The purpose of this guide is to describe to you what I feel are the central ideas that we learned in this course. The exam will have 10 questions, but of course these questions may have multiple parts. I’ve broken the course into three components; there will be 3 problems from the first two parts and 4 problems from the last part. In particular, the material from the last week of class will appear prominently on the exam, since this is what we’ve been building towards.

Let me now describe the three components of the course, which I view as arranged vertically. On the bottom, we have Part I, where we studied the basic structure of the complex numbers and the behavior of analytic functions. We noticed there was something interesting going on with derivatives in the complex numbers when we proved the Cauchy–Riemann equations.

On the next level, we have Part II, where we introduced the definition of contour integrals and proved the Cauchy-Goursat theorem. We then proved the Cauchy integral formula, which was a generalization of the Cauchy-Goursat theorem. It had the amazing consequence that a function that is analytic is actually infinitely differentiable. Furthermore, it related the value of a derivative of a function at a point to the value of a certain contour integral. We proved the maximum modulus principle and Louisville’s Theorem.

One the top level, we have Part III, where we developed advanced techniques to evaluate integrals. We realized that a good way to represent functions in a neighborhood of a point is via a power series or Laurent series. This led to a discussion of singular points and the definition of the residue of a function. The highlight of this part was the Cauchy Residue Theorem. We concluded by applying the methods we developed to calculate certain improper integrals from ordinary calculus of the real numbers. We realized these integrals as a portion of an integral around a closed contour in the complex plane, then we calculated those contour integrals using the Cauchy Residue Theorem. A key part in these computations is showing that the integral along a semicircle goes to zero as the radius goes to infinity.

Hopefully you’ll agree that by the end of the course, we really accomplished something concrete — there are certain integrals that you didn’t know how to compute from calculus before that you can solve using complex analysis now. Of course, complex analysis is beautiful in its own right, and I hope you continue to explore this field in the future. Unfortunately, we were limited by time in how far we could get. While we stopped at a very natural ending point, I recommend that if you are interested pursuing complex analysis further, you should read through the remainder of Chapters 6 and 7 as well as Chapters 8 and 9. This will prepare you for learning graduate level Complex Analysis in the future. Now onto the detailed outline.
Part I: Basic structure of the complex numbers (aka pre-midterm material)

1. Arithmetic in the complex numbers, conjugates, absolute values, the triangle inequality.
   - Calculate \( \frac{2}{3} + (4 + 2i)(3 - i) \)
   - Prove that for complex numbers \( z_1 \) and \( z_2 \), we have \( |z_1 z_2| = |z_1| \cdot |z_2| \).
   - Prove that \( f(z) = 2z^9 + 2z^4 - 2z^3 - 2iz + 9 \) has no zeroes on or inside the unit circle (centered at the origin).

2. Arguments of complex numbers, exponential form, and finding roots of complex numbers.
   - Explain why \( \text{arg}(z_1 z_2) = \text{arg}(z_1) + \text{arg}(z_2) \) always holds, but \( \text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) \) might not hold.
   - Find all roots of \( z^4 = -8 - 8\sqrt{3}i \).

3. Definition of limits.
   - Prove that \( \lim_{z \to i}(z^2 + 4iz + 2i) = -5 + 2i \) using a rigorous \( \epsilon/\delta \) proof.

   - Prove, using the definition of continuity, that the sum of two continuous functions is continuous.

5. Cauchy-Riemann equations, harmonic functions.
   - At what complex numbers is \( f(z) = x^2 + iy^2 \) is differentiable? What is the derivative at those points?
   - Where is the function \( f(z) = xy + iy \) analytic?
   - Show that \( u(x, y) = 2x - x^3 + 3xy^2 \) is harmonic, and find a harmonic conjugate.

6. Exponential function and trigonometric functions.
   - #8, #10, #12 in Section 30.
   - #4(b) and #8 in Section 38.

7. Logarithms, branch cuts and branch points.
   - #1, #2, Section 33.
   - Definitions of branch cuts, branch points and examples (logarithm and power functions \( z^c \) when \( c \) is not an integer).

Part II: Contour Integration and applications

1. Definition of a contour integral.
   - Know the definition (Section 44, equation (2)).
   - Review the first question on the quiz.

2. Upper bounds for absolute values of integrals.
   - If \( f(z) \leq M \) on a contour \( C \) of length \( L \), then \( |\int_C f(z)dz| \leq ML \).
Show, without evaluating the integral, that

\[ \left| \int_C \frac{dz}{z^2 - 1} \right| \leq \frac{\pi}{3} \]

where \( C \) is the quarter-circle centered at the origin traversing counter-clockwise from \( z = 2 \) to \( z = 2i \) in the first quadrant.

   - Know the statement. (Theorem in Section 50.)
   - Use the Cauchy–Goursat Theorem to evaluate
     \[ \int_C \frac{z^2}{z - 3} \, dz \]
     where \( C \) is the unit circle centered at the origin.

4. Cauchy Integral Formula (including general form for derivatives).
   - Statement (Theorems in Sections 54 and 55). You need to know the setup and not just the equations!
   - Review the second question on the quiz.

5. Liouville’s Theorem and the maximum modulus principle.
   - Know the statements (Theorem 1 in Section 58 and Theorem in Section 59).
   - Prove that any polynomial with complex coefficients has at least one complex root (Theorem 2 in Section 58).

Part III: Series, Residues, and evaluation of improper integrals.

1. Taylor series and Laurent series.
   - Know the statements of Taylor’s Theorem and Laurent’s Theorem (theorems in sections 62 and 66, respectively), as well as the uniqueness of these series (both theorems in section 72).
   - Know some basic series (geometric, exponential, trig functions).

   - Know that you can integrate and differentiate term by term in Taylor series (theorems in section 71). Example: Section 72, #1.
   - Section 73, #1.

3. Isolated singular points versus non-isolated singular points. Three types of isolated singular points. Description in terms of local behavior of function. Example: Label each of these singular points of functions as non-isolated, removable, pole, or essential:
   - \( f(z) = \frac{(z + 1)}{\sin(\pi z)}, z_0 = -1 \).
   - \( f(z) = \sin(\pi/(z + 1)), z_0 = -1 \).
   - \( f(z) = \log(z) \cdot (z + 1), z_0 = -1 \).
   - \( f(z) = g(z) / h(z) \) where \( g \) and \( h \) are analytic at \( z_0, g(z_0) = h(z_0) = 0 \) and \( g'(z_0) = 0, h'(z_0) \neq 0 \).
   - \( f(z) = g(z) / h(z) \) where \( g \) and \( h \) are analytic at \( z_0, g(z_0) = h(z_0) = 0 \) and \( g'(z_0) \neq 0, h'(z_0) = 0 \).

4. Definition of residues.
• Know the setup of the definition \( f(z) \) has an isolated singularity at \( z_0 \), so it has a Laurent series expansion in a punctured neighborhood of \( z_0 \) as well as the definition itself (first equation below figure 90 in section 75, and equation (3) on that same page; be able to explain what \( C \) is in this latter equation, don’t just blindly memorize the formula!). Know why these two numbers (i.e. the two definitions) are equal.

• Study 3 examples in section 75 (try to evaluate these examples yourself without reading the text first.)

5. Method to calculate residues at poles (Section 80).

• If \( g(z) \) is analytic at \( z_0 \), and \( f(z) = g(z)/(z - z_0) \), then \( \text{Res}_{z=z_0} f(z) = g(0) \).

• More generally, if \( g(z) \) is analytic at \( z_0 \), and \( f(z) = g(z)/(z - z_0)^n \), then

\[
\text{Res}_{z=z_0} f(z) = g^{(n-1)}(0)/(n-1)!
\]

• Examples in Section 81 (try to evaluate these examples yourself without reading the text first.)

6. Calculating real improper integrals using contours in the complex plane. (The important example is section 86.)

• Key idea: showing that the integral along the semicircle \( E_R \) of radius \( R \) goes to 0 as \( R \to \infty \) using the “Upper bounds for absolute values of integrals” section (number 2 in part I above).

• Calculate

\[
\int_{0}^{\infty} \frac{dx}{x^4 + 1}
\]