \end{bmatrix}$$

and

$$Z_t \equiv \begin{bmatrix} X_t \\ x_t \end{bmatrix}.$$

The general solution to systems such as (21) under rational expectations was derived by Blanchard and Kahn (1980).

We can write A as $Q^{-1}\Lambda Q$, where Λ is a diagonal matrix of the eigenvalues of A and Q is the corresponding matrix of eigenvectors. Order Λ so that λ_1 is the smallest and $\lambda_{n_1+n_2}$ is the largest eigenvalue. If we premultiply (21) by Q,

$$Q\mathrm{E}_t Z_{t+i+1} = QA\mathrm{E}_t Z_{t+i} + Q\mathrm{E}_t \varepsilon_{t+i+1} = \Lambda Q\mathrm{E}_t Z_{t+i} + Q\mathrm{E}_t \varepsilon_{t+i+1}$$

or

$$z_{t+i+1} = \begin{bmatrix} \mathrm{E}_t Y_{t+i+1} \\ \mathrm{E}_t P_{t+i+1} \end{bmatrix} = \Lambda_t z_{t+i} + \mathrm{E}_t \xi_{t+i+1},$$

where

$$z_{t+i+1} = \begin{bmatrix} \mathrm{E}_t Y_{t+i+1} \\ \mathrm{E}_t P_{t+i+1} \end{bmatrix} = Q \begin{bmatrix} \mathrm{E}_t X_{t+i+1} \\ \mathrm{E}_t x_{t+i+1} \end{bmatrix}$$

and $\mathrm{E}_t \xi_{t+1} = Q\mathrm{E}_t \varepsilon_{t+1}$.

Eigenvalues and determinacy

$$z_{t+i+1} = \Lambda_t z_{t+i} + E_t \xi_{t+i+1},$$

can be written as

$$\begin{bmatrix} E_t Y_{t+i+1} \\ E_t P_{t+i+1} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} E_t Y_{t+i} \\ E_t P_{t+i} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} E_t \psi_{t+i+1} \\ 0 \end{bmatrix}$$

where Λ_1 consists of the $\tilde{n}_1$ eigenvalues on or inside the unit circle and Λ_2 consists of the $\tilde{n}_2$ eigenvalues outside the unit circle.

Eigenvalues and determinacy

$$\begin{bmatrix} E_t Y_{t+i+1} \\ E_t P_{t+i+1} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} E_t Y_{t+i} \\ E_t P_{t+i} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} E_t \psi_{t+i+1} \\ 0 \end{bmatrix}$$

Rewrite this as

$$E_t Y_{t+i+1} = \Lambda_1 E_t Y_{t+i} + Q_{11} E_t \psi_{t+i+1}$$

and

$$E_t P_{t+i+1} = \Lambda_2 E_t P_{t+i} + Q_{21} E_t \psi_{t+i+1}$$

Eigenvalues and determinacy

$$E_t P_{t+i+1} = \Lambda_2 E_t P_{t+i} + Q_{21} E_t \psi_{t+i+1}$$

Since this set of equations is explosive (the elements of Λ_2 are outside the unit circle), it must be the case in any non-explosive equilibrium that

$$P_t = -\sum_{i=0}^{\infty} \Lambda_2^{-i-1} Q_{21} E_t \psi_{t+i}$$

which uniquely determines P_t.

For $\|\lambda_i\| > 1$, solve forward: $z_{it+1} = \lambda_i z_{it} + \xi_{it+1}$ so

$$\begin{aligned} z_{it} &= \left(\frac{1}{\lambda_i}\right) E_t \left(z_{it+1} - \xi_{it+1}\right) \\ &= -\left(\frac{1}{\lambda_i}\right) E_t \xi_{it+1} + \left(\frac{1}{\lambda_i}\right)^2 E_t \left(z_{it+2} - \xi_{it+2}\right) \\ &= -\sum_{j=0}^{\infty} \left(\frac{1}{\lambda_i}\right)^j E_t \xi_{it+j} \end{aligned}$$

Eigenvalues and determinacy

$$E_t Y_{t+i+1} = \Lambda_1 E_t Y_{t+i} + Q_{11} E_t \psi_{t+i+1}$$

Since elements of Λ_1 are inside or one the unit circle, solve backward

- For $\|\lambda_i\| < 1$, solve backward: $z_{it+1} = \lambda_i z_{it} + \xi_{it+1}$ so

$$\begin{aligned} z_{it} &= \lambda_i z_{it-1} + \xi_{it} = \lambda_i \left(\lambda_i z_{it-2} + \xi_{it-1}\right) + \lambda_i \xi_{it} \\ &= \sum_{j=0}^{\infty} \lambda_i^j \xi_{it-j}. \end{aligned}$$

Eigenvalues and determinacy

From

$$\begin{bmatrix} Y_t \\ P_t \end{bmatrix} = Q \begin{bmatrix} X_t \\ x_t \end{bmatrix},$$

$$\begin{bmatrix} X_t \\ x_t \end{bmatrix} = Q^{-1} \begin{bmatrix} Y_t \\ P_t \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} Y_t \\ P_t \end{bmatrix}.$$

The first $\tilde{n}_1$ equations (corresponding to the λ_i inside or on the unit circle) can be written

$$X_t = Q_{11} Y_t + Q_{12} P_t.$$

$\tilde{n}$ equations to determine the n elements of X.

Eigenvalues and determinacy

$$X_t = Q_{11}Y_t + Q_{12}P_t. \tag{22}$$

- If $\tilde{n}_1 < n_1$, or $\tilde{n}_2 = (n_1 + n_2) - \tilde{n}_1 > n_2$, more roots outside unit circle that there are forward looking variables. Eq. (22) must satisfy

  $$X_0 = Q_{11}Y_0 + Q_{12}P_0$$

  for the initial conditions on X_0. But this imposes more that $\tilde{n}_1$ conditions on Y_0 and so there will generally be no solution.

- If $\tilde{n}_1 > n_1$, or $\tilde{n}_2 = (n_1 + n_2) - \tilde{n}_1 < n_2$, fewer roots outside unit circle that there are forward looking variables. Eq. (22) is underdetermined. Generally will ahve multiple solutions.

- If $\tilde{n}_1 = n_1$, or $\tilde{n}_2 = n_2$, unique solution.

5 Numerically solving the MIU model

The first step is to write the model in state space form:

$$A\begin{bmatrix} Z_{t+1} \\ E_t z_{t+1} \end{bmatrix} = B\begin{bmatrix} Z_t \\ z_t \end{bmatrix} + \begin{bmatrix} u_{t+1} \\ 0 \end{bmatrix},$$

where Z_t is an $n_1 \times 1$ vector of predetermined variables, z_t is an $n_2 \times 1$ vector of non-predetermined variables, u_{t+1} is an $n_1 \times 1$ vector of exogenous stochastic innovations, and A and B are conformal matrices (i.e., they are both $n_1 + n_2 \times n_1 + n_2$). The elements of z are often also called the forward-looking variables. By predetermined, we mean that Z_t is known at time t and not jointly determined with z_t, while z_t are endogenously determined at time t.

Assuming A is nonsingular, premultiply by the inverse of A to obtain

$$\begin{bmatrix} Z_{t+1} \\ E_t z_{t+1} \end{bmatrix} = M\begin{bmatrix} Z_t \\ z_t \end{bmatrix} + \begin{bmatrix} u_{t+1} \\ 0 \end{bmatrix}.$$

Blanchard and Kahn (Econometrica 1980) show that a unique, stationary, rational expectations solution exists if and only if the number of eigenvalues of M outside the unit circle is equal to n_2, the number of non-predetermined variables.

We will write the model in the form

$$\begin{bmatrix} X_{t+1} \\ \mathrm{E}_t x_{t+1} \end{bmatrix} = A\begin{bmatrix} X_t \\ x_t \end{bmatrix} + \begin{bmatrix} \psi_{t+1} \\ 0 \end{bmatrix},$$

where X is a vector of predetermined variables, x is the vector of forward-looking or non-predetermined variables, and ψ is a vector of exogenous, serially uncorrelated disturbances. To write the MIU model in this form, it will be convenient to first eliminate $E_t y_{t+1}$ from the equation defining the real return.

We can do so by utilizing the production function and the AR(1) process for the productivity shock, which implies

$$\mathrm{E}_t\hat{y}_{t+1} = \alpha\hat{k}_t + (1-\alpha)\mathrm{E}_t\hat{n}_{t+1} + \rho_z z_t.$$

From the first-order condition for the household's supply of labor and the Euler equation,

$$\left[1+\eta\left(\frac{n^{ss}}{l^{ss}}\right)\right]\mathrm{E}_t\hat{n}_{t+1} = \mathrm{E}_t\hat{y}_{t+1} + \mathrm{E}_t\lambda_{t+1} = \mathrm{E}_t\hat{y}_{t+1} + \lambda_t - r_t.$$

Therefore,

$$E_t\hat{y}_{t+1} = \alpha\hat{k}_t + (1-\alpha)\frac{(E_t\hat{y}_{t+1} + \lambda_t - r_t)}{1+\eta n^{ss}/l^{ss}} + \rho_z z_t,$$

or

$$E_t\hat{y}_{t+1} = \left(\frac{1+\eta n^{ss}/l^{ss}}{\alpha+\eta n^{ss}/l^{ss}}\right)\alpha\hat{k}_t + \left(\frac{1-\alpha}{\alpha+\eta n^{ss}/l^{ss}}\right)(\lambda_t - r_t) + \left(\frac{1+\eta n^{ss}/l^{ss}}{\alpha+\eta n^{ss}/l^{ss}}\right)\rho_z z_t.$$

Hence,

$$E_t\hat{y}_{t+1} - \hat{k}_t = (\alpha-1)\left(\frac{\eta n^{ss}/l^{ss}}{\alpha+\eta n^{ss}/l^{ss}}\right)\hat{k}_t + \left(\frac{1-\alpha}{\alpha+\eta n^{ss}/l^{ss}}\right)(\lambda_t - r_t) + \left(\frac{1+\eta n^{ss}/l^{ss}}{\alpha+\eta n^{ss}/l^{ss}}\right)\rho_z z_t.$$

Substituting this into the real return equation and rearranging yields

$$\begin{aligned}
\hat{r}_t &= \alpha\left(\frac{y^{ss}}{k^{ss}}\right)\left(\mathrm{E}_t\hat{y}_{t+1} - \hat{k}_t\right) \\
&= \alpha\left(\frac{y^{ss}}{k^{ss}}\right)(\alpha-1)\left(\frac{\eta n^{ss}/l^{ss}}{\alpha+\eta n^{ss}/l^{ss}}\right)\hat{k}_t + \alpha\left(\frac{y^{ss}}{k^{ss}}\right)\left(\frac{1-\alpha}{\alpha+\eta n^{ss}/l^{ss}}\right)(\lambda_t - r_t) + \\
&\quad \alpha\left(\frac{y^{ss}}{k^{ss}}\right)\left(\frac{1+\eta n^{ss}/l^{ss}}{\alpha+\eta n^{ss}/l^{ss}}\right)\rho_z z_t.
\end{aligned}$$

Collecting the terms in $\hat{r}_t$,

$$\begin{aligned}
\left[1+\alpha\left(\frac{y^{ss}}{k^{ss}}\right)\left(\frac{1-\alpha}{\alpha+\eta n^{ss}/l^{ss}}\right)\right]\hat{r}_t &= \alpha\left(\frac{y^{ss}}{k^{ss}}\right)(\alpha-1)\left(\frac{\eta n^{ss}/l^{ss}}{\alpha+\eta n^{ss}/l^{ss}}\right)\hat{k}_t \\
&\quad +\alpha\left(\frac{y^{ss}}{k^{ss}}\right)\left(\frac{1-\alpha}{\alpha+\eta n^{ss}/l^{ss}}\right)\lambda_t \\
&\quad +\alpha\left(\frac{y^{ss}}{k^{ss}}\right)\left(\frac{1+\eta n^{ss}/l^{ss}}{\alpha+\eta n^{ss}/l^{ss}}\right)\rho_z z_t
\end{aligned}$$

Let

$$\theta = \left[\alpha+\eta\left(\frac{n^{ss}}{l^{ss}}\right) + \alpha(1-\alpha)\left(\frac{y^{ss}}{k^{ss}}\right)\right]$$

Then

$$\theta\hat{r}_t = \alpha\left(\frac{y^{ss}}{k^{ss}}\right)\left\{-(1-\alpha)\eta\left(\frac{n^{ss}}{l^{ss}}\right)\hat{k}_t + (1-\alpha)\lambda_t + \left[1+\eta\left(\frac{n^{ss}}{l^{ss}}\right)\right]\rho_z z_t\right\}$$

5.1 Putting the model in matrix form

$$z_{t+1} = \rho_z z_t + e_{t+1},$$

$$u_{t+1} = \phi z_t + \rho_u u_t + \varphi_{t+1}$$

$$\hat{y}_t - (1-\alpha)\hat{n}_t = z_t + \alpha \hat{k}_{t-1}$$

$$\left(\frac{y^{ss}}{k^{ss}}\right)\hat{y}_t - \left(\frac{c^{ss}}{k^{ss}}\right)\hat{c}_t - \delta \hat{x}_t = 0$$

$$\hat{k}_t - \delta \hat{x}_t = (1-\delta)\hat{k}_{t-1}$$

$$\left[1 + \eta\left(\frac{n^{ss}}{l^{ss}}\right)\right]\hat{n}_t - \hat{y}_t = \lambda_t$$

$$\Omega_1 \hat{c}_t + \Omega_2 \hat{m}_t = \hat{\lambda}_t$$

$$\theta \hat{r}_t + \alpha\left(\frac{y^{ss}}{k^{ss}}\right)(1-\alpha)\eta\left(\frac{n^{ss}}{l^{ss}}\right)\hat{k}_t = \alpha\left(\frac{y^{ss}}{k^{ss}}\right)\left[1 + \eta\left(\frac{n^{ss}}{l^{ss}}\right)\right]\rho_z z_t + \alpha\left(\frac{y^{ss}}{k^{ss}}\right)(1-\alpha)\lambda_t$$

$$\hat{m}_t = u_t + \hat{m}_{t-1} - \pi_t$$

$$\hat{m}_t - \hat{c}_t + \left(\frac{1}{b}\right)\left(\frac{1}{i}\right)\hat{\imath}_t = 0$$

$$\hat{r}_t + \mathrm{E}_t \hat{\lambda}_{t+1} = \hat{\lambda}_t$$

$$\hat{r}_t - \hat{\imath}_t + \mathrm{E}_t \pi_{t+1} = 0$$

Let

$$X_{t+1} =_1 \begin{bmatrix} \hat{z}_{t+1} \\ \hat{u}_{t+1} \\ \hat{y}_t \\ \hat{c}_t \\ \hat{k}_t \\ \hat{x}_t \\ \hat{n}_t \\ \hat{r}_t \\ \hat{m}_t \\ \hat{\imath}_t \end{bmatrix}, \text{ and } \mathrm{E}_t x_{t+1} = \begin{bmatrix} \mathrm{E}_t \hat{\lambda}_{t+1} \\ \mathrm{E}_t \pi_{t+1} \end{bmatrix}$$

Then

$$
A_1 \begin{bmatrix} \hat{z}_{t+1} \\ \hat{u}_{t+1} \\ \hat{y}_t \\ \hat{c}_t \\ \hat{k}_t \\ \hat{x}_t \\ \hat{n}_t \\ \hat{r}_t \\ \hat{m}_t \\ \hat{\imath}_t \\ \mathrm{E}_t\hat{\lambda}_{t+1} \\ \mathrm{E}_t\pi_{t+1} \end{bmatrix} = A_2 \begin{bmatrix} \hat{z}_t \\ \hat{u}_t \\ \hat{y}_{t-1} \\ \hat{c}_{t-1} \\ \hat{k}_{t-1} \\ \hat{x}_{t-1} \\ \hat{n}_{t-1} \\ \hat{r}_{t-1} \\ \hat{m}_{t-1} \\ \hat{\imath}_{t-1} \\ \hat{\lambda}_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} e_{t+1} \\ \varphi_{t+1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

where

$$
A_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & \alpha-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{y^{ss}}{k^{ss}} & -\frac{c^{ss}}{k^{ss}} & 0 & -\delta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -\delta & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1+\eta\left(\frac{n^{ss}}{l^{ss}}\right) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Omega_1 & 0 & 0 & 0 & 0 & \Omega_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha\frac{y^{ss}}{k^{ss}}(1-\alpha)\eta\frac{n^{ss}}{l^{ss}} & 0 & 0 & \theta & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & \frac{1}{bi^{ss}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1
\end{bmatrix}
$$

$$
A_2 = \begin{bmatrix}
\rho_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\phi & \rho_u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1-\delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\alpha\frac{y^{ss}}{k^{ss}}\left(1+\eta\frac{n^{ss}}{l^{ss}}\right)\rho_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha(1-\alpha)\frac{y^{ss}}{k^{ss}} & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

and

$$
A = A_1^{-1} A_2.
$$

6 Other approaches

6.1 Uhlig's toolkit (Uhlig 1999)

Let $x_{1t} = (k_t, m_t)'$ be the vector of endogenous state variables, and let $x_{2t} = (y_t, c_t, n, \pi_t, i, r_t, \lambda_t)'$ be the vector of other endogenous variables. The equilibrium conditions of the MIU model can be written in the form

$$Ax_{1t} + Bx_{1t-1} + Cx_{2t} + D\psi_t = 0$$

$$F\mathrm{E}_t x_{1t+1} + Gx_{1t} + Hx_{1t-1} + J\mathrm{E}_t x_{2t+1} + Kx_{2t} + M\psi_t = 0$$

$$\psi_{t+1} = N\psi_t + \varepsilon_{t+1},$$

where $\psi_t = (z_t, u_t)'$. It is assumed that C is of full column rank and that the eigenvalues of N are all within the unit circle.

Then if an equilibrium solution to this system of equations exists, it takes the form of stable laws of motion

$$x_{1t} = Px_{1t-1} + Q\psi_t$$

$$x_{2t} = Rx_{1t-1} + S\psi_t$$

for x_{1t} and x_{2t}. When C is a square invertible matrix, Uhlig proves that P satisfies the quadratic matrix equation

$$(F - JC^{-1}A)P^2 - (JC^{-1}B - G + KC^{-1}A)P - KC^{-1}B + H = 0$$

and the equilibrium is stable if and only if all the eigenvalues of P are less than unity in absolute value. The matrix R is given by

$$R = -C^{-1}(AP + B),$$

while Q and S are given by

$$\begin{aligned} & \left(N' \otimes (F - JC^{-1}A) + I_k \otimes (JR + FP + G\_KC^{-1}A)\right) vec(Q) \\ = \quad & vec\left(\left(JC^{-1}D - L\right) N + KC^{-1}D - M\right) \end{aligned}$$

and

$$S = -C^{-1}\left(AQ + D\right).$$

Uhlig provides a fuller discussion and treats the case in which C is $l \times n$ with $l > n$.

6.1.1 More on eigenvalues

Consider general model of form

$$y_{t+1} = Ay_t + C\varepsilon_{t+1}. \tag{23}$$

Suppose y_t is $n \times 1$. Write A as $Q\Lambda Q^{-1}$ where Λ is a diagonal matrix of the eigenvalues of A and Q is the corresponding matrix of eigenvectors. Then premultiple (23) by Q^{-1}, obtaining

$$Q^{-1}y_{t+1} = \Lambda Q^{-1} y_t + Q^{-1} C \varepsilon_{t+1},$$

or

$$z_{t+1} = \Lambda z_t + \xi_{t+1}, \tag{24}$$

where $z_{t+1} = Q^{-1}y_{t+1}$ and $\xi_{t+1} = Q^{-1}C\varepsilon_{t+1}$.

Denote the diagonal elements of the matrix Λ by λ_i. Then (24) consists of n equations of the form

$$z_{i,t+1} = \lambda_i z_{i,t} + \xi_{i,t+1}$$

For all $\|\lambda_i\| > 1$, rewrite the equation as $z_{i,t} = \lambda_i^{-1} E_t \left(z_{i,t+1} - \xi_{i,t+1}\right)$ and solve forward:

$$\begin{aligned} z_{it} &= \left(\frac{1}{\lambda_i}\right) E_t \left(z_{it+1} - \xi_{it+1}\right) \\ &= -\left(\frac{1}{\lambda_i}\right) E_t \xi_{it+1} + \left(\frac{1}{\lambda_i}\right)^2 E_t \left(z_{it+2} - \xi_{it+2}\right) \\ &= -\sum_{j=0}^{\infty} \left(\frac{1}{\lambda_i}\right)^j E_t \xi_{it+j} \end{aligned}$$

For $\|\lambda_i\| < 1$, solve $z_{i,t+1} = \lambda_i z_{i,t} + \xi_{it+1}$ backward:

$$\begin{aligned} z_{it} &= \lambda_i z_{it-1} + \xi_{it} = \lambda_i \left(\lambda_i z_{it-2} + \xi_{it-1}\right) + \lambda_i \xi_{it} \\ &= \sum_{j=0}^{\infty} \lambda_i^j \xi_{it-j}. \end{aligned}$$

Choose the unique locally-bounded equilibrium be setting the variables and shocks associated with the forward-looking variables to zero. Write the variables associated with the eigenvalues less than 1 as

$$z_t^* = \Lambda^* z_{t-1}^* + \xi_t^*. \tag{25}$$

Let Q^* be the columns of Q corresponding to the eigenvalues less than 1. Then the solution is (25) and

$$y_t = Q^* z_t^*.$$

6.2 The optimal linear regulator approach

Gerali and Lippi (2XXX) discuss an approach to solving linear rational expectations models that build on the optimal linear regulator approach of Sargent

(XXXX). Consider a system of equations that can be written in the following state-space form:

$$AA \begin{bmatrix} X_{t+1} \\ E_t x_{t+1} \end{bmatrix} = AB \begin{bmatrix} X_t \\ x_t \end{bmatrix} + \begin{bmatrix} \psi_{t+1} \\ 0 \end{bmatrix},$$

where X consists of the n_1 predetermined variables, x the n_2 forward looking variables, and ψ the stochastic innovation.

The system can be re-written as

$$\begin{bmatrix} X_{t+1} \\ E_t x_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_t \\ x_t \end{bmatrix} + \begin{bmatrix} e_{t+1} \\ 0 \end{bmatrix} \tag{26}$$

where

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = A = (AA)^{-1} AB.$$

and A_{11} is $n_1 \times n_1$, A_{12} is $n_1 \times n_2$, A_{21} is $n_2 \times n_1$, and A_{22} is $n_2 \times n_2$.

In equilibrium,

$$x_{t+1} = G_{t+1} X_{t+1}.$$

Since (26) implies

$$X_{t+1} = A_{11} X_t + A_{12} x_t + e_{t+1}$$

and

$$E_t x_{t+1} = A_{21} X_t + A_{22} x_t,$$

it follows that

$$E_t x_{t+1} = A_{21} X_t + A_{22} x_t = G_{t+1} X_{t+1} = G_{t+1} (A_{11} X_t + A_{12} x_t).$$

Solving for x_t,

$$x_t = (A_{22} - G_{t+1} A_{12})^{-1} (G_{t+1} A_{11} - A_{21}) X_t,$$

where we assume $A_{22} - G_{t+1} A_{12}$ is non-singular. Hence,

$$x_t = G_t X_t = (A_{22} - G_{t+1} A_{12})^{-1} (G_{t+1} A_{11} - A_{21}) X_t.$$

The solution is given by the fixed point G of

$$G_t = (A_{22} - G_{t+1} A_{12})^{-1} (G_{t+1} A_{11} - A_{21}). \tag{27}$$

Given the solution G to (27), the equilibrium is

$$x_t = G X_t$$

and

$$\begin{aligned} X_{t+1} &= A_{11} X_t + A_{12} x_t + e_{t+1} = A_{11} X_t + A_{12} G X_t + e_{t+1} \\ &= (A_{11} + A_{12} G) X_t + e_{t+1} \\ &= H X_t + e_{t+1}, \end{aligned}$$

where $H = A_{11} + A_{12} G$.

The model is solved by iterating on (27) to obtain G. Given G, H can be easily calculated.