10.5.1 Let
\[ \begin{array}{ccc}
A & \xrightarrow{\psi} & B \\
\downarrow{\alpha} & & \downarrow{\beta} \\
A' & \xrightarrow{\psi'} & B'
\end{array} \quad \begin{array}{ccc}
B & \xrightarrow{\varphi} & C \\
\downarrow{\gamma} & & \downarrow{\delta} \\
C' & \xrightarrow{\varphi'} & D
\end{array} \]
be a commutative diagram with exact rows.

10.5.1a We show that if \( \varphi \) and \( \alpha \) are surjective, and \( \beta \) is injective, then \( \gamma \) is injective. Let \( c \in \ker \gamma \). Since \( \varphi \) is surjective, \( c = \varphi(b) \) for some \( b \in B \). By commutativity of the diagram, \( \varphi'(\beta(b)) = \gamma(\varphi(b)) = 0 \). Thus, \( \beta(b) \in \ker \varphi' \) and by exactness, \( \beta(b) = \psi'(a') \) for some \( a' \in A' \). Since \( \alpha \) is surjective, \( a' = \alpha(a) \) for some \( a \in A \). By commutativity of the diagram, \( \beta(b) = \psi'(\alpha(a)) = \beta(\psi(a)) \). Since \( \beta \) is injective, \( b = \psi(a) \).

Thus, \( c = \varphi(\psi(a)) = 0 \) by exactness of the upper row, so \( \gamma \) is injective.

10.5.1b We show that if \( \psi', \alpha, \) and \( \gamma \) are injective, then \( \beta \) is injective. Let \( b \in \ker \beta \). Then \( \varphi'(\beta(b)) = \gamma(\varphi(b)) = 0 \) by the commutativity of the second square. Since \( \gamma \) is injective, this means that \( \varphi(b) = 0 \). In particular, \( b \in \ker \varphi \) so \( b = \psi(a) \) for some \( a \in A \) by exactness of the upper row. Then \( \beta(\psi(a)) = \psi'(\alpha(a)) = 0 \) by commutativity of the left square so \( \alpha(a) = 0 \) by the injectivity of \( \psi' \) and thus \( a = 0 \) by the injectivity of \( \alpha \). So, \( b = \psi(0) = 0 \).

10.5.1c We show that if \( \varphi, \alpha, \) and \( \gamma \) are surjective, then \( \beta \) is surjective. Let \( b' \in B' \). Then \( \varphi'(b') \in C' \). By surjectivity of \( \gamma \), \( \varphi'(b') = \gamma(c) \) for some \( c \in C \). By surjectivity of \( \varphi \), \( c = \varphi(b) \) for some \( b \in B \). By the commutativity of the right square, \( \gamma(\varphi(b)) = \varphi'(\beta(b)) = \varphi'(b') \). But this means that \( \varphi'(\beta(b) - b') = 0 \) so that \( \beta(b) - b' \in \ker \varphi' \). Then \( \beta(b) - b' = \psi'(a') \) for some \( a' \in A' \). By surjectivity of \( \alpha \), \( a' = \alpha(a) \) for some \( a \in A \). Thus, \( \beta(b) - b' = \psi'(\alpha(a)) = \beta(\psi(a)) \) by the commutativity of the left square. Thus, \( b' = \beta(b - \psi(a)) \) so \( \beta \) is surjective.

10.5.1d Let \( \beta \) be injective and \( \alpha \) and \( \varphi \) be surjective. We show \( \gamma \) is injective. Let \( c \in \ker \gamma \). Then by surjectivity of \( \varphi \), \( c = \varphi(b) \) for some \( b \in B \). Then \( \gamma(\varphi(b)) = \varphi'(\beta(b)) = 0 \) by commutativity of the right square. So, \( \beta(b) \in \ker \varphi' \). Thus, \( \beta(b) = \psi'(a') \) for some \( a' \in A' \) by exactness of the second row. By surjectivity of \( \alpha \), \( a' = \alpha(a) \) for some \( a \in A \). Then \( \psi'(\alpha(a)) = \beta(\psi(a)) = \beta(b) \) by commutativity of the left square. By injectivity of \( \beta \), this means \( \psi(a) = b \). Thus, \( c = \varphi(\psi(a)) = 0 \) so \( \gamma \) is injective.

10.5.1e We show that if \( \beta \) is surjective and \( \gamma \) and \( \psi' \) are injective, then \( \alpha \) is surjective. Let \( a' \in A' \). By surjectivity of \( \beta \), \( \psi'(a') = \beta(b) \) for some \( b \in B \). Note that \( \varphi'(\psi'(a')) = 0 \) by exactness of the lower row. Then \( \varphi'(\beta(b)) = \gamma(\varphi(b)) = 0 \). Since \( \gamma \) is injective, \( \varphi(b) = 0 \). So, \( b = \psi(a) \) for some \( a \in A \). Thus, \( \psi'(a') = \beta(\psi(a)) = \psi'(\alpha(a)) \). By injectivity of \( \psi' \), \( \alpha(a) = a' \) so \( \alpha \) is surjective.

10.5.2 Let
\[ \begin{array}{ccc}
A & \xrightarrow{\psi} & B \\
\downarrow{\alpha} & & \downarrow{\beta} \\
A' & \xrightarrow{\psi'} & B'
\end{array} \quad \begin{array}{ccc}
B & \xrightarrow{\varphi} & C \\
\downarrow{\gamma} & & \downarrow{\delta} \\
C' & \xrightarrow{\varphi'} & D
\end{array} \]
be a commutative diagram with exact rows.

10.5.2a Let \( \alpha \) be surjective and \( \beta \) and \( \delta \) be injective. We show \( \gamma \) is injective. Let \( c \in \ker \gamma \). Then \( \theta'(\gamma(c)) = 0 = \delta(\theta(c)) \). By injectivity of \( \delta \), \( \theta(c) = 0 \) so \( c = \varphi(b) \) for some \( b \in B \). This means \( \gamma(\varphi(b)) = \varphi'(\beta(b)) = 0 \). By exactness of the second row, \( \beta(b) = \psi'(a') \) for some \( a' \in A' \). Since \( \alpha \) is surjective, \( a' = \alpha(a) \) for some \( a \in A \). Thus, \( \psi'(\alpha(a)) = \beta(\psi(a)) = \beta(b) \). But \( \beta \) is injective so \( b = \psi(a) \). But the first row is exact, so \( c = \varphi(\psi(a)) = 0 \) and \( \gamma \) is injective.

10.5.2b Let \( \delta \) be injective, and let \( \alpha \) and \( \gamma \) be surjective. We show \( \beta \) is surjective. Let \( b' \in B' \). Then \( \varphi'(b') = \gamma(c) \) for some \( c \in C \) since \( \gamma \) is surjective. By exactness of the second row, \( \theta'(\varphi'(b')) = \theta'(\gamma(c)) = 0 \). By the commutativity of the right-most square, \( \delta(\theta(c)) = \theta'(\gamma(c)) = 0 \). But, \( \delta \) is injective so that \( \theta(c) = 0 \). Thus, \( c = \varphi(b) \) for some \( b \in B \). By commutativity of the central square, \( \varphi'(b') = \gamma(\varphi(b)) = \varphi'(\beta(b)) \). Thus, \( \varphi'(\beta(b) - b') = 0 \). By exactness of the second row, \( \psi'(a') = \beta(b) - b' \)
for some \( a' \in A' \). By surjectivity of \( \alpha, a' = \alpha(a) \) for some \( a \in A \). By commutativity of the first square, 
\[
\psi'(\alpha(a)) = \beta(\psi(a)) = \beta(b) - b'
\]
so that \( b' = \beta(\psi(a) - b) \). Thus, \( \beta \) is surjective.

10.5.12 Let \( A \) be an \( R \)-module and let \( I \) be any nonempty index set and for each \( i \in I \), let \( B_i \) be an \( R \)-module.

10.5.12a. We show there is an isomorphism of abelian groups \( \text{Hom}(\oplus_{i \in I} B_i, A) \cong \prod_{i \in I} \text{Hom}(B_i, A) \). Consider the inclusion map \( \psi : \oplus_{i \in I} B_i \to \prod_{i \in I} B_i \). We define \( \varphi : \text{Hom}(\oplus_{i \in I} B_i, A) \to \prod_{i \in I} \text{Hom}(B_i, A) \) via 
\[
\varphi(\psi) = \{ \psi \circ \iota_i \}_{i \in I}.
\]
We check this is a homomorphism of abelian groups. Take \( \psi_1, \psi_2 \in \text{Hom}(\oplus_{i \in I} B_i, A) \). Then 
\[
\varphi(\psi_1 + \psi_2) = \{ (\psi_1 + \psi_2) \circ \iota_i \}_{i \in I} = \{ \psi_1 \circ \iota_i + \psi_2 \circ \iota_i \}_{i \in I} = \varphi(\psi_1) + \varphi(\psi_2).
\]

For surjectivity, note that for \( \{ f_i \}_{i \in I} \in \prod_{i \in I} \text{Hom}(B_i, A) \), each \( f_i \in \text{Hom}(B_i, A) \). Thus by the universal property of direct sum, we have a map \( g : \oplus_{i \in I} B_i \to A \) with \( f_i = g \circ \iota_i \). Hence \( \varphi(g) = \{ f_i \}_{i \in I} \). Note that \( g \) is given explicitly by \( g(b_i)_{i \in I} = \sum_{i \in I} f_i(b_i) \). This sum, which looks infinite, is actually finite since all but finitely many of the \( b_i \) are zero by the definition of the direct sum.

For injectivity, let \( f \in \ker \varphi \). Then for all \( b_1, b_2 \in \oplus_{i \in I} B_i \), \( b = \sum_i i \cdot b_i \) with \( b_i \in B_i \). Then 
\[
f(b) = \sum_i f \circ \iota_i(b_i) = 0
\]
for all \( b \), so \( f = 0 \). The second equality in the displayed equation follows since \( f \circ \iota_i = 0 \) for all \( i \), since \( f \in \ker \varphi \).

Therefore, \( \varphi \) is an isomorphism of abelian groups.

Now suppose that \( R \) is commutative. To show that \( \varphi \) is an \( R \)-module isomorphism, it remains only to show that \( \varphi \) respects the \( R \)-module action. For \( r \in R \), \( \varphi(r \psi) = \{ (r \psi) \circ \iota_i \}_{i \in I} = r \{ \psi \circ \iota_i \}_{i \in I} = r \varphi(\psi) \).

Therefore, \( \varphi \) is an \( R \)-module homomorphism.

10.5.12b. We show \( \text{Hom}(A, \prod_{i \in I} B_i) \cong \prod_{i \in I} \text{Hom}(A, B_i) \). Similarly, we define \( \pi_i : \prod_{i \in I} B_i \to B_i \) to be the projection and let 
\[
\varphi : \text{Hom}(A, \prod_{i \in I} B_i) \to \prod_{i \in I} \text{Hom}(A, B_i)
\]
be defined by \( \varphi(\psi) = \{ \pi_i \circ \psi \}_{i \in I} \). Then this is a homomorphism of abelian groups since for \( \psi_1, \psi_2 \in \text{Hom}(A, \prod_{i \in I} B_i) \), we have 
\[
\varphi(\psi_1 + \psi_2) = \{ \pi_i \circ (\psi_1 + \psi_2) \}_{i \in I} = \{ \pi_i \circ \psi_1 + \pi_i \circ \psi_2 \}_{i \in I} = \varphi(\psi_1) + \varphi(\psi_2).
\]

For surjectivity, take \( \{ f_i \}_{i \in I} \in \prod_{i \in I} \text{Hom}(A, B_i) \). Then \( f_i \in \text{Hom}(A, B_i) \). By the universal property of the direct product, we have a map \( g : A \to \prod_{i \in I} B_i \) with \( f_i = \pi_i \circ g \). Then \( \varphi(g) = \{ f_i \}_{i \in I} \). Explicitly, \( g \) is given by \( g(a) = \{ f_i(a) \}_{i \in I} \).

For injectivity, let \( f \in \ker \varphi \). Then \( \varphi(f) = \{ \pi_i \circ f \}_{i \in I} = 0 \). So for any \( a \in A \), we have \( \pi_i(f(a)) = 0 \) for all \( i \), which implies \( f(a) = 0 \). Thus \( f = 0 \).

Therefore \( \varphi \) is an isomorphism of abelian groups. If \( R \) is commutative, we can check that \( \varphi \) respects the \( R \)-action: \( \varphi(r \psi) = \{ \pi_i \circ (r \psi) \}_{i \in I} = r \{ \pi_i \circ \psi \}_{i \in I} = r \varphi(\psi) \).

10.5.14 Let 
\[
0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0
\]
be a sequence of \( R \)-modules.
10.5.14a We show

$$0 \longrightarrow \text{Hom}_R(D, L) \xrightarrow{\psi'} \text{Hom}_R(D, M) \xrightarrow{\varphi'} \text{Hom}_R(D, N) \longrightarrow 0$$

is a short exact sequence of abelian groups for all $R$-modules $D$ if and only if the original sequence is a split short exact sequence.

($\Leftarrow$). Let the module sequence be a split short exact sequence. Then, there exists a homomorphism $u : N \rightarrow M$ such that $\varphi \circ u = \text{id}$ and similarly $\lambda : M \rightarrow L$ such that $\lambda \circ \psi = \text{id}$. Let $\alpha : D \rightarrow N$ be a module homomorphism. Then $u \circ \alpha : D \rightarrow M$ is a module homomorphism satisfying $\varphi'(u \circ \alpha) = (\varphi \circ u) \circ \alpha = \text{id} \circ \alpha = \alpha$. So, $\varphi'$ is surjective.

Theorem 28 gives the rest of the exactness we want.

($\Rightarrow$). Suppose the Hom sequence is a short exact sequence for all $D$. Then for $D = N$, $\varphi'$ is surjective so we can lift the identity map, as in the hint. We show that the lift of the identity map in $\text{Hom}_R(N, N)$ to $\text{Hom}_R(N, M)$ is a splitting homomorphism for $\varphi$. In particular, let $\lambda$ be the lift of the identity map on $N$. Then $\varphi \circ \lambda = \text{id}$. This is the splitting homomorphism we wanted. Since $\varphi(\alpha(n)) = n$ for all $n \in N$, we see that $\varphi$ is surjective.

Next we show that $\psi$ is injective. Let $D = \ker \psi$ and let $f \in \text{Hom}_R(D, L)$ be the canonical inclusion. Then $\psi(f)$ is easily seen to be 0. But $\psi'$ is injective by assumption, so $f = 0$, which implies that $D = \ker \psi = 0$.

It remains to show that $\text{im} \psi = \ker \varphi$. First, to see that $\varphi \circ \psi = 0$, let $D = L$ and consider the identity map in $\text{Hom}_R(L, L)$. Its image in $\text{Hom}_R(L, N)$ under $\varphi' \circ \psi'$ is precisely $\varphi \circ \psi$, by definition. Yet $\varphi' \circ \psi' = 0$, so $\varphi \circ \psi = 0$. Therefore $\text{im} \psi \subset \ker \varphi$.

Finally, let $D = \ker \varphi$ and consider the canonical inclusion $g \in \text{Hom}_R(D, M)$. By definition, $\varphi'(g) = 0$. Then by exactness, there exists $f \in \text{Hom}_R(D, L)$ such that $\psi'(f) = g$, i.e. such that $\psi \circ f = g$. Taking images of both sides, we see that $\text{im} \psi = \text{im} \psi \circ f \subset \text{im} \psi$. But $\text{im} \psi = \ker \varphi$ by the definition of $g$, hence $\ker \varphi \subset \text{im} \psi$ as desired. This completes the proof.

10.5.14b We show

$$0 \longrightarrow \text{Hom}_R(N, D) \xrightarrow{\psi'} \text{Hom}_R(M, D) \xrightarrow{\varphi'} \text{Hom}_R(L, D) \longrightarrow 0$$

is a short exact sequence of abelian groups for all $R$-modules $D$ if and only if the original sequence is a split short exact sequence.

($\Leftarrow$) Suppose the original sequence is split. By Theorem 33, all we need to show is that $\psi'$ is surjective.

Let $\alpha : L \rightarrow D$ be a module homomorphism. Then $\alpha \circ \lambda : M \rightarrow D$ is a module homomorphism (with $\lambda \circ \psi = \text{id}$ defined as in (a)) such that $\psi'(\alpha \circ \lambda) = \alpha \circ \lambda \circ \psi = \alpha \circ \text{id} = \alpha$. So, we get surjectivity.

($\Rightarrow$) Conversely, suppose that the induced sequence is short exact for all $D$. We first show that the original sequence is short exact. First let $D = L$ and consider the identity in $\text{Hom}_R(L, D)$. By surjectivity, there exists $g \in \text{Hom}_R(M, D)$ such that $\psi'(g) = \text{id}_L$, i.e. such that $g \circ \psi = \text{id}_L$. If $\ell \in \ker \psi$, then applying this equality to $\ell$ we obtain $g(0) = \ell$, i.e. $\ell = 0$. Therefore $\ker \psi = 0$, i.e. $\psi$ is injective.

For the surjectivity of $\varphi$, let $D = N/\text{im} \varphi$ and let $f \in \text{Hom}_R(N, D)$ be the canonical projection. Then clearly $\varphi'(f) \in \text{Hom}_R(M, D)$ is zero, hence by injectivity of $\varphi'$ we see that $f = 0$. This implies that $D = 0$, i.e. that $\varphi$ is surjective.

To see that $\varphi \circ \psi = 0$, let $D = N$ and consider the identity $\text{id}_N \in \text{Hom}_R(N, D)$. Then $\psi \circ \varphi(\text{id}_N) = 0$ implies that $\varphi \circ \psi = 0$. To conclude the proof of exactness, we must show that $\ker \varphi \subset \text{im} \psi$. For this, let $D = M/\text{im} \psi$ and consider the canonical projection $f \in \text{Hom}_R(M, D)$. By construction, $\psi'(f) = 0$, hence there exists $g \in \text{Hom}(N, D)$ such that $\varphi'(g) = f$, i.e. such that $g \circ \varphi = f$. Hence if $m \in \ker \varphi$, it follows that $f(m) = f(\varphi(m)) = f(0) = 0$. By the definition of $f$ and $D$, this implies that $m \in \text{im} \psi$. Thus $\ker \varphi \subset \text{im} \psi$ as desired.

Finally, to see the original sequence is split, let $D = L$ and consider the identity $\text{id}_L \in \text{Hom}_R(L, L)$. Then by surjectivity, there exists $g \in \text{Hom}_R(M, L)$ such that $\varphi'(g) = \text{id}_L$. This $g$ is a splitting of the original short exact sequence.
10.5.27a Clearly, $X \subset A \oplus B$. We show that $X = \{(a,b) \mid a \in A, b \in B, f(a) = g(b)\}$ is an $R$-submodule of the direct sum $A \oplus B$ with $f : A \to M$ and $g : B \to M$. First, note that $(0,0) \in X$. Then, note that for $(a_1,b_1)$ and $(a_2,b_2)$ in $X$ and $r \in R$, we have $f(a_1 + ra_2) = f(a_1) + rf(a_2) = g(b_1) + rg(b_2) = g(b_1 + rb_2)$ so that $X$ is a submodule.

For the commutativity, $g \circ \pi_2(a,b) = g(b) = f(a) = f(\pi_1(a,b))$ for any $(a,b) \in X$. In particular, we construct the diagram by naturally considering $\pi_1, \pi_2$ and then using the maps $f$ and $g$.

10.5.27b Let $\{f'(m),-g'(m)\mid m \in M\} = Q$. We only need to show $Q$ is a submodule of $A \oplus B$. Clearly, $Q \subset A \oplus B$. Also, $(0,0) \in Q$ since $f'(0) = 0 = -g'(0)$. Now, for $m_1, m_2 \in M$ and $r \in R$, we have

$$f'(m_1),-g'(m_1)) + r(f'(m_2),-g'(m_2)) = (f'(m_1) + r f'(m_2), -g'(m_1) - rg'(m_2)) \in Q$$

via the property that $f',g'$ are $R$-module homomorphisms and adding componentwise. So, $Q$ is a submodule and $Y$ is an $R$-module.

For the commutativity, we want to show $\pi'_2 \circ g'(m) = \pi'_1 \circ f'(m)$ in the quotient. We have

$$\pi'_2 \circ g'(m) = (0,g'(m))$$
$$\pi'_1 \circ f'(m) = (f'(m),0)$$

Subtracting the two, we get $f'(m),-g'(m)$ so that the diagram commutes.

11.1.5 We show that the space of continuous real-valued functions on the closed interval $[a,b]$ is an infinite dimensional vector space over $\mathbb{R}$, where $a < b$.

Note that polynomials are continuous real-valued functions, so we define $U_n = \{x^k \mid 0 \leq k \leq n\}$ for each integer $n \geq 0$. $U_n$ is linearly independent since

$$\sum_{i=0}^{n} r_i x^i = 0 \implies \frac{d^n}{dx^n} \sum_{i=0}^{n} r_i x^i = n! r_n = 0.$$  

So, $r_n = 0$. Then, we take the $n-1$th derivative to show $r_{n-1} = 0$. Inductively, this shows that $r_i = 0$ for each $i$ so $B_n$ is a linearly independent set of continuous functions. In particular, this polynomial space is infinite dimensional and a subset of the set of continuous functions, so that our original space is infinite dimensional as well.

11.1.6 Let $V$ be a FDVS (finite-dimensional vector space). Let $\varphi : V \to V$ be a linear transformation. We show there is an integer $m$ such that the intersection of the image of $\varphi^m$ and the kernel of $\varphi^m$ is $\{0\}$. Note that if $\varphi^k(x) = 0$, then $\varphi^{k+1}(x) = 0$ as well. Hence we get a chain $\ker \varphi \subset \ker \varphi^2 \subset \ker \varphi^3 \subset \ldots$ By the finite-dimensionality of $V$, this chain stabilizes at some point, i.e., there exists an $m \geq 1$ such that $\ker \varphi^m = \ker \varphi^{m+i}$ for all $i \geq 0$. We show that $\ker \varphi^m \cap \operatorname{im} \varphi^m = \{0\}$. To show this, let $x \in \ker \varphi^m \cap \operatorname{im} \varphi^m$. Then $x = \varphi^m(y)$ for some $y \in V$. So, $\varphi^{2m}(y) = \varphi^m(x) = 0$. In particular, $y \in \ker \varphi^m = \ker \varphi^m$ which means that $x = \varphi^m(y) = 0$, so we are done.

11.1.8 Let $V$ be a vector space over $F$ and let $\varphi : V \to V$ be a linear transformation. We show that for any fixed $\lambda \in F$, the collection of eigenvectors of $\varphi$ with eigenvalue $\lambda$ together with $0$ forms a subspace of $V$.

Denote the space $V_\lambda$. Note that $0 \in V_\lambda$ by assumption. Let $x, y \in V_\lambda$ and let $\alpha \in F$. Then $\varphi(x + \alpha y) = \varphi(x) + \alpha \varphi(y) = \lambda x + \alpha \lambda y = \lambda(x + \alpha y) \in V_\lambda$. So, $V_\lambda$ is a subspace.

11.1.9 We show that eigenvectors with distinct eigenvalues form a linearly independent set. We proceed with induction. For one vector $v_1$, this vector is linearly independent. Now, let $v_i$ for $1 \leq i \leq k$ be any set of $k$ distinct eigenvectors with distinct eigenvalues $\lambda_i$. Let these be linearly independent by the inductive step. Now, suppose we have a linear dependence

$$\sum_{i=1}^{k+1} a_i v_i = 0$$  

(1)
with not all $a_i$ equal to 0. If $a_{k+1} = 0$, we obtain a linear dependence of $v_1, \ldots, v_k$, contradicting the induction hypothesis. But

$$0 = \varphi(0) = \sum_{i=1}^{k+1} a_i \lambda_i v_i. \quad (2)$$

Multiplying the original dependence (1) by $\lambda_{k+1}$ and subtracting (2) we obtain

$$\sum_{i=1}^{k} a_i (\lambda_{k+1} - \lambda_i) v_i = 0.$$ 

But the $\lambda$'s are distinct and the vectors $v_1, \ldots, v_k$ are linearly independent, so this implies that $a_1 = a_2 = \cdots = a_k = 0$. Thus our original dependence has the form $a_{k+1} v_{k+1} = 0$, which is a contradiction since $a_{k+1}$ and $v_{k+1}$ are both nonzero. Therefore, eigenvectors with distinct eigenvalues must be linearly independent.

Now, the dimension of $V$ is just the number of elements in a maximal linearly independent set. In particular, for a vector space of dimension $n$, we have just shown that there are at most $n$ distinct eigenvalues.

11.1.10 We show that any vector space $V$ has a basis. We follow the hint in the book. Let $S$ be the set of subsets of $V$ consisting of linearly independent vectors, partially ordered under inclusion. If $V = 0$, take a basis to be $\emptyset$. Let $C \subset S$ be a chain. Consider $\cup C$. Every finite subset of $\cup C$ is a subset of an element of $C$, so is a linearly independent subset of $V$. Then $\cup C \in S$ is an upper bound for $C$. Zorn’s lemma says that there is a maximal element $U \in S$. We claim $U$ is our basis. $U$ is already linearly independent by definition.

If $U$ does not span $V$, there would exist $w \in V$ such that $w \notin U$. Then consider $U \cup \{w\}$. We claim that $U \cup \{w\}$ is linearly independent. Indeed, any linear dependence not involving $w$ would be a linear dependence of elements in $U$ and hence be a contradiction; any linear dependence involving $w$ can be rewritten to express $w$ as a linear combination of elements of $U$, which contradicts the definition of $w$. Therefore $U \cup \{w\}$ is linearly independent. But this contradicts the maximality of $U$. Thus, $U$ spans $V$ and is a basis as desired.