1 Introduction

This note provides more details on the derivation of the linear approximation used in Chapter 3 (3rd ed.) to study dynamics in the basic cash-in-advance (CIA) model. See miu_dynamics_3e.pdf for additional details on linearizing the equations common to both the CIA and MIU models and for a discussion of techniques for solving linear, rational expectations models.

2 The CIA model

The basic equilibrium conditions of the CIA model of Chapter 3 are given by

\[ y_t = e^x_t k_{t-1}^\alpha n_t^{1-\alpha} \], \hspace{1cm} (1)
\[ y_t = c_t + x_t \], \hspace{1cm} (2)
\[ x_t = k_t - (1-\delta)k_{t-1} \] \hspace{1cm} (3)
\[ \Psi(1-n_t)^{-\eta} = (1-\alpha) \left( \frac{y_t}{n_t} \right) \lambda_t \] \hspace{1cm} (4)
\[ R_t = \left[ \alpha E_t \left( \frac{y_{t+1}}{k_t} \right) + 1 - \delta \right] \] \hspace{1cm} (5)
\[ \lambda_t = \beta E_t R_t \lambda_{t+1} \] \hspace{1cm} (6)
\[ c_t^{\Phi} = \lambda_t(1+i_t) \] \hspace{1cm} (7)
\[ \lambda_t = \beta E_t \left( \frac{c_{t+1}^{\Phi}}{\Pi_{t+1}} \right) \] \hspace{1cm} (8)
\[ c_t = m_t \] \hspace{1cm} (9)
\[ m_t = \left( \frac{1+u_t}{1+\pi_t} \right) m_{t-1}, \] \hspace{1cm} (10)
\[ z_t = \rho_z z_{t-1} + e_t \] \hspace{1cm} (11)
\[ u_t = \rho_u u_{t-1} + \phi z_{t-1} + \varphi_t, \hspace{0.5cm} 0 \leq \gamma < 1, \] \hspace{1cm} (12)
3 The linear approximation

The next step is to derive first-order linear approximations to the model’s equilibrium conditions.

3.1 Functional forms

The utility function:

\[ u(c_t, m_t, 1 - n_t) = \frac{[aC_t^{1-b} + (1 - a)m_t^{1-b}]}{1 - \Phi} + \Psi \left[ \frac{(1 - n_t)^{1-\eta}}{1 - \eta} \right]. \]

The production function:

\[ y_t = e^{z_t} k_{t-1}^{\alpha} n_t^{1-\alpha} \]

4 The linear approximation

The next step is to derive first-order linear approximations to the model’s equilibrium conditions.

4.1 Functional forms

The utility function:

\[ u(c_t, m_t, 1 - n_t) = \frac{C_t^{1-\Phi}}{1 - \Phi} + \Psi \left[ \frac{(1 - n_t)^{1-\eta}}{1 - \eta} \right]. \]

The production function:

\[ y_t = e^{z_t} k_{t-1}^{\alpha} n_t^{1-\alpha} \]

4.2 Linearization

The production function, the goods clearing condition, the capital accumulation equation, the labor-leisure optimality condition, the real return equation, and the Euler condition are the same as in the MIU model. See miu_dynamics_3e.pdf for details. The equations for the evolution of real money balances and for the exogenous disturbances are also identical to those used for the MIU model.

4.2.1 Marginal utility of consumption

We can rewrite

\[ c_t^{-\Phi} = \lambda_t (1 + i_t) \]

as

\[ (c^{ss})^{-\Phi} (1 + \tilde{c}_t)^{-\Phi} \approx (c^{ss})^{-\Phi} (1 - \Phi \tilde{c}_t) = \lambda_t (1 + i_t) = \lambda^{ss} (1 + \hat{\lambda}_t) (1 + i_t) \approx \lambda^{ss} \left( 1 + \hat{\lambda}_t + i_t \right). \]
Since \((c^{ss})^{-\phi} = \lambda^{ss}(1 + \dot{i}^{ss})\), we have
\[-(c^{ss})^{-\phi} \Phi \hat{\lambda}_t \approx \lambda^{ss} \left(\lambda_t + \dot{i}_t - \dot{i}^{ss}\right) \Rightarrow -\Phi \hat{\lambda}_t = \lambda_t + \dot{i}_t.

### 4.2.2 Money holdings

From the first order condition with respect to money holdings and the result that \(c_{t}^{-\phi} = \lambda_t(1 + \dot{i}_t)\),
\[
\lambda_t = \beta E_t \left(\frac{c_{t+1}^{-\phi}}{\Pi_{t+1}}\right).
\]

Rewrite this as
\[
\lambda^{ss} \left(1 + \lambda_t\right) \approx \beta E_t \left[\frac{(c^{ss})^{-\phi} (1 - \Phi \hat{c}_{t+1})}{(1 + \pi_{t+1})}\right] \approx \beta (c^{ss})^{-\phi} E_t \left(1 - \Phi \hat{c}_{t+1} - \pi_{t+1}\right).
\]

Noting that
\[
\lambda^{ss} = \beta \left[\frac{(c^{ss})^{-\phi}}{1 + \pi^{ss}}\right],
\]
we have
\[
\lambda^{ss} \left(1 + \lambda_t\right) \approx \lambda^{ss}(1 + \pi^{ss}) E_t \left(1 - \Phi \hat{c}_{t+1} - \pi_{t+1}\right) \approx \lambda^{ss} E_t \left(1 - \Phi \hat{c}_{t+1} - \pi_{t+1} + \pi^{ss}\right).
\]
This implies that
\[
\hat{\lambda}_t = -E_t \left(\Phi \hat{c}_{t+1} + \hat{\pi}_{t+1}\right).
\]

### 4.2.3 Cash-in-advance constraint

From
\[
e_t = m_t
\]
we obtain
\[
\hat{c}_t = \hat{m}_t.
\]

### 4.2.4 The Fisher equation

From
\[-\Phi \hat{c}_t = \hat{\lambda}_t + \hat{i}_t
\]
and
\[
\hat{\lambda}_t = -E_t \left(\Phi \hat{c}_{t+1} + \hat{\pi}_{t+1}\right),
\]
we can derive the consumption Euler equation:
\[
\hat{c}_t = E_t \hat{c}_{t+1} - \left(\frac{1}{\Phi}\right) (\hat{i}_t - E_t \hat{\pi}_{t+1}).
\]
We can also use these same two equations, together with the Euler condition expressed in terms of \( \lambda_t \) to obtain
\[
\hat{\lambda}_t = -E_t (\Phi \hat{c}_{t+1} + \hat{\pi}_{t+1}) = \hat{\lambda}_t = E_t \left( \lambda_{t+1} + \hat{i}_{t+1} - \hat{\pi}_{t+1} \right),
\]
or
\[
E_t \hat{i}_{t+1} = \hat{r}_t + E_t \hat{\pi}_{t+1}.
\]

### 4.3 Some simplifications

As before, we will eliminate expected future output from the equation for the real rate of return to obtain
\[
\hat{r}_t = \alpha \left( \frac{y^{ss}}{k^{ss}} \right) \left\{ -(1 - \alpha) \eta \left( \frac{n^{ss}}{l^{ss}} \right) \hat{k}_t + (1 - \alpha) \lambda_t + \left[ 1 + \eta \left( \frac{n^{ss}}{l^{ss}} \right) \right] \rho_z z_t \right\}
\]
where
\[
\theta = \left[ \alpha + \eta \left( \frac{n^{ss}}{l^{ss}} \right) + \alpha (1 - \alpha) \left( \frac{y^{ss}}{k^{ss}} \right) \right]
\]

Notice that in contrast to the MIU model, there remain three forward-looking variables: the Lagrangian multiplier \( \lambda \), inflation \( \hat{\pi} \), and consumption \( \hat{c} \). However, we can use the cash-in-advance constraint to eliminate \( E_t \hat{c}_{t+1} \) by noting that
\[
E_t \hat{c}_{t+1} = E_t \hat{\pi}_{t+1} = E_t (u_{t+1} - \pi_{t+1} + \hat{m}_t) = \rho_u u_t + \hat{m}_t - E_t \hat{\pi}_{t+1}.
\]
Hence,
\[
\hat{\lambda}_t = -\Phi E_t \hat{c}_{t+1} - E_t \hat{\pi}_{t+1} = -\Phi \hat{m}_t - (1 - \Phi) E_t \hat{\pi}_{t+1} - \Phi \rho_u u_t.
\]

### 4.4 Collecting all equations

Unknowns: \( \hat{y}_t, \hat{k}_t, \hat{n}_t, \hat{x}_t, \hat{c}_t, \hat{\lambda}_t, \hat{r}_t, \hat{i}_t, \pi_t, \hat{m}_t \) – 10 variables.

Ten equations, plus the specification of the processes governing the exogenous productivity and money growth disturbances.

\[
z_t = \rho_z z_{t-1} + \epsilon_t
\]
\[
u_t = \rho_u u_{t-1} + \phi z_{t-1} + \varphi_t
\]
\[
\hat{y}_t = z_t + \alpha \hat{k}_{t-1} + (1 - \alpha) \hat{m}_t
\]
\[
\left( \frac{y^{ss}}{k^{ss}} \right) \hat{y}_t = \left( \frac{c^{ss}}{k^{ss}} \right) \hat{c}_t + \delta \hat{x}_t
\]
\[
\hat{k}_t = (1 - \delta) \hat{k}_{t-1} + \delta \hat{x}_t
\]
\[
\left[ 1 + \eta \left( \frac{n^{ss}}{l^{ss}} \right) \right] \hat{n}_t = \hat{y}_t + \lambda_t
\]
\[
\theta \hat{r}_t = \alpha \left( \frac{y^{ss}}{k^{ss}} \right) \left\{ - (1 - \alpha) \eta \left( \frac{n^{ss}}{l^{ss}} \right) \hat{k}_t + (1 - \alpha) \lambda_t + \left[ 1 + \eta \left( \frac{n^{ss}}{l^{ss}} \right) \right] \rho_z z_t \right\} \\
-\Phi \hat{c}_t = \hat{\lambda}_t + \hat{\nu}_t \\
\hat{c}_t = \hat{m}_t \\
\hat{m}_t = u_t - \pi_t + \hat{m}_{t-1} \\
\hat{\lambda}_t = \hat{\nu}_t + E_t \hat{\lambda}_{t+1} \\
\hat{\lambda}_t = -\Phi \hat{m}_t - (1 - \Phi) E_t \hat{\nu}_{t+1} - \Phi \rho_u u_t.
\]