On the Gross–Stark Conjecture and Refinements, I

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Outline

1. $L$-functions, Shintani’s Method, Cohomological Approach
2. $p$-adic $L$-functions, Gross–Stark conjecture
Let $F$ = totally real field, $\mathfrak{f} \subset \mathcal{O}_F$ ideal.

Ray class group

$$G_f = I(\mathfrak{f})/P(\mathfrak{f})$$

where

$$I(\mathfrak{f}) = \text{fractional ideals prime to } \mathfrak{f}$$

$$P(\mathfrak{f}) = \langle (\alpha) : \alpha \gg 0, \alpha \equiv 1 \pmod{\mathfrak{f}} \rangle.$$

Ray class character

$$\chi : G_f \rightarrow \overline{\mathbb{Q}}^*.$$
$L(\chi, s) = \sum_{a \subset \mathcal{O}_F, (a, f) = 1} \frac{\chi(a)}{N\alpha^s}$

$= \sum_{[b] \in G_f} \chi(b) \sum_{a \sim b \text{ in } G_f} N\alpha^{-s}$

$= \sum_{[b] \in G_f} \chi(b) Nb^{-s} \sum_{\alpha \in (b^{-1}f + 1)/E(f)} N\alpha^{-s}$

by the change of variables $a = b(\alpha)$. Here

$E(f) = \{ \epsilon \in \mathcal{O}_F^* : \epsilon \gg 0, \epsilon \equiv 1 \pmod{f} \}.$
Embed $F \subset \mathbb{R}^n$, where $n = [F : \mathbb{Q}]$.

**Theorem (Shintani)**

*There exists a fundamental domain $\mathcal{D}$ for the action of $E(f)$ on $(\mathbb{R}^*)^n$ consisting of a union of simplicial cones generated by elements of $F$.*

Simplicial cone:

$$C(v_1, \ldots, v_k) = \{t_1 v_1 + \ldots + t_k v_k : t_i > 0\}.$$
Example of Shintani domain, $n = 2$

$$\mathcal{D} = C(1, \epsilon) \cup C(1)$$
Shintani’s Theorems, cont’d

\[ L(\chi, s) = \sum_{[b] \in G_f} \chi(b) N b^{-s} \sum_{\alpha \in b^{-1} f + 1} N \alpha^{-s} \]

**Theorem (Shintani)**

The function

\[ \zeta(b, D, s) = \sum_{\alpha \in b^{-1} f + 1} N \alpha^{-s} \]

converges absolutely for \( \text{Re}(s) > 1 \) and extends by analytic continuation to a meromorphic function on \( \mathbb{C} \). Its values at non-positive integers \( s \) are rational. Therefore \( L(\chi, s) \in \mathbb{Q}(\chi) \) for such \( s \).
Let $\epsilon_1, \ldots, \epsilon_{n-1}$ be a basis for $E(f)$. For each $\sigma \in S_{n-1}$, let

$$v_{i,\sigma} = \epsilon_{\sigma(1)} \cdots \epsilon_{\sigma(i-1)} \in E(f), \quad i = 1, \ldots, n.$$  

**Theorem (Colmez)**

*Under certain sign conditions*

$$\bigsqcup_{\sigma \in S_{n-1}} C(v_1,\sigma, \ldots, v_n,\sigma)$$

*together with some boundary faces is a fundamental domain for the action of $E(f)$ on $(\mathbb{R}^*_+)^n$. In other words, for all $x \in (\mathbb{R}^*_+)^n$ we have*

$$\sum_{u \in E(f)} \sum_{\sigma \in S_{n-1}} 1_{C^*(v_1,\sigma, \ldots, v_n,\sigma)}(u \ast x) = 1$$
Examples

\[ n = 2: \]
\[ \mathcal{D} = C^*(1, \epsilon). \]

\[ n = 3: \]
\[ \mathcal{D} = C^*(1, \epsilon_1, \epsilon_1\epsilon_2) \cup C^*(1, \epsilon_2, \epsilon_1\epsilon_2). \]
Generalization and alternate proof

Theorem (Diaz y Diaz–Friedman, Charollois–D–Greenberg)

There is an explicit sign $\pm$ such that

$$
\pm \bigcup_{\sigma \in S_{n-1}} \sgn(\sigma) \sgn(\det(v_\sigma)) C^*(v_{1,\sigma}, \ldots, v_{n,\sigma})
$$

is a “signed fundamental domain” for the action of $E(\mathfrak{f})$ on $(\mathbb{R}_+\mathbb{R})^n$, i.e.,

$$
\pm \sum_{u \in E(\mathfrak{f})} \sum_{\sigma \in S_{n-1}} \sgn(\sigma) \sgn(\det(v_\sigma))1_{C^*(v_{1,\sigma}, \ldots, v_{n,\sigma})}(u \ast x) = 1
$$

for all $x \in (\mathbb{R}_+\mathbb{R})^n$.  

Clue from group homology

\[ \eta = \sum_{\sigma \in S_{n-1}} \text{sgn}(\sigma)[v_{1,\sigma}, \ldots, v_{n,\sigma}] \in \mathbb{Z}[E(f)^n] \]

represents a generator ("fundamental class") of

\[ H_{n-1}(E(f), \mathbb{Z}) \cong \mathbb{Z}. \]
Cocycle Property of Shintani’s Construction

\[ C(\mathbf{v}_1, \mathbf{v}_2) \cup C(\mathbf{v}_2, \mathbf{v}_3) = C(\mathbf{v}_1, \mathbf{v}_3) \]

Need to worry about:
- orientations
- boundaries
- degenerate situations
Let $v_1, \ldots, v_n \in F_+$. Let $C^*(v_1, \ldots, v_n) = \text{union of } C(v_1, \ldots, v_n) \text{ and its boundary faces that move into the interior when translated infinitesimally in the direction } (0, \ldots, 0, 1)$.

**Theorem (Charollois–D–Greenberg)**

The function

$$\Psi(v_1, \ldots, v_n) = \text{sgn}(\det(v_1, \ldots, v_n))1_{C^*(v_1, \ldots, v_n)}$$

is a homogeneous cocycle representing a class in

$$H^{n-1}(F_+^*, \text{Func}(\mathbb{R}_+^n, \mathbb{Z})).$$
Picture for \( n = 2 \)

\[
C^*(v_1, v_3) \\
C^*(v_1, v_2) \\
C^*(v_2, v_3)
\]
Signed version of Colmez’s Theorem, revisited

Cap product provides a pairing

\[ H^{n-1}(E(f), \text{Func}((R^*_+)^n, \mathbb{Z})) \times H_{n-1}(E(f), \mathbb{Z})) \longrightarrow H_0(E(f), \text{Func}((R^*_+)^n, \mathbb{Z})). \]

The signed version of Colmez’s Theorem simply says that if \( f \in \text{Func}((R^*_+)^n, \mathbb{Z}) \) represents \( \Psi \cap \eta \), then

\[ \sum_{u \in E(f)} f(u \ast x) = 1 \]

for all \( x \in (R^*_+)^n \).

Here \( \eta \) is an appropriate choice of generator for \( H_{n-1}(E(f), \mathbb{Z}) \cong \mathbb{Z} \).
Define

\[ \kappa_b(v_1, \ldots, v_n)(s) = Nb^{-s} \sum_{\alpha \in b^{-1}f + 1} \frac{\Psi(v_1, \ldots, v_n)(\alpha)}{(N\alpha)^s}. \]

By Shintani’s Theorem, this has a meromorphic continuation to \( \mathbb{C} \). Class of \( \kappa_b(s) \) in \( H^{n-1}(E(f), \mathbb{C}) \) depends only on \([b] \in G_f\). Define

\[ \kappa_{\chi}(s) = \sum_{[b] \in G_f} \chi(b)\kappa_b(s) \in H^{n-1}(E(f), \mathbb{C}). \]
The cap product provides a pairing

\[ H^{n-1}(E(f), C) \times H_{n-1}(E(f), \mathbb{Z}) \rightarrow C. \]

**Proposition**

*For the appropriate generator \( \eta \in H_{n-1}(E(f), \mathbb{Z}) \cong \mathbb{Z} \), we have*

\[ L(\chi, s) = \kappa_\chi(s) \cap \eta. \]

This amounts to an unraveling of definitions together with the signed version of Colmez’s Theorem stated earlier.
Cassou–Noguès gave a $p$-adic refinement of Shintani’s method.

**Theorem (Cassou-Noguès, Deligne–Ribet)**

*Suppose that $\chi$ is totally real. There exists a $p$-adic analytic (as long as $\chi \neq 1$, otherwise meromorphic)*

\[ L_p(\chi, s) : \mathbb{Z}_p \rightarrow \mathbb{Q}_p(\chi) \]

*such that*

\[ L_p(\chi, n) = L^*(\chi \omega^{n-1}, n) \]

*for all $n \in \mathbb{Z}, n \leq 0$.*

Here $^*$ denotes removing all Euler factors above $p$. 
The Gross–Stark Conjecture

Let $\chi$ be totally odd. Let $R = \{p \mid p: \chi(p) = 1\}$, $r = \#R$.

Conjecture (Gross 1981)

1. We have

$$\text{ord}_{s=0} L_p(\chi \omega, s) = r.$$ 

2. We have

$$\frac{L_p^{(r)}(\chi \omega, 0)}{r! L(\chi, 0)} = R_p(\chi) \prod_{\chi(p) \neq 1} (1 - \chi(p))$$

where $R_p(\chi) = \text{det}(M_p(\chi))$ is a certain $p$-adic regulator.
Results

Theorem (Wiles, Charollois–D, Spiess)
We have
\[ \text{ord}_{s=0} L_p(\chi \omega, s) \geq r. \]

Theorem (D–Kakde–Ventullo 2016)
The leading term formula in the Gross–Stark conjecture is true, i.e. we have
\[ \frac{L_p^{(r)}(\chi \omega, 0)}{r! L(\chi, 0)} = R_p(\chi) \prod_p (1 - \chi(p)). \]

What remains is to prove that \( R_p(\chi) \neq 0 \).
For $p \in R$, let

$$U_p := \{ u \in K^* : \text{ord}_\mathfrak{P}(u) = 0 \text{ for all } \mathfrak{P} \nmid p \}. $$

Write

$$U_{p,\chi} := (U_p \otimes \overline{Q})^{\chi^{-1}}$$

$$= \{ u \in U_p \otimes \overline{Q} : \sigma(u) = u^{\chi^{-1}(\sigma)} \text{ for all } \sigma \in G \}. $$

Then

$$\dim_{\overline{Q}} U_{p,\chi} = 1.$$ 

Let $u_{p,\chi}$ denote any generator (i.e., non-zero element) of $U_{p,\chi}$. 

Explicit description of $M_p$
Define

\[ o_p = \text{ord}_p : F_p^* \rightarrow \mathbb{Z} \]

\[ \ell_p = \log_p \circ \text{Norm}_{F_p/Q_p} : F_p^* \rightarrow \mathbb{Z}_p, \]

where \( \log_p : \mathbb{Q}_p^* \rightarrow \mathbb{Z}_p \) denotes Iwasawa’s \( p \)-adic logarithm.
Explicit description of $M_p$, cont’d

For each $p \in R$ choose $\mathfrak{p}_p$ of $K$ lying above $p$. Then for $p, q \in R$:

$$U_p \subset K \subset K_{\mathfrak{p}_q} \cong F_q.$$ 

Extend $o_q, \ell_q$ by linearity:

$$o_q, \ell_q : U_{p, \chi} \longrightarrow C_p.$$

Define

$$\mathcal{L}_{\text{alg}}(\chi)_{p, q} = - \frac{\ell_q(u_{p, \chi})}{o_p(u_{p, \chi})}.$$

$$M_p := (\mathcal{L}_{\text{alg}}(\chi)_{p, q})_{p, q \in R}.$$
Ingredients In Proof of Gross–Stark

- Reinterpret the relevant unit group in terms of $p$-adic Galois cohomology.
- Use $p$-adic families of Hilbert modular forms. Eisenstein series have $L$-functions as constant terms.
- Use Galois representations associated to modular forms to define the necessary Galois cohomology classes.
- Show the cohomology classes satisfy the necessary formulae via their explicit construction with Eisenstein series.

Previous works that served as background for this proof (and for D–Darmon–Pollack): Ribet, Mazur–Wiles, Greenberg–Stevens. However, this is a particularly delicate case that involves many new features, e.g., the introduction of an $r \times r$ determinant.
First Step—Galois Cohomology

Let $E = \mathbb{Q}_p(\chi)$, finite extension of $\mathbb{Q}_p$.

Let $H^1_R(G_F, E(\chi^{-1})) = \text{cohomology classes unramified outside } R$.

Given $p \in R$, we have

$$\text{res}_p : H^1_R(G_F, E(\chi^{-1})) \longrightarrow H^1(G_p, E) = \text{Hom}_{\text{cont}}(G_p, E) = \text{Hom}_{\text{cont}}(\hat{F}_p^*, E).$$
Poitou–Tate Duality

Via

$$\mathcal{O}_K[1/p]^* \subset K^*_p = F^*_p,$$

we can view $\text{res}_p \kappa$ as giving a homomorphism $\mathcal{O}_K[1/p]^* \to E$. Extend by $E$-linearity to

$$\text{res}_p \kappa : (\mathcal{O}_K[1/p]^* \otimes E)^{\chi^{-1}} \to E.$$

Proposition

If $\kappa \in H^1_R(G_F, E(\chi^{-1}))$ and $u \in (\mathcal{O}_K[1/p]^* \otimes E)^{\chi^{-1}}$, then

$$\sum_{p \in R} (\text{res}_p \kappa)(u) = 0.$$
The $p$-units $u_{p, \chi}$ for $p \in R$ form an $E$-basis for $(\mathcal{O}_K[1/p]^* \otimes E)^{\chi^{-1}}$. We have $r$ equations

$$\sum_{p \in R} (\text{res}_p \kappa)(u_{q, \chi}) = 0.$$ 

This implies

$$\det((\text{res}_p \kappa)(u_{q, \chi}))_{p, q} = 0.$$ 

In our application, the cocycle $\kappa$ will take values in a complicated ring, where the vanishing of this determinant yields the desired result. Details will be given in the next talk.
Goal for remainder of talk

We would like to refine the Gross–Stark conjecture by describing the characteristic polynomial of $M_p(\chi)$.

Even more, if $J \subset R$, we would like to give a conjectural formula for the principal minor of $M_p(\chi)$ associated to $J$.

This is the determinant of the $j \times j$ matrix given by the intersection of the rows and columns indexed by $J$, where $j = \#J$.

For example, the diagonal elements are the $1 \times 1$ principal minors.
Let

\[ F_R := \prod_{p \in R} F_p, \quad E_R(f) = \{ \epsilon \in \mathcal{O}_{F,R}^* : \epsilon \gg 0, \epsilon \equiv 1 \pmod{f} \}. \]

We will define a \( p \)-adic version of the Shintani–Eisenstein Cocycle

\[ \psi_{\chi} \in H^{n-1}(E_R(f), \text{Meas}(F_R, \mathcal{O}_E)). \]

For each \( J \subset R \), we will define

\[ \eta_J \in H_{n-1}(E_R(f), C_c(F_R, E)) \]

and conjecture that the principal minor of \( M_p(\chi) \) is equal to

\[ \frac{\psi_{\chi} \cap \eta_J}{\psi_{\chi} \cap \eta_\phi} \in E \]
Integration Pairing

The integration pairing

\[ \text{Meas}(F_R, \mathcal{O}_E) \times C_c(F_R, E) \longrightarrow E \]

\[ (\mu, f) \longmapsto \int_{F_R} f \, d\mu \]

induces the cap product pairing

\[ H^{n-1}(E_R(f), \text{Meas}(F_R, \mathcal{O}_E)) \times H_{n-1}(E_R(f), C_c(F_R, E)) \rightarrow E \]
For $U \subset F_R$ compact open, let

$$
\psi_{\chi}(v_1, \ldots, v_n)(U) = \sum_{[b] \in G_f} \chi(b) \left( \sum_{\substack{\alpha \in b^{-1}_R f + 1 \\ \alpha \in U}} \frac{\psi(v_1, \ldots, v_n)(\alpha)}{(N\alpha)^s} \right) |_{s=0}.
$$

Here $b^{-1}_R := b^{-1} \mathcal{O}_{F, R}$. 
The function $\psi_\chi$ is a homogeneous $(n - 1)$-cocycle representing a class

$$\psi_\chi \in H^{n-1}(E_R(f), \text{Dist}(F_R, E)).$$

After smoothing using an auxiliary prime $\lambda$ of $F$, we can define

$$\psi_{\chi, \lambda} \in H^{n-1}(E_R(f), \text{Meas}(F_R, O_E)).$$

We ignore $\lambda$ for notational simplicity in this talk.
Spiess’s $c$-construction

Let $g : F^*_p \to E$ be a continuous homomorphism.

Define

$$c_g(a) = " (1 - a)(g \cdot 1_{\mathcal{O}_p})"$$

$$= 1_{a\mathcal{O}_p} \cdot g(a) + (1_{\mathcal{O}_p} - 1_{a\mathcal{O}_p}) \cdot g.$$ 

Then $c_g \in H^1(F^*_p, C_c(F_p, E))$. 
Spiess’s $c$-construction, cont’d

For $J \subset R$, let

$$g_i = \begin{cases} 
\ell_{p_i} & \text{if } p_i \in J \\
o_{p_i} & \text{if } p_i \not\in J.
\end{cases}$$

Obtain $c_{g_i} \in H^1(F_{p_i}^*, C_c(F_{p_i}, E))$ for $i = 1, \ldots, r$.

Define

$$c_J = c_{g_1} \cup c_{g_2} \cdots \cup c_{g_r} \in H^r(F_R^*, C_c(F_R, E)).$$
Let $\vartheta \in H_{n+r-1}(E_R(f), \mathbb{Z}) \cong \mathbb{Z}$ be a generator.

**Conjecture (D–Spiess)**

The principal minor of $M_p(\chi)$ associated to a subset $J \subset R$ is given by

$$\det (\mathcal{L}_{\text{alg}}(\chi)_{p,q})_{p,q \in J} = \frac{c_J \cap (\psi_{\chi} \cap \vartheta)}{c_{\phi} \cap (\psi_{\chi} \cap \vartheta)}.$$

Therefore, the characteristic polynomial of $M_p(\chi)$ is given by

$$\det(x \cdot I_r - M_p(\chi)) = \frac{\bigcup_{p \in R} (xc_{op} + c_{lp}) \cap (\psi_{\chi} \cap \vartheta)}{\bigcup_{p \in R} c_{op} \cap (\psi_{\chi} \cap \vartheta)}.$$
The conjecture is well-formed:

- It does not depend on the choices of representatives $\mathfrak{b}$ for $G_f$.
- It does not depend on the choice of auxiliary prime $\lambda$.
- For the appropriate generator $\vartheta \in H^{n+r-1}(E_R(f), \mathbb{Z})$, we have

\[
c_\phi \cap (\psi \chi \cap \theta) = L(\chi, 0) \prod_{\chi(p) \neq 1} (1 - \chi(p)) \neq 0.
\]
Results, cont’d

**Theorem**

For the appropriate generator $\vartheta \in H^{n+r-1}(E_R(f), \mathbb{Z})$, we have

$$c_R \cap (\psi \chi \cap \vartheta) = \frac{L_p^{(r)}(\chi, 0)}{r!},$$

so the conjecture holds for $J = R$ by the theorem of D–Kakde–Ventullo.
For \( \#J = 1 \), our conjecture for

\[ \mathcal{L}_{\text{alg}}(\chi)_{p,p} \]

should agree with the conjecture that I made in a previous paper (Duke 2008).

**Theorem**

*This is true (i.e. the two conjectures are equivalent) if \( F \) is a real quadratic field.*

My student Shawn Tsosie is currently working on the general case.
Is there an analytic formula for the eigenvalues of $M_p(\chi)$ similar to the D–Spiess formula for the characteristic polynomial?

Specifically:

- Are the eigenvalues indexed by the primes $p \in S$ in a natural way?
- Are the eigenvalues algebraic linear combinations of $p$-adic logarithms of algebraic integers? (Likely not.)