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Geometric Phases for the Free Rigid Body with Variable Inertia Tensor

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ABSTRACT

The Hannay-Berry connection was introduced by Montgomery to provide a geometric framework for the classical Hannay angles associated to an integrable Hamiltonian which depends on a slowly varying parameter. The main goal of this paper is to compute the parallel transport, curvature, and holonomy of this connection for a free rigid body whose inertia tensor varies with time. We consider a symplectic fiber bundle whose base is the space of possible inertia tensors and whose fiber is the union of the regions in phase space which admit local action angle charts. Each element in the base induces dynamics in the corresponding fiber which are completely integrable. The Hannay-Berry connection is defined by averaging the trivial connection over the $\mathbb{T}^3$ action induced by this local set of parameter dependent action-angle variables. One result which is proved states that for any loop in the base along which the moments of inertia vary, while the principal axes of inertia are fixed, the holonomy is trivial, and hence the Hannay angles are zero. Another result identifies certain loops for which the moments of inertia are constant, while the principal axes undergo a rotation, and whose holonomy is non-trivial. The curvature form is found to have fifteen terms, all but three of which are computed explicitly. In addition, an action integral is found which does not seem to appear in the standard literature on the rigid body. For general multifrequency systems the actions are known to be almost adiabatic invariants. However this new action is seen to be a true adiabatic invariant of the system.

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Dedication

This work is dedicated to the memory of

Don Nicholas Tantalo

1. Introduction

Consider a family of classically integrable Hamiltonians H_m, depending on a parameter $m \in M$, where M is a smooth manifold. When such a system undergoes an adiabatic circuit $t \mapsto m(\epsilon t)$, $0 \le t \le c/\epsilon$, in M, the shift in the angle variables splits naturally into an obvious dynamical part

$$\Delta\boldsymbol{\theta}_{\mathrm{dyn}} = \int_0^{c/\epsilon} \boldsymbol{\omega}(\mathbf{I}(t), \epsilon t)dt,$$

which depends on the duration of the excursion; and a geometric part

$$\Delta\boldsymbol{\theta}_{\mathrm{geom}} = \int_\gamma \langle d_M \boldsymbol{\theta} \rangle,$$

depending only on the image γ, of the circuit. Here $(\mathbf{I}, \boldsymbol{\theta})$ denote parameter dependent action-angle variables, $\boldsymbol{\omega} = \partial H_m/\partial \mathbf{I}$ the frequency vector, d_M the exterior derivative with respect to the parameters, and $\langle\cdot\rangle$ the operation of averaging over invariant phase space tori. We call $\Delta\boldsymbol{\theta}_{\mathrm{dyn}}$ the *dynamic phase* while $\Delta\boldsymbol{\theta}_{\mathrm{geom}}$ is called the *geometric phase* or the *classical Hannay angles.* Hannay[14] and Berry[5] explain the extra term by noting that the generating function which gives the canonical transformation to action-angle variables depends on the parameter. When one writes Hamilton's equations in parameter dependent action-angle coordinates, there is an extra term in the equation giving the angle rate of change. Hannay averages this equation to get, in the adiabatic limit of slow parameter variation, the total angle shift:

$$\Delta\boldsymbol{\theta} = \Delta\boldsymbol{\theta}_{\mathrm{dyn}} + \Delta\boldsymbol{\theta}_{\mathrm{geom}}.$$

Berry showed that $\Delta\boldsymbol{\theta}_{\mathrm{geom}}$ is the classical analog of a phase occurring in quantum mechanics.

In the quantum case, Berry[4] considers a parameter dependent family of self adjoint operators and calculates the phase factor acquired by an eigenstate which undergoes an adiabatic cycle. Simon[27] gave a geometric framework for Berry's phase factor, interpreting it as the holonomy of a natural connection on a Hermitian line bundle. Montgomery[24], Golin, Knauf, and Marmi[11], and later Marsden, Montgomery, and Ratiu[21] have given an analogous geometric account for the classical Hannay angles, which extends to non- integrable Hamiltonians that are invariant under parameter dependent phase space symmetries. The connection in this case is called the Hannay-Berry (HB) connection. For integrable Hamiltonians, it is defined by averaging the trivial connection on $M \times P$ (where P is the phase space) over the $\mathbb{T}^n$ action induced by the local action angle coordinates.

The Hannay angles (or equivalently the holonomy of the HB connection) have been computed in many simple examples such as: families of harmonic oscillators [24,4], the Foucault pendulum [24,16,14], and the ball in the rotating hoop [21,16]. The main goal of the present work is to compute the parallel transport, curvature, and holonomy of the HB connection in an example which exhibits more complexity: the free rigid body. In this case the parameter is the inertia tensor, a positive definite, symmetric matrix whose eigenvalues satisfy certain inequalities. Each such matrix induces a completely integrable system on the phase space $T^*SO(3) \cong SO(3) \times \mathbb{R}^3$, via the corresponding kinetic energy function. By allowing the inertia matrix to vary with time, we obtain a non-autonomous Hamiltonian system called the *rigid body with variable inertia tensor.* A related system is that of a *deformable body*, which models the motion of a free space structure with flexible attachments under robotic control. Interestingly, these two systems are not identical, as one might at first expect. Their relationship is discussed in §2.5.

Two natural types of parameter variations suggest themselves: (I) those in which the principal moments of inertia remain fixed, while the principal axes undergo a rotation; and (II) those in which the principal axes are fixed and the moments of inertia vary. Let M denote the inertia tensors for the rigid body with distinct eigenvalues. We find in §4.1 that M has the structure of a trivial fiber bundle whose base B is a contractible open set in $\mathbb{R}^3$ and whose fiber F is the quotient of $SO(3)$ by a discrete subgroup. Paths in $M \cong B \times F$ of type (I) are those which remain in a single fiber, while those of type (II) lie in the base. The complexity of the parameter space is one feature which gives this example its added interest.

The HB connection in this example is an Ehresman connection defined on a bundle whose base is M, and whose fiber is a certain submanifold of the rigid body phase space, namely the union of those regions where action-angle coordinates exist.

Among our results is the fact that the holonomy of any loop of type (II) is trivial. This result follows from an expression derived in §2.3 for an action integral which does not seem to appear in the standard literature on the rigid body (e.g. Landau and Lifshitz[17], Lawden[18].) This expression shows that the parallel transport along curves of type (II) depends on a single real parameter, so even though $\dim(B) = 3$, the connection behaves as if B were one dimensional.

For curves of type (I) we compute parallel transport and holonomy only in special cases. Among these are four loops whose holonomy is non-trivial. Since $\dim(M) = 6$, there are fifteen terms in the curvature form. Of these, we compute twelve and three remain unknown. Among the twelve known terms ten are zero, indicating the presence of many loops with trivial holonomy.

The contents of this work is summarized as follows. Chapter two deals with the rigid body and related systems. In §2.1 we review basic facts regarding integrable systems. In §2.2 we introduce the body coordinate system on $T^*SO(3)$ and give the explicit solution to the Euler equations. In §2.3 and §2.4 we discuss a complete system of action integrals for the rigid body including one which was previously unknown. We also calculate the frequencies of the motion. In §2.5 we consider the deformable body and discuss its relationship to the rigid body with variable inertia tensor.

Chapter three summarizes the work of Montgomery[24] and Marsden, Montgomery, and Ratiu[21]. In §3.1 and §3.2 we give the basic notation and definitions concerning the HB connection. In §3.3 we show that the holonomy of the HB connection in the case of integrable systems is the classical Hannay angles. In §3.4 we give a formula, due to Montgomery, for the curvature.

Chapter four deals with the HB connection as it applies to the rigid body. In §4.1 we study the parameter space in detail. In §4.2 we compute functions whose Hamiltonian vector fields give the horizontal lift. In §4.3 we compute the curvature form, and in §4.4 we compute the holonomy of various curves in M. In §4.5 we draw conclusions on the adiabatic invariance of the actions, and discuss averaging over a natural $\mathbb{T}^2$ action.

2. Dynamics of Rigid and Deformable Bodies

The free rigid body is a well studied classical system which, nevertheless, possesses some new and interesting features. Recently Marsden and Holm[15] showed that the body angular momentum space of this system (which is identified with $\mathbb{R}^3$) is foliated into invariant elliptical cylinders, and the dynamic trajectories on these cylinders are those of a simple pendulum. Montgomery[23] has given a formula for the angle $\Delta\Theta$, by which the rigid body rotates in space as the body angular momentum vector executes one period of its motion. In this chapter, we find a previously unknown action integral for this system. We show that

$$I_3 := \frac{A}{2\pi\|\mathbf{J}\|}, \tag{2.1}$$

has Hamiltonian flow which is 2π-periodic, where A is the oriented surface area on the momentum sphere enclosed by the trajectory of the body angular momentum vector, and $\mathbf{J}$ is the angular momentum vector in space. Putting $I_2 := \|\mathbf{J}\|$ and $I_1 := \langle \mathbf{J}, \mathbf{e}_1 \rangle$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ denotes an inertial frame in $\mathbb{R}^3$, and $\langle\ ,\ \rangle$ the standard inner product, we obtain a complete set of independent action integrals: $\mathbf{I} = (I_1, I_2, I_3)$. That is, their differentials dI_i are linearly independent at almost every point of phase space, their Poisson brackets $\{I_i, I_j\}$ are pairwise zero, and their Hamiltonian flows are 2π-periodic. Note the definition of I_1 is arbitrary in that we could have taken $I_1 = \langle \mathbf{J}, \mathbf{u} \rangle$ with $\mathbf{u}$ any unit vector in $\mathbb{R}^3$. We choose $\mathbf{u} = \mathbf{e}_1$ merely for definiteness. This ambiguity implies that a typical trajectory lies on many *different* invariant 3-tori in the phase space $SO(3) \times \mathbb{R}^3$, and hence on their intersection, an invariant 2-torus. Therefore the system has a proper resonance (i.e. a resonance independent of initial conditions.) This fact will be verified directly in §2.4.

In §2.1 we review the basic facts concerning integrable systems. In §2.2 we introduce body coordinates on $SO(3) \times \mathbb{R}^3$ and summarize the explicit solutions to the Euler equations. In §2.3 we identify those regions in $SO(3) \times \mathbb{R}^3$ which support action-angle variables for the rigid body, and prove that I_3 defined above is an action. In Appendix A we compute the area A, in terms of complete elliptic integrals whose modulus is a function of $\|\mathbf{J}\|$, the energy H, and the principal moments of inertia. Using this expression and Montgomery's formula for the angle $\Delta\Theta$, we compute in §2.4, the frequencies of the system with respect to the above actions. This allows us to write the rigid body equations in action-angle coordinates. Written in this way the equations are linear, and their solution is trivial. Also, we see directly that the motion has an extra resonance (i.e. is periodic) precisely when $\Delta\Theta$ is a rational multiple of 2π. Finally in §2.5 we derive the Euler equations for a deformable body, and discuss the relationship between deformable bodies and the rigid body with variable inertia tensor.

2.1 Integrable Systems

In this section we review basic facts concerning completely integrable systems. We state the Arnold-Liouville theorem and a method, due to Arnold, for computing the local action coordinates. We also give a criterion of Duistermaat for the existence of global action-angle variables, and a corollary which we later apply to the rigid body.

Let (P, ω) be a symplectic manifold with $\dim(P) = 2n$, and $H \in C^\infty(P)$. The Hamiltonian system on P with energy H is called *completely integrable* if there exist n functions $f_1, \ldots, f_n = H \in C^\infty(P)$ in involution which are independent almost everywhere (with respect to Liouville volume) on P. This means that the Poisson brackets vanish: $\{f_i, f_j\} = 0$ for $1 \le i, j \le n$, and for almost every $x \in P$, the differentials $\{df_1(x), \ldots, df_n(x)\}$ are linearly independent (equivalently $df_1(x) \wedge \ldots \wedge df_n(x) \neq 0$.)

Theorem 2.1.1 (Arnold-Liouville): *Let $f_1, \ldots, f_n = H$ be an integrable system on (P, ω), and $c \in \mathbb{R}^n$ a regular value of $f = (f_1, \ldots, f_n) : P \to \mathbb{R}^n$. Then for each compact connected component, F_c, of $f^{-1}(c)$, there is an open neighborhood $U \subset P$ about F_c invariant under the flow of X_{f_i} $(1 \le i \le n)$, a diffeomorphism*

$$(I, \theta) : U \to V \times \mathbb{T}^n$$

with $V \subset \mathbb{R}^n$ *open, and a diffeomorphism* $\chi : f(U) \to V$ *such that* $I = \chi \circ f$. *Furthermore, we have on* U *the expression*

$$\omega = \sum_{i=1}^{n} dI_i \wedge d\theta_i.$$

The canonical coordinate system $I_1, \dots, I_n, \theta_1, \dots, \theta_n$ is called a set of *action-angle variables.* The proof can be found in Arnold[2], Duistermaat[9], or Markus and Meyer[19]. Using the diffeomorphism χ, H can be expressed in the coordinates (I, θ) as a function of $(I_1, \dots, I_n)$ alone. Thus the Hamiltonian system with energy H becomes:

$$\left\{ \begin{array}{lcl} \dot{\theta}_i & = & \omega_i(I) = \frac{\partial H}{\partial I_i}(I) \\ \dot{I}_i & = & 0, \end{array} \right. \tag{2.2}$$

which is linear and easily solved. The generic motion of an integrable system is therefore a quasiperiodic trajectory on the invariant n-torus $F_c \subset P$. If the frequencies $\omega_i(I)$ satisfy some integer relations $\langle \boldsymbol{\omega},\ \mathbf{k} \rangle = 0$, where $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ and $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$, then this motion takes place on a torus of smaller dimension. (See Arnold[ ].) To be precise, if $\mathbf{k}^1, \dots, \mathbf{k}^l \in \mathbb{Z}^n$ are $\mathbb{Z}$-linearly independent, and $\langle \boldsymbol{\omega}, \mathbf{k}^j \rangle = 0$ for $1 \le j \le l$, then the trajectory (2.2) is dense on a torus of dimension $(n - l)$ contained in $F_c \cong \mathbb{T}^n$.

The actions I_i can be obtained by the following construction due to Arnold[2]. A proof can also be found in Duistermaat[9].

Theorem 2.1.2 (Arnold): *Let* $\gamma_i(c)$, $(1 \le i \le n)$ *be closed curves in* F_c, *which depend smoothly on* $c \in f(U)$, *and whose homology classes form a basis for* $H_1(F_c, \mathbb{Z})$. *Suppose* ω *is exact on* U, *say* $\omega = -d\beta$. *Then*

$$I_i(x) = \frac{1}{2\pi} \int_{\gamma_i(f(x))} \beta \qquad\qquad (x \in U),$$

forms a set of action coordinates.

For systems with one degree of freedom (i.e. $n = 1$) it follows from Stokes Theorem that the action integral is $(2\pi)^{-1}$ times the phase plane area enclosed by a level contour of the energy H. Once the actions are obtained, the conjugate angle variables are constructed by finding a Lagrangian submanifold of U which is transversal to the tori $F_c,\ c \in f(U)$, and transporting it by the flow of X_{I_i}. The angle θ_i is then the time taken to reach a given point. Not all action coordinates are of the form given by Theorem 2.1.2. Those which are of this form are called *standard actions.*

Suppose for convenience that $f : P \to \mathbb{R}^n$ is everywhere regular, and that the fibers $f^{-1}(c),\ c \in \mathbb{R}^n$, are compact and connected. (If not, just remove the points at which f is singular, and for which $f^{-1}(c)$ is not compact. By Theorem 2.1.1 what remains is an open submanifold of P, to which we can restrict f. We can further restrict f so that the fibers are connected.) Let $B = f(P) \subset \mathbb{R}^n$, which is open. By Theorem 2.1.1 $f : P \to B$ is a locally trivial fiber bundle whose fibers are Lagrangian tori. The bundle charts are given by the local action-angle variables. If these coordinates are defined globally on P, then the bundle f is globally trivial. With an additional assumption, the converse is also true. The following is a paraphrase of Theorem 2.2 in Duistermaat[9].

Theorem 2.1.3: *Suppose the symplectic form ω on P is exact. Then $f : P \to B$ is trivial if and only if P admits global action-angle variables.*

Corollary 2.1.1: *Let (P, ω) be a symplectic manifold, and $f = (f_1, \ldots, f_n) : P \to \mathbb{R}^n$ a completely integrable system. Suppose $P_1 \subset P$ is an open set satisfying:*

1. *ω is exact on P_1.*
2. *$f|_{P_1}$ is submersive and $f^{-1}(c) \cap P_1$ is compact and connected for $c \in f(P_1)$.*
3. *$f(P_1) \subset \mathbb{R}^n$ is contractible.*

Then P_1 admits a set of action angle variables.

Proof: By (3), the bundle $f : P_1 \to f(P_1)$ is trivial. The result follows immediately from Theorem 2.1.3, and the preceding discussion. ■

In this case the so called monodromy, described in Duistermaat[9], is not present. As we shall see, this situation arises in the case of the rigid body.

2.2 Left Trivialization of $SO(3)$

The configuration space of the rigid body is the Lie group $SO(3)$ consisting of 3×3 orientation preserving orthogonal matrices. Each such matrix represents a rigid rotation of the body in space from some fixed reference configuration. To be precise, let $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ denote a coordinate frame in $\mathbb{R}^3$ attached to the body, and let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an inertial frame. We require that the origin of both frames coincide with the center of mass of the body. The relationship is given by $g \cdot \mathbf{E}_i = \mathbf{e}_i$, $(1 \leq i \leq 3)$ for some $g \in SO(3)$. The phase space is then the cotangent bundle $T^*SO(3)$ of $SO(3)$. Since $SO(3)$ is a Lie group, $T^*SO(3)$ is naturally diffeomorphic to $SO(3) \times so(3)^*$, where $so(3)$ is the Lie algebra of $SO(3)$ consisting of 3×3 skew symmetric matrices and $so(3)^*$ is its dual. Specifically, the left action of $SO(3)$ on itself given by left translation:

$$L_g h := gh \qquad\qquad (g, h \in SO(3)),$$

lifts to a left $SO(3)$ action on $T^*SO(3)$:

$$g \cdot \alpha_h := T^* L_{g^{-1}} \cdot \alpha_h \in T^*_{gh} SO(3) \qquad\qquad (\alpha_h \in T^*_h SO(3)), \tag{2.3}$$

so that $g^{-1} \cdot \alpha_g \in T^*_{\mathrm{id}} SO(3) = so(3)^*$. The required diffeomorphism is then

$$\alpha_g \in T^*_g SO(3) \mapsto (g, g^{-1} \cdot \alpha_g) \in SO(3) \times so(3)^*. \tag{2.4}$$

Further, we can identify $\mathbb{R}^3$ with $so(3)$ via the map

$$a \in \mathbb{R}^3 \mapsto \hat{a} := \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \in so(3).$$

One checks the formulas $\hat{a} \cdot b = a \times b$, and $\widehat{g \cdot a} = g\hat{a}g^{-1}$ for $a, b \in \mathbb{R}^3$, and $g \in SO(3)$. The fact that $\hat{}: (\mathbb{R}^3, \times) \to (so(3), [\ ,\ ])$ is a Lie algebra isomorphism is also readily verified. We denote the inverse of this map by $\check{}: so(3) \to \mathbb{R}^3$. Also $so(3)^*$ is identified with $so(3)$ via the inner product $\langle\langle \xi, \eta \rangle\rangle = \frac{1}{2}\mathrm{trace}(\xi\eta^T)$ on $so(3)$. Note that for $a, b \in \mathbb{R}^3$, $\langle a, b\rangle = \langle\langle \hat{a}, \hat{b}\rangle\rangle$. Therefore we may identify $T^*SO(3)$ with $SO(3) \times \mathbb{R}^3$. Equation (2.4), combined with the above identifications, is called the body coordinate system, or left trivialization of $T^*SO(3)$. It should be noted that this is not a coordinate system in the ordinary sense, since $SO(3)$ is topologically non-trivial. In many texts it is customary to coordinatize $T^*SO(3)$ by Euler angles and their conjugate momenta, forming a canonical coordinate system. In this paper we use body coordinates exclusively. We now write a few relevant expressions in left trivialization.

(i) Lifted left action:

$$g \cdot (h, \alpha) = (gh, \alpha) \qquad\qquad (g, h \in SO(3),\ \ \alpha \in \mathbb{R}^3).$$

This is (2.3) written in left trivialization.

(ii) Poisson bracket:

$$\{f_1, f_2\}(g, \alpha) = \langle (g^{-1} \cdot d_g f_1)\check{}\,, \nabla_\alpha f_2\rangle - \langle (g^{-1} \cdot d_g f_2)\check{}\,, \nabla_\alpha f_1\rangle - \langle \alpha, \nabla_\alpha f_1 \times \nabla_\alpha f_2\rangle.$$

Here $g^{-1} \cdot d_g f$ denotes the left translate of $d_g f \in T_g^* SO(3)$ to $so(3)^*$. This expression can be computed from the formula, due to Cushman, for the canonical symplectic form on T^*G (G a Lie group) given in Proposition 4.4.1 of Abraham and Marsden[1]. If f_1, f_2 happen to be invariant under (i), i.e. functions of α only, then this reduces to

$$\{f_1, f_2\}(\alpha) = -\langle \alpha, \nabla_\alpha f_1 \times \nabla_\alpha f_2\rangle,$$

which is the standard left Lie Poisson structure on $\mathbb{R}^3$.

(iii) Hamiltonian vector fields:

$$X_f(g, \alpha) = (g \cdot \widehat{\nabla_\alpha f}, \alpha \times \nabla_\alpha f - (g^{-1} \cdot d_g f)\check{}\,).$$

This expression follows directly from (ii). For left invariant functions we have

$$X_f(g, \alpha) = (g \cdot \widehat{\nabla_\alpha f}, \alpha \times \nabla_\alpha f).$$

We denote by $\Phi_t^f(g,\alpha)$ the flow of X_f with initial point (g,α).

(iv) Kinetic energy:

$$H_m(\alpha) = \frac{1}{2}\langle \alpha, m^{-1}\alpha\rangle.$$

Here m denotes the inertia tensor, a positive definite symmetric matrix (see §2.5.) If m is diagonal with respect to $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, say $m = diag(\lambda_1, \lambda_2, \lambda_3)$, then

$$H_m(\alpha) = \frac{1}{2}\left(\frac{\alpha_1^2}{\lambda_1} + \frac{\alpha_2^2}{\lambda_2} + \frac{\alpha_3^2}{\lambda_3}\right).$$

Note that H_m is independent of g, hence invariant under the action (i).

(v) Momentum Map:

$$\mathbf{J}(g,\alpha) = g\cdot\alpha.$$

$\mathbf{J} : P \to \mathbb{R}^3$ is an equivariant momentum map for the action (i), as one verifies. Its value is the angular momentum of the system in the inertial frame. By definition of a momentum map we have, for $\xi \in \mathbb{R}^3$,

$$\begin{aligned} X_{\langle \mathbf{J},\xi\rangle}(g,\alpha) &= \xi_P(g,\alpha) = \left.\frac{d}{dt}\right|_{t=0} \exp t\hat{\xi}\cdot(g,\alpha) \\ &= \left.\frac{d}{dt}\right|_{t=0} \Big((\exp t\hat{\xi})g, \alpha\Big) \\ &= (\hat{\xi}g, 0). \end{aligned}$$

This can be checked directly using (iii). The flow of $\langle \mathbf{J},\xi\rangle$ is given by $\Phi_t^{\langle \mathbf{J},\xi\rangle}(g,\alpha) = (\exp t\hat{\xi}\cdot g, \alpha)$.

Using (iii) and (iv) one finds that $X_{H_m}(g,\alpha) = (g\cdot \widehat{m^{-1}\alpha}, \alpha\times m^{-1}\alpha)$. Put

$$b_1 = \frac{1}{\lambda_3} - \frac{1}{\lambda_2}, \qquad b_2 = \frac{1}{\lambda_1} - \frac{1}{\lambda_3}, \qquad b_3 = \frac{1}{\lambda_2} - \frac{1}{\lambda_1},$$

then $\alpha\times m^{-1}\alpha = (b_1\alpha_2\alpha_3, b_2\alpha_1\alpha_3, b_3\alpha_1\alpha_2)$. Thus the flow, $\Phi_t^{H_m}(g_0,\alpha_0)$ of X_{H_m} is obtained by solving:

$$\dot{\alpha} = \alpha\times m^{-1}\alpha \qquad \text{i.e.} \quad \left\{\begin{aligned} \dot{\alpha}_1 &= b_1\alpha_2\alpha_3 \\ \dot{\alpha}_2 &= b_2\alpha_1\alpha_3 \\ \dot{\alpha}_3 &= b_3\alpha_1\alpha_2, \end{aligned}\right. \tag{2.5}$$

and

$$\dot{g} = g \cdot \widehat{m^{-1}\alpha}, \tag{2.6}$$

with initial conditions $g(0) = g_0$ and $\alpha(0) = \alpha_0$. $\Phi_t^{H_m}(g_0, \alpha_0) = (g(t), \alpha(t))$ then gives the dynamic evolution of the rigid body in left trivialization. Equations (2.5) are the Euler equations for a rigid body, and their solution, $\alpha(t)$, is called a reduced trajectory. The solution $g(t)$, of (2.6), is called the reconstructed trajectory, and gives the attitude of the body with respect to the inertial coordinate frame. See [1,20,22] for a discussion of Marsden-Weinstein reduction and reconstruction.

For later reference we give the explicit solution to the Euler equations (2.5). Assume, for definiteness, that $\lambda_1 \geq \lambda_2 \geq \lambda_3$, so that $b_1 \geq 0,\ b_2 \leq 0,\ b_3 \geq 0$. The functions H_m and $\mathbf{J}$ are conserved quantities for this system, so we have that

$$r^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \|\mathbf{J}(g, \alpha)\|^2,$$

and

$$h = \frac{1}{2}\left(\frac{\alpha_1^2}{\lambda_1} + \frac{\alpha_2^2}{\lambda_2} + \frac{\alpha_3^2}{\lambda_3}\right) = H_m(\alpha),$$

are constant along the solution $\alpha(t)$, of (2.5). Put $S_r^2 = \{\alpha \in \mathbb{R}^3 \mid \|\alpha\| = r\}$ and let $h > 0$. Then $\alpha(t) \in S_r^2 \cap H_m^{-1}(h)$ for all t. One checks that the condition $S_r^2 \cap H_m^{-1}(h) \neq \emptyset$ implies $\lambda_1 \geq r^2/2h \geq \lambda_3$.

Case 1. Let $\lambda_1 = \lambda_2 = \lambda_3 =: \lambda$. Then $b_1 = b_2 = b_3 = 0$ and (2.5) becomes $\dot{\alpha} = 0$, whence $\alpha(t) = \alpha(0)$. In this case (2.6) is $\dot{g} = g \cdot \lambda^{-1}\widehat{\alpha(0)}$, so that $g(t) = g(0) \cdot \exp t\hat{\xi}$ where $\xi = \lambda^{-1}\alpha(0) \in \mathbb{R}^3$. Thus the absolute motion of the body in space is steady rotation about the vector ξ, with period $2\pi/\|\xi\|$.

Case 2. Let $\lambda_1 = \lambda_2 > \lambda_3$. Then $b_3 = 0$ and $b_1 = -b_2 > 0$. We have $\dot{\alpha}_3 = 0$, so $\alpha_3(t) = \alpha_3(0)$ and (2.5) becomes

$$\begin{cases} \dot{\alpha}_1 &= c\alpha_2 \\ \dot{\alpha}_2 &= -c\alpha_1, \end{cases}$$

where $c = b_1\alpha_3(0)$. The solution is

$$\begin{cases} \alpha_1(t) &= \alpha_1(0)\cos(ct) + \alpha_2(0)\sin(ct) \\ \alpha_2(t) &= -\alpha_1(0)\sin(ct) + \alpha_2(0)\cos(ct). \end{cases}$$

Case 3. $\lambda_1 > \lambda_2 = \lambda_3$ is similar to case 2 above. Interchange the subscripts 1 and 3 to obtain the solution in this case.

Case 4. Let $\lambda_1 > \lambda_2 > \lambda_3$. . We have five subcases.

Case 4a: $r^2/2h = \lambda_1$. Then $r^2 = 2\lambda_1 h$ implies

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \alpha_1^2 + \frac{\lambda_1}{\lambda_2}\alpha_2^2 + \frac{\lambda_1}{\lambda_3}\alpha_3^2,$$

so that

$$\left(\frac{\lambda_1 - \lambda_2}{\lambda_2}\right)\alpha_2^2 + \left(\frac{\lambda_1 - \lambda_3}{\lambda_3}\right)\alpha_3^2 = 0.$$

Since $\lambda_1 - \lambda_2 > 0$ and $\lambda_1 - \lambda_3 > 0$, $\alpha_2 = \alpha_3 = 0$. Equation (2.5) becomes $\dot{\alpha_1} = 0$, so $\alpha_1(t) = \alpha_1(0)$. The solution is $\alpha(t) = (\alpha_1(0), 0, 0)$. In this case (2.6) is $\dot{g} = g \cdot \lambda_1^{-1}\widehat{\alpha(0)}$ and the absolute motion is

$$g(t) = g(0) \cdot \exp(ct\hat{\mathbf{E}}_1),$$

where $c = \lambda_1^{-1}\alpha_1(0)$.

Case 4b: $r^2/2h = \lambda_3$. This case is similar to (4a) above. We get $\alpha_1 = \alpha_2 = 0$, whence $\alpha(t) = (0, 0, \alpha_3(0))$, and $g(t) = g(0) \cdot \exp(ct\hat{\mathbf{E}}_3)$ where $c = \lambda_3^{-1}\alpha_3(0)$.

Case 4c: $r^2/2h = \lambda_2$. We have $r^2 = 2\lambda_2 h$ implies

$$\left(\frac{\lambda_1 - \lambda_2}{\lambda_1}\right)\alpha_1^2 - \left(\frac{\lambda_2 - \lambda_3}{\lambda_3}\right)\alpha_3^2 = 0,$$

so that

$$\alpha_3 = \pm\sqrt{\eta}\,\alpha_1,$$

where $\eta = \lambda_3(\lambda_1 - \lambda_2)/\lambda_1(\lambda_2 - \lambda_3)$. We see that $\alpha(t)$ lies in one of the planes $\alpha_3 = \sqrt{\eta}\,\alpha_1$ or $\alpha_3 = -\sqrt{\eta}\,\alpha_1$. If $\alpha(0)$ lies in the intersection of these two planes (namely the α_2-axis) then $\dot{\alpha}(0) = 0$ by (2.5), and $\alpha(t) = (0, \alpha_2(0), 0)$ is an equilibrium. In this case

$g(t) = g(0) \cdot \exp(ct\hat{\mathbf{E}}_2)$, where $c = \lambda_2^{-1}\alpha_2(0)$. If $\alpha(0)$ does not lie on the α_2-axis, the solution is

$$\begin{aligned}\alpha_1(t) &= \pm P\,\mathrm{sech}(s(t-t_0))\\ \alpha_2(t) &= \mp Q\,\tanh(s(t-t_0))\\ \alpha_3(t) &= \pm R\,\mathrm{sech}(s(t-t_0)),\end{aligned}$$

where P, Q, R, s are given by

$$\begin{aligned}P^2 &= \frac{2h\lambda_1(\lambda_2-\lambda_3)}{(\lambda_1-\lambda_3)}\\ Q^2 &= 2h\lambda_2\\ R^2 &= \frac{2h\lambda_3(\lambda_1-\lambda_2)}{(\lambda_1-\lambda_3)}\\ s^2 &= \frac{2h(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}{\lambda_1\lambda_2\lambda_3}.\end{aligned}$$

The constants r, h, t_0 as well as the signs are chosen so as to agree with the initial point $\alpha(0)$.

Case 4d: $\lambda_2 > r^2/2h > \lambda_3$. The solution is given by

$$\begin{aligned}\alpha_1(t) &= P\,\mathrm{cn}(s(t-t_0),k)\\ \alpha_2(t) &= -Q\,\mathrm{sn}(s(t-t_0),k)\\ \alpha_3(t) &= R\,\mathrm{dn}(s(t-t_0),k),\end{aligned}$$

where P, Q, R, s, k are given by

$$\begin{aligned}P^2 &= \frac{\lambda_1(r^2-2\lambda_3 h)}{(\lambda_1-\lambda_3)}\\ Q^2 &= \frac{\lambda_2(r^2-2\lambda_3 h)}{(\lambda_2-\lambda_3)}\\ R^2 &= \frac{\lambda_3(2\lambda_1 h-r^2)}{(\lambda_1-\lambda_3)}\\ s^2 &= \frac{(\lambda_2-\lambda_3)(2\lambda_1 h-r^2)}{\lambda_1\lambda_2\lambda_3}\\ k^2 &= \frac{(\lambda_1-\lambda_2)(r^2-2\lambda_3 h)}{(\lambda_2-\lambda_3)(2\lambda_1 h-r^2)}.\end{aligned}$$

Here $\mathrm{cn}(\cdot,k)$, $\mathrm{sn}(\cdot,k)$, $\mathrm{dn}(\cdot,k)$ denote the Jacobi elliptic functions of modulus k. Again the constants r, h, t_0 are chosen in accordance with initial conditions.

Case 4e: $\lambda_1 > r^2/2h > \lambda_2$. We have solutions:

$$\begin{aligned} \alpha_1(t) &= P\,\mathrm{dn}(s(t-t_0),k) \\ \alpha_2(t) &= -Q\,\mathrm{sn}(s(t-t_0),k) \\ \alpha_3(t) &= R\,\mathrm{cn}(s(t-t_0),k), \end{aligned}$$

where P, Q, R, s, k are given by

$$\begin{aligned} P^2 &= \frac{\lambda_1(r^2-2\lambda_3 h)}{(\lambda_1-\lambda_3)} \\ Q^2 &= \frac{\lambda_2(2\lambda_1 h - r^2)}{(\lambda_2-\lambda_3)} \\ R^2 &= \frac{\lambda_3(2\lambda_1 h - r^2)}{(\lambda_1-\lambda_3)} \\ s^2 &= \frac{(\lambda_1-\lambda_2)(r^2-2\lambda_3 h)}{\lambda_1\lambda_2\lambda_3} \\ k^2 &= \frac{(\lambda_2-\lambda_3)(2\lambda_1 h - r^2)}{(\lambda_1-\lambda_2)(r^2-2\lambda_3 h)}. \end{aligned}$$

Note we have given the solution to the group equations (2.6) in only a few special cases. In fact (2.6) can be solved explicitly in all cases by transforming to Euler angles on $SO(3)$. The details can be found in Landau and Lifshitz[17] or Lawden[18].

If we take the limit of the solution in case (4d) as $(\lambda_1-\lambda_2)\to 0$ we get $k\to 0$, so that $\mathrm{sn}(\cdot,k)\to\sin(\cdot)$, $\mathrm{cn}(\cdot,k)\to\cos(\cdot)$, $\mathrm{dn}(\cdot,k)\to 1$. The solution then approaches that of case (2) (with $\lambda_2 > r^2/2h > \lambda_3$), which can therefore be subsumed under case (4d). Similar remarks hold for cases (4e) and (3) respectively. Cases (4a), (4b), and (4c) (with $\alpha_1(0)=\alpha_2(0)=0$) give the six relative equilibria for the Euler equations, two of which (4c) are unstable. Case (4c)(with $\alpha_1(0)\neq 0\neq\alpha_3(0)$) represents the four well known heteroclinic orbits, which connect the two unstable equilibria and are diffeomorphic to $\mathbb{R}$. Cases (4d) and (4e) describe four families of closed orbits separated by the planes $\alpha_3=\pm\sqrt{\eta}\,\alpha_1$.

2.3 Action-Angle Coordinates for the Rigid Body

The goals of this section are first, to determine those regions in $SO(3) \times \mathbb{R}^3$ which admit action angle coordinates, and second, to establish equation (2.1) giving the third action. Throughout we will assume that the inertia tensor is diagonal with respect to the body frame $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, say $m = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 > \lambda_2 > \lambda_3$. As in the previous section $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ denotes an inertial frame in $\mathbb{R}^3$. The first two actions, which are well known, are given by $I_1 = \langle \mathbf{J}, \mathbf{e}_1 \rangle$ and $I_2 = \|\mathbf{J}\|$. (Recall we could just as well take $I_1 = \langle \mathbf{J}, \mathbf{u} \rangle$, where $\mathbf{u}$ is any other unit vector in $\mathbb{R}^3$.) We show below that I_1, I_2, and H_m are in involution and that their differentials are independent on an open dense subset of $SO(3) \times \mathbb{R}^3$, verifying that the rigid body is indeed completely integrable.

From §2.2(v) we have

$$X_{I_1}(g, \alpha) = (\hat{\mathbf{e}}_1 g, 0),$$

and

$$\Phi_t^{I_1}(g, \alpha) = ((\exp t\hat{\mathbf{e}}_1)g, \alpha)\,.$$

For $\alpha \neq 0$, $\nabla_\alpha I_2 = \alpha/\|\alpha\|$ so by §2.2(iii)

$$X_{I_2}(g, \alpha) = \left(g\frac{\hat{\alpha}}{\|\alpha\|}, \alpha \times \frac{\alpha}{\|\alpha\|}\right) = \left(g\frac{\hat{\alpha}}{\|\alpha\|}, 0\right),$$

whence

$$\Phi_t^{I_2}(g, \alpha) = \left(g \exp t\frac{\hat{\alpha}}{\|\alpha\|}, \alpha\right) = \left((\exp t\frac{\widehat{g\alpha}}{\|\alpha\|})g, \alpha\right).$$

Here we have used $\widehat{g\alpha} = g\hat{\alpha}g^{-1}$ and $\exp(g\hat{\xi}g^{-1}) = g(\exp\hat{\xi})g^{-1}$, for $\alpha, \xi \in \mathbb{R}^3,\ g \in SO(3)$. Note that $\Phi_t^{I_1}$ and $\Phi_t^{I_2}$ are 2π-periodic. Now observe that

$$\begin{aligned} \{I_2, I_1\} &= dI_2 \cdot X_{I_1} \\ &= \langle (d_g I_2, \nabla_\alpha I_2), (\hat{\mathbf{e}}_1 g, 0) \rangle \\ &= 0, \end{aligned}$$

since $d_g I_2 = 0$. Since $\mathbf{J}$ is conserved along the flow of H_m we have $\{I_1, H_m\} = \{I_2, H_m\} = 0$.

In the following lemma we determine where $I_1,\ I_2,\ H_m$ are independent.

Lemma 2.3.1: *Define*

$$W := \{(g,\alpha) \in SO(3) \times \mathbb{R}^3 \mid g\alpha \times \mathbf{e}_1 \neq 0, \text{ and } \alpha \times m^{-1}\alpha \neq 0\},$$

and

$$f := (I_1, I_2, H_m) : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3.$$

Then $f|_W$ is a submersion.

Proof: We must show that $dI_1,\ dI_2,\ dH_m$ are linearly independent on W, which is equivalent to showing the same for $X_{I_1},\ X_{I_2},\ X_{H_m}$. Let $c_i \in \mathbb{R},\ 1 \leq i \leq 3$, and suppose

$$c_1 X_{I_1} + c_2 X_{I_2} + c_3 X_{H_m} = 0$$

at $(g,\alpha) \in W$. This is the same as

$$\begin{cases} c_1 g^{-1}\mathbf{e}_1 + c_2 \frac{\alpha}{\|\alpha\|} + c_3 m^{-1}\alpha &= 0 \\ c_3(\alpha \times m^{-1}\alpha) &= 0. \end{cases}$$

Now $c_3 = 0$ since $\alpha \times m^{-1}\alpha \neq 0$. If $c_1 \neq 0$ then the first equation becomes $\mathbf{e}_1 = cg\alpha$ where $c = -c_2/c_1\|\alpha\|$, whence $g\alpha \times \mathbf{e}_1 = 0$, contrary to our choice of (g,α). Thus $c_1 = c_2 = 0$ as required. ∎

One also checks that $dI_1,\ dI_2,\ dH_m$ are linearly *dependent* on $(SO(3) \times \mathbb{R}^3) \setminus W$. Note that W is obtained by removing two codimension one submanifolds from $SO(3) \times \mathbb{R}^3$, and hence W is an open dense submanifold.

Next we remove those points $(g,\alpha) \in W$ for which $f^{-1}(f(g,\alpha)) \cap W$ is non-compact. Recall from §2.2 Case (4c) that the separatrix planes in $\mathbb{R}^3$ are given by $\alpha_3 = \pm\sqrt{\eta}\,\alpha_1$ where

$$\eta = \frac{\lambda_3(\lambda_1 - \lambda_2)}{\lambda_1(\lambda_2 - \lambda_3)}.$$

Define

$$\begin{aligned} U_1 &:= \{\alpha \in \mathbb{R}^3 \mid \alpha_3^2 < \eta\alpha_1^2, \text{ and } \alpha_2^2 + \alpha_3^2 \neq 0\} \\ &= \{\alpha \in \mathbb{R}^3 \mid \lambda_1 > r^2/2h > \lambda_2\}, \end{aligned}$$

where $r = \|\alpha\|$ and $h = H_m(\alpha)$ as in §2.2. A short calculation, which we omit, verifies the equality of the above sets. Similarly set

$$\begin{aligned} U_3 &:= \{\alpha \in \mathbb{R}^3 \mid \alpha_3^2 > \eta\alpha_1^2, \text{ and } \alpha_1^2 + \alpha_2^2 \neq 0\} \\ &= \{\alpha \in \mathbb{R}^3 \mid \lambda_2 > r^2/2h > \lambda_3\}. \end{aligned}$$

Note that $U_1 \subset \mathbb{R}^3$ (respectively $U_3 \subset \mathbb{R}^3$) is open and contains the trajectories given by Case (4e) (respectively (4d)) of §2.2, which are closed orbits about the α_1-axis (respectively α_3-axis). Put

$$\begin{aligned} U_i^+ &= \{\alpha \in U_i \mid \alpha_i > 0\} \\ U_i^- &= \{\alpha \in U_i \mid \alpha_i < 0\}, \end{aligned}$$

for $i = 1, 3$. Each U_i^ν ($i = 1, 3$; $\nu = +, -$) is open, connected, and $U_i = U_i^+ \cup U_i^-$. Define the projection $\pi : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$, $(g, \alpha) \mapsto \alpha$, and put

$$\begin{aligned} P_i &= \pi^{-1}(U_i) \cap W \\ P_i^\nu &= \pi^{-1}(U_i^\nu) \cap W, \end{aligned}$$

for $i = 1, 3$ and $\nu = +, -$. Then $P_i^\nu \subset SO(3) \times \mathbb{R}^3$ is open, connected, and $P_i = P_i^+ \cup P_i^-$. We shall see that each P_i^ν admits a single action angle chart.

Note that for $(g, \alpha) \in W$, we have $|I_1(g, \alpha)| = |\langle g\alpha, \mathbf{e}_1\rangle| < \|g\alpha\| = I_2(g, \alpha)$. (The inequality is strict since $g\alpha \times \mathbf{e}_1 \neq 0$.) Now set

$$\begin{aligned} V_1 &= \{(c, r, h) \in \mathbb{R}^3 \mid |c| < r, \text{ and } \lambda_1 > r^2/2h > \lambda_2\} \\ V_3 &= \{(c, r, h) \in \mathbb{R}^3 \mid |c| < r, \text{ and } \lambda_2 > r^2/2h > \lambda_3\}. \end{aligned}$$

It follows from the preceding definitions that

$$f(P_i^+) = f(P_i^-) = f(P_i) = V_i$$

for $i = 1, 3$.

Lemma 2.3.2: *Let* $(c, r, h) \in V_i$ *($i = 1$ or 2). Then*

$$f^{-1}(c, r, h) \cap W \subset SO(3) \times \mathbb{R}^3$$

is compact.

Proof: We see that $(g, \alpha) \in f^{-1}(c, r, h) \cap W$ if and only if

$$\begin{cases} \text{(i)} & \langle g\alpha, \mathbf{e}_1 \rangle = c \\ \text{(ii)} & \|\alpha\| = r \\ \text{(iii)} & H_m(\alpha) = h \\ \text{(iv)} & g\alpha \times \mathbf{e}_1 \neq 0 \\ \text{(v)} & \alpha \times m^{-1}\alpha \neq 0 \end{cases} \tag{2.7}$$

Let $\{(g_n, \alpha_n)\}_{n=1}^{\infty}$ be a sequence in $SO(3) \times \mathbb{R}^3$ whose points satisfy (2.7). We must find a subsequence which converges to a point satisfying the same conditions. Since $SO(3)$ is compact we may, by passing to a subsequence, assume that $g_n \to g \in SO(3)$. Condition (ii) says α_n lies in the sphere $S_r^2 \subset \mathbb{R}^3$ which is also compact. Passing to a further subsequence we have $\alpha_n \to \alpha \in S_r^2$. Note that conditions (i)-(iii) define closed sets, so the limit point (g, α) automatically satisfies them.

Conditions (ii) and (iii) together say that $\alpha_n \in S_r^2 \cap H_m^{-1}(h)$. Since either $\lambda_1 > r^2/2h > \lambda_2$ (for $i = 1$) or $\lambda_2 > r^2/2h > \lambda_3$ (for $i = 3$), $S_r^2 \cap H_m^{-1}(h)$ is one of the closed orbits described in Case (4e) or (4d) of §2.2. These orbits are compact so that $\alpha \in S_r^2 \cap H_m^{-1}(h)$, and therefore α does not lie on one of the coordinate axes. Thus α is not an eigenvector of the inertia tensor, showing that $\alpha \times m^{-1}\alpha \neq 0$, which is condition (iv).

Let θ_n, θ denote the acute angles that $g_n\alpha_n$ and $g\alpha$ make with the vector $\mathbf{e}_1$, respectively. Since $\|g_n\alpha_n\| = \|g\alpha\| = r$ and $\langle g_n\alpha_n, \mathbf{e}_1 \rangle = \langle g\alpha, \mathbf{e}_1 \rangle = c$, we have $\cos\theta_n = c/r = \cos\theta$, whence $\theta_n = \theta$. Thus $\|g\alpha \times \mathbf{e}_1\| = r\sin\theta = r\sin\theta_n = \|g_n\alpha_n \times \mathbf{e}_1\| \neq 0$, which proves (v). We've shown that (g, α) satisfies (2.7) so that $(g, \alpha) \in f^{-1}(c, r, h) \cap W$. Therefore $f^{-1}(c, r, h) \cap W$ is compact as required. ∎

It now follows from the Arnold-Liouville Theorem 2.1.1 that for $(c, r, h) \in V_i$ ($i = 1, 2$), $f^{-1}(c, r, h) \cap P_i$ is a disjoint union of Lagrangian tori, and in a neighborhood of such a torus we have action-angle variables. In fact $f^{-1}(c,r,h)\cap P_i$ consists of two copies of $\mathbb{T}^3$, one in P_i^+, the other in P_i^-.

Proposition 2.3.1: *Each of the sets* $P_1^+, P_1^-, P_3^+, P_3^- \subset SO(3)\times\mathbb{R}^3$ *admits a single action-angle chart.*

Proof: We verify that the sets $V_i \subset \mathbb{R}^3$ $(i=1,2)$ are contractible. The result then follows directly from Corollary 2.1.1. For definiteness we assume $i=1$. Note that

$$V_1 = \{(c,r,h) \mid |c| < r,\ \lambda_1 > r^2/2h > \lambda_2\}$$

retracts onto its intersection with the r,h-plane via the map $t \mapsto ((1-t)c, r, h),\ 0 \le t \le 1$. This intersection $\{(r,h) \mid \lambda_1 > r^2/2h > \lambda_2\}$, is simply the region in the first quadrant lying between the parabolas $h = r^2/2\lambda_2$ and $h = r^2/2\lambda_1$, which is obviously contractible. ■

Set $U = U_1 \cup U_3$, $P = P_1 \cup P_3$, and $V = V_1 \cup V_3$, so that $P = \pi^{-1}(U)\cap W$, and $f(P) = V$. We've shown that $U \subset \mathbb{R}^3$ and $P \subset SO(3)\times\mathbb{R}^3$ are open, dense, and that P is the disjoint union of exactly four action-angle charts.

We now turn our attention to the proof of formula (2.1) for the third action I_3. Fix $\mu \in \mathbb{R}^3$, a regular value of $\mathbf{J} : SO(3)\times\mathbb{R}^3 \to \mathbb{R}^3$, $(g,\alpha)\mapsto g\alpha$. A straightforward check shows that $\mathbf{J}^{-1}(\mu) \subset SO(3)\times\mathbb{R}^3$ is invariant under the action of the coadjoint isotropy subgroup $SO(3)_\mu = \{g \in SO(3) \,|\, g\mu = \mu\} \cong S^1$, and that this action is free and proper, whence $\mathbf{J}^{-1}(\mu)$ is a principal S^1 bundle over the quotient. This quotient is naturally identified with the sphere $S_r^2 \subset \mathbb{R}^3$ (where $r = \|\mu\|$) via the map $[g,\alpha] \in \mathbf{J}^{-1}(\mu)/S^1 \mapsto \alpha \in S_r^2$. With this identification the bundle projection $\pi_\mu : \mathbf{J}^{-1}(\mu) \to S_r^2$ is $(g,\alpha)\mapsto\alpha$, which is just $\pi : SO(3)\times\mathbb{R}^3 \to \mathbb{R}^3$ restricted to $\mathbf{J}^{-1}(\mu)$. By the Marsden-Weinstein reduction theorem [1,22], there is a symplectic form ω_μ on S_r^2 such that $\pi_\mu^*\omega_\mu = i_\mu^*\omega$, where ω is the canonical symplectic form on $T^*SO(3) \cong SO(3)\times\mathbb{R}^3$ and $i_\mu : \mathbf{J}^{-1}(\mu) \to SO(3)\times\mathbb{R}^3$ is inclusion. In this case the induced form ω_μ is given by

$$\omega_\mu(u)(u \times v_1, u \times v_2) = -\langle u, v_1 \times v_2 \rangle,$$

where $u \in S_r^2$, and $v_i \in \mathbb{R}^3$, so $u \times v_i \in T_u S_r^2$ $(i = 1, 2.)$ This is a special case of Example 4.3.4(v) (p.302) in Abraham and Marsden[1]. The above form ω_μ, is $-r^{-1}$ times the standard oriented area form on S_r^2, as is easily checked.

Consider a dynamic trajectory $(g(t), \alpha(t)) = \Phi_t^H(g_0, \alpha_0)$, with $\mathbf{J}(g_0, \alpha_0) = \mu$, and $(g_0, \alpha_0) \in P$. Let T be the period of $\alpha(t)$ and D be the region on S_r^2 enclosed by $\alpha(t)$ for $0 \leq t \leq T$. We consider the above S^1 bundle restricted to D. Since D is diffeomorphic to a disc, $\pi_\mu : \pi_\mu^{-1}(D) \to D$ is trivial, i.e. $\pi_\mu^{-1}(D) \cong D \times S^1$. Let $\sigma : D \to \pi_\mu^{-1}(D)$ be a section of this bundle and set $\gamma_3 = \partial\sigma(D)$. Note that $\pi_\mu(\gamma_3) = \partial D$ in S_r^2.

Lemma 2.3.3: *Let β denote the canonical one form in phase space, and A the area of D oriented by the direction of the trajectory $\alpha(t)$. Then*

$$\int_{\gamma_3} \beta = \frac{A}{r}.$$

Proof:

$\omega = -d\beta$ is the canonical two form so by Stokes theorem, and the fact that $\sigma^*\omega = \omega_\mu$ we have,

$$\begin{aligned} \int_{\gamma_3} \beta &= -\int\int_{\sigma(D)} \omega \\ &= -\int\int_D \sigma^*\omega \\ &= -\int\int_D \omega_\mu \\ &= -\left(-\frac{A}{r}\right). \end{aligned}$$

■

Maintaining the above notation, let γ_1, and γ_2 denote the images of $\Phi_t^{I_1}(g_0, \alpha_0)$, and $\Phi_t^{I_2}(g_0, \alpha_0)$ respectively. Let $(c, r, h) = f(g_0, \alpha_0)$, so that $(c, r, h) \in V$ since $(g_0, \alpha_0) \in P$. The Liouville torus passing through (g_0, α_0) is then the connected component $\mathcal{C}$, of

$f^{-1}(c,r,h)$ containing (g_0,α_0). One readily verifies that $\pi(\mathcal{C}) = \pi(f^{-1}(c,r,h)) = \partial D = S_r^2 \cap H_m^{-1}(h)$.

Lemma 2.3.4: *The homology classes of* $\gamma_1,\ \gamma_2,\ \gamma_3$ *form a basis for* $H_1(\mathcal{C},\mathbb{Z})$.

Proof: Note that for $(g,\alpha) \in \mathbf{J}^{-1}(\mu)$ we have

$$\Phi_t^{I_2}(g,\alpha) = \left((\exp t\frac{\hat{\mu}}{\|\mu\|})g, \alpha\right),$$

showing that $\Phi_t^{I_2}|_{\mathbf{J}^{-1}(\mu)}$ parametrizes the action of the coadjoint isotropy subgroup $SO(3)_\mu \subset SO(3)$. Thus the orbits of $\Phi_t^{I_2}|_{\mathbf{J}^{-1}(\mu)}$ are exactly the fibers of the reduction map $\pi_\mu : \mathbf{J}^{-1}(\mu) \to S_r^2$. One also checks that the orbits of $\Phi_t^{I_1}$ which intersect $\mathbf{J}^{-1}(\mu)$, do so only once, and furthermore the flow $\Phi_t^{I_1}$ is nowhere tangent to $\mathbf{J}^{-1}(\mu)$.

From these observations we see that γ_3 can be obtained from $\mathcal{C}$ by a two step reduction. First, factor $\mathcal{C}$ by the flow of I_1 to obtain $\mathcal{C} \cap \mathbf{J}^{-1}(\mu)$. The resulting S^1 bundle is then trivial. (We can obtain a global section by transporting $\mathcal{C} \cap \mathbf{J}^{-1}(\mu)$ along the flow of I_1.) Second, factor $\mathcal{C} \cap \mathbf{J}^{-1}(\mu)$ by the flow of I_2 (which is just Marsden-Weinstein reduction) to get ∂D. The section $\sigma|_{\partial D} : \partial D \to \gamma_3$ selects a representative from each equivalence class in the quotient. The second S^1 bundle is also trivial since σ can be extended to D, which is contractible. We now have a tower of trivial S^1 bundles:

$$\mathcal{C} \longrightarrow \mathcal{C} \cap \mathbf{J}^{-1}(\mu) \longrightarrow \gamma_3,$$

proving that $\mathcal{C} \cong \gamma_1 \times \gamma_2 \times \gamma_3$. Therefore the homology classes of $\gamma_1,\ \gamma_2,\ \gamma_3$ form a basis for $H_1(\mathcal{C},\mathbb{Z})$ as required. ■

Its not hard to show that $\gamma_1,\ \gamma_2,\ \gamma_3$ can be chosen to vary smoothly with $(g_0,\alpha_0) \in P$. Using a local section of $f : P \to V$, one can also show that $\gamma_1,\ \gamma_2,\ \gamma_3$ depend smoothly on the value $f(g_0,\alpha_0) = (c,r,h) \in V$. Lemma 2.3.3, Lemma 2.3.4, Theorem 2.1.2, and the above discussion together prove

Proposition 2.3.2: $I_3 = A/2\pi\|\mathbf{J}\|$ *is a standard action integral for the rigid body.*

In fact one can also compute

$$I_i(g_0, \alpha_0) = \frac{1}{2\pi} \int_{\gamma_i} \beta$$

for $i = 1, 2$, showing that I_1 and I_2 are also standard actions.

In retrospect the formula for I_3 is not surprising. We saw in the proof of Lemma 2.3.3 that $(-A/r)$ is the symplectic area in the reduced manifold S_r^2 ($r = \|\mathbf{J}\|$), enclosed by a reduced trajectory. Therefore I_3 is actually an action integral for the reduced system.

2.4 Frequencies

In this section we show by direct calculation that the Hamiltonian flow of I_3 is 2π-periodic. We also compute the frequencies of the system with respect to the actions $\mathbf{I} = (I_1, I_2, I_3)$. It is determined that H_m depends only on I_2 and I_3, showing that the system has a "proper" resonance (i.e. a resonance independent of initial conditions.) An additional resonance is seen to occur (making the system periodic) precisely when $\Delta\Theta$ is a rational multiple of 2π.

Let $(g(t), \alpha(t)) = \Phi_t^{H_m}(g_0, \alpha_0)$, $(g_0, \alpha_0) \in P$, and $\mu = \mathbf{J}(g_0, \alpha_0)$ a regular value. Denote the period of $\alpha(t)$ by T. Since $\mathbf{J}$ is preserved by the flow of X_{H_m} we have

$$\mu = g_0\alpha_0 = g(T)\alpha(T) = g(T) \cdot \alpha_0,$$

so that $g_0^{-1}\mu = g(T)^{-1}\mu$, and $g(T)g_0^{-1}\mu = \mu$, showing that $g(T)g_0^{-1}$ is a rotation about $\mu \in \mathbb{R}^3$. We denote the angle of this rotation by $\Delta\Theta$. On this trajectory, let us put $h = H_m$, and $r = \|\mathbf{J}\|$. In [23], Montgomery proves

$$\Delta\Theta = -\frac{A}{r^2} + \frac{2hT}{r}, \tag{2.8}$$

where A denotes the oriented area on the sphere $S_r^2 = \{\alpha \in \mathbb{R}^3 \mid \|\alpha\| = r\}$ enclosed by the loop $\alpha(t)$. In Landau and Lifshitz [ ], it is shown that T is given by a complete elliptic integral whose modulus is a function of r, h, and the moments of inertia λ_i, $1 \leq i \leq 3$. We demonstrate in Appendix A that the area is also given by a combination of complete elliptic integrals, and hence so is $\Delta\Theta$. Using these expressions for T and $\Delta\Theta$ one can show that

$$\frac{\partial}{\partial r} T = -\frac{\partial}{\partial h} \Delta\Theta.$$

Thus $T\, dh - \Delta\Theta\, dr$ is a closed one-form on phase space, and hence is locally exact. In Appendix B we prove that I_3 defined by

$$2\pi I_3 = \frac{A}{r}$$

satisfies $d(2\pi I_3) = T\, dh - \Delta\Theta\, dr = T\, dH_m - \Delta\Theta\, dI_2$, so that

$$X_{2\pi I_3} = T\, X_{H_m} - \Delta\Theta\, X_{I_2}.$$

Our aim is to show that the flow $\Phi_t^{I_3}$, of X_{I_3} is 2π-periodic. We claim that

$$\Phi_t^{2\pi I_3} = \Phi_{-t\Delta\Theta}^{I_2} \circ \Phi_{tT}^{H}.$$

Since $\{I_2, H_m\} = 0$ the flows of X_{I_2} and X_H commute, i.e. $\Phi_t^{I_2} \circ \Phi_s^H = \Phi_s^H \circ \Phi_t^{I_2}$ for any $t, s \in \mathbb{R}$. Put $\gamma(t) = \Phi_{-t\Delta\Theta}^{I_2} \circ \Phi_{tT}^{H}(g_0, \alpha_0)$, then using this fact we get

$$\frac{d}{dt}\gamma(t) = T\, X_H(\gamma(t)) - \Delta\Theta\, X_{I_2}(\gamma(t)) = X_{2\pi I_3}(\gamma(t)),$$

and $\gamma(0) = (g_0, \alpha_0)$, showing that $\Phi_t^{2\pi I_3}(g_0, \alpha_0) = \gamma(t)$ as claimed. Since $g(T)g_0^{-1}$ is a rotation about μ by $\Delta\Theta$ radians we have

$$\begin{aligned} g(T)g_0^{-1} &= \exp\left(\Delta\Theta \frac{\widehat{\mu}}{\|\mu\|}\right) \\ &= \exp\left(\Delta\Theta \frac{\widehat{g_0\alpha_0}}{\|g_0\alpha_0\|}\right) \\ &= g_0 \exp\left(\Delta\Theta \frac{\widehat{\alpha_0}}{\|\alpha_0\|}\right) g_0^{-1}, \end{aligned}$$

whence

$$g(T) \exp\left(-\Delta\Theta \frac{\widehat{\alpha_0}}{\|\alpha_0\|}\right) = g_0.$$

Thus

$$\begin{aligned}
\gamma(1) &= \Phi^{I_2}_{-\Delta\Theta} \circ \Phi^{H_m}_{T}(g_0, \alpha_0) \\
&= \Phi^{I_2}_{-\Delta\Theta}(g(T), \alpha_0) \\
&= \left(g(T) \exp\left(-\Delta\Theta \frac{\widehat{\alpha}_0}{\|\alpha_0\|}\right), \alpha_0\right) \\
&= (g_0, \alpha_0) \\
&= \gamma(0).
\end{aligned}$$

We've shown that $\Phi^{2\pi I_3}_t = \Phi^{I_3}_{2\pi t}$ is 1-periodic, whence $\Phi^{I_3}_t$ is 2π-periodic as required.

Now from $d(2\pi I_3) = T\,dH_m - \Delta\Theta\,dI_2$ (see Appendix B) we get

$$dH_m = \frac{\Delta\Theta}{T} dI_2 + \frac{2\pi}{T} dI_3.$$

The frequencies are therefore

$$\begin{aligned}
\omega_1(\mathbf{I}) &= \frac{\partial H_m}{\partial I_1} = 0 \\
\omega_2(\mathbf{I}) &= \frac{\partial H_m}{\partial I_2} = \frac{\Delta\Theta}{T} \\
\omega_3(\mathbf{I}) &= \frac{\partial H_m}{\partial I_3} = \frac{2\pi}{T}.
\end{aligned}$$

In action-angle coordinates Hamilton's equations become

$$\begin{aligned}
\dot{I}_i &= 0 \qquad\qquad (1 \leq i \leq 3) \\
\dot{\theta}_1 &= 0 \\
\dot{\theta}_2 &= \frac{\Delta\Theta}{T} \\
\dot{\theta}_3 &= \frac{2\pi}{T},
\end{aligned}$$

with solution

$$\begin{aligned}
I_i(t) &= I_i(0) \\
\theta_1(t) &= \theta_1(0) \\
\theta_2(t) &= \theta_2(0) + \frac{\Delta\Theta}{T} t \\
\theta_3(t) &= \theta_3(0) + \frac{2\pi}{T} t.
\end{aligned}$$

If $\omega_2/\omega_3 = \Delta\Theta/2\pi$ is irrational then we see immediately that the trajectory is a dense winding on the 2-torus parametrized by $\theta_2,\ \theta_3$. Otherwise the solution is a periodic orbit lying on the same torus.

2.5 Deformable Bodies

Consider a closed system of particles (discrete or continuous) whose motion is known with respect to some coordinate frame in $\mathbb{R}^3$. We emphasize that this frame is not necessarily an inertial one. To say that the system is closed means that the only forces acting on the particles are those of interaction so that angular momentum is conserved. Suppose that the center of mass remains at the origin of the given frame throughout the motion. Then there exists an inertial frame whose origin coincides with that of the given one. The purpose of this section is to show that this data (i.e. the motion of the system with respect to the non-inertial or *moving* frame) is sufficient to determine the attitude of the moving frame relative to the inertial frame, and to derive the equations which govern that relationship. This attitude is specified by a matrix in $SO(3)$, which is therefore the configuration space of the system. The phase space is $T^*SO(3)$, which as before, is identified with $SO(3) \times \mathbb{R}^3$.

We have in mind a system such as a spacecraft with flexible attachments under robotic control. Here the moving frame is affixed to the relatively massive spacecraft, and the motion of the flexible structures is programmed by the designer. If the data consists of knowing that the particles (e.g. flexible attachments) are fixed in the moving frame we recover the rigid body, and indeed we find that the equations reduce to the Euler (2.5) and group (2.6) equations in this case. Occasionally we will refer to the moving frame as the *body* frame, even though there is no longer necessarily any body to which it is attached.

We denote points in the rotating frame by R and points in the inertial frame by r. Let $\rho_t = \rho(\cdot, t) : \mathbb{R}^3 \to \mathbb{R},\ 0 \leq t \leq c$ be the mass density of the system at time t, which we assume has compact support. Suppose that the motion of the system in the rotating frame is governed by

$$\dot{R}(t) = Y(R(t), t) \qquad 0 \leq t \leq c \tag{2.9}$$

where Y is a time dependent vector field defined on $\mathrm{supp}(\rho_t)$ and c is a fixed constant.

Let $g(t) \in SO(3)$ be the matrix which, at time t, gives the transformation from the moving frame to the inertial one. Then the position of a typical particle with respect to the inertial frame is $r(t) = g(t)R(t)$, and its velocity is $\dot{r} = \dot{g}R + g\dot{R} = \dot{g}R + gY(R,t)$. Returning the velocity to the moving frame we get

$$g^{-1}\dot{r} = g^{-1}\dot{g}R + Y(R,t).$$

One may check that $g^{-1}\dot{g} \in so(3)$ so there is a vector $\Omega \in \mathbb{R}^3$ such that $\hat{\Omega} = g^{-1}\dot{g}$, and hence

$$g^{-1}\dot{g}R = \Omega \times R.$$

Thus

$$\begin{aligned} \|\dot{r}\|^2 &= \|g^{-1}\dot{r}\|^2 = \|\Omega \times R + Y(R,t)\|^2 \\ &= \|\Omega \times R\|^2 + 2\langle \Omega \times R, Y(R,t)\rangle + \|Y(R,t)\|^2, \end{aligned}$$

and the total kinetic energy of the system is

$$\frac{1}{2}\int_{\mathbb{R}^3} \rho(R,t)\|\dot{r}\|^2 d^3R = \frac{1}{2}\int_{\mathbb{R}^3} \rho(R,t)\|\Omega \times R\|^2 d^3R$$

$$+\langle \Omega, \int_{\mathbb{R}^3} \rho(R,t)(R \times Y(R,t))d^3R\rangle + \frac{1}{2}\int_{\mathbb{R}^3} \rho(R,t)\|Y(R,t)\|^2 d^3R.$$

Here d^3R denotes Lebesgue measure on $\mathbb{R}^3$ and $\langle \cdot, \cdot \rangle$ the standard inner product.

The third term above depends on t only, hence is the total time derivative of some function, and therefore need not be included in the Lagrangian of the system. (See Landau and Lifshitz[17].) We set

$$F(t) = \int_{\mathbb{R}^3} \rho(R,t)(R \times Y(R,t))d^3R, \tag{2.10}$$

and the second term becomes $\langle \Omega, F(t) \rangle$. Note that F can be interpreted as the angular momentum of the system calculated *as if* the rotating frame were inertial. For the first term we define a time dependent, symmetric, bilinear form by

$$\langle\langle a, b \rangle\rangle = \int_{\mathbb{R}^3} \rho(R,t) \langle a \times R, b \times R \rangle d^3 R \tag{2.11}$$

for $a, b \in \mathbb{R}^3$. Equivalently this form can be given by a 3×3 symmetric matrix $m(t)$: $\langle\langle a, b \rangle\rangle = \langle a, m(t) b \rangle$. The first term is then $\frac{1}{2}\langle \Omega, m(t)\Omega \rangle$, and we may take as our Lagrangian

$$L(\Omega, t) = \frac{1}{2} \langle \Omega, m(t) \Omega \rangle + \langle \Omega, F(t) \rangle.$$

The matrix $m(t)$ (as well as the form $\langle\langle \cdot, \cdot \rangle\rangle$) is called the inertia tensor of the system.

Let our moving frame be given by the basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$. The entries of m with respect to this frame are then

$$\begin{aligned} m_{i,j}(t) &= \langle \mathbf{E}_i, m(t) \mathbf{E}_j \rangle = \langle\langle \mathbf{E}_i, \mathbf{E}_j \rangle\rangle \\ &= \int_{\mathbb{R}^3} \rho(R,t) \langle \mathbf{E}_i \times R, \mathbf{E}_j \times R \rangle d^3 R \\ &= \begin{cases} \int_{\mathbb{R}^3} \rho(R,t)(\|R\|^2 - R_i^2) d^3 R & \text{if } i = j \\ -\int_{\mathbb{R}^3} \rho(R,t) R_i R_j d^3 R & \text{if } i \neq j, \end{cases} \end{aligned}$$

where we have written R_i for the i^{th} component of R in this frame. Being symmetric, $m(t)$ is also diagonalizable. Let us suppose for the moment that m is diagonal with respect to the above basis. Then $m(t) = diag(\lambda_1(t), \lambda_2(t), \lambda_3(t))$ where

$$\lambda_i(t) = \int_{\mathbb{R}^3} \rho(R,t)(\|R\|^2 - R_i^2) d^3 R \qquad (1 \leq i \leq 3),$$

so for instance if $i = 1$:

$$\lambda_1(t) = \int_{\mathbb{R}^3} \rho(R,t)(R_2^2 + R_3^2) d^3 R.$$

If $\lambda_1(t) = 0$ for some t, then we see $R_2 = R_3 = 0$ for all $R \in \operatorname{supp}(\rho_t)$, so that all the mass of the system is concentrated in a line. From now on we assume that this is not the case and hence $\lambda_i(t) > 0$ for all t. With this stipulation, we have that $m(t)$ is invertible for all t. Observe that

$$\begin{aligned}\lambda_1 + \lambda_2 &= \int_{\mathbb{R}^3} \rho(R,t)(R_1^2 + R_2^2 + 2R_3^2)d^3R \\ &= \lambda_3 + 2\int_{\mathbb{R}^3} \rho(R,t)R_3^2 d^3R \\ &\geq \lambda_3.\end{aligned}$$

Similarly $\lambda_1 + \lambda_3 \geq \lambda_2$ and $\lambda_2 + \lambda_3 \geq \lambda_1$. Equality holds above if and only if $R_3 = 0$ for all $R \in \mathrm{supp}(\rho_t)$, whence all the mass of the system is concentrated in a plane.

The above discussion shows that the inertia matrix must lie among the 3×3, positive definite, symmetric matrices whose eigenvalues satisfy $\lambda_i + \lambda_j \geq \lambda_k$ $(i, j, k$ cyclic.) The set of possible inertia tensors is thus a subset of the six dimensional vector space of 3×3 symmetric matrices. If we further insist that the mass not be concentrated in a plane then the above inequalities become strict, and the set of inertia tensors, which we denote by M_1, forms an open subset of this vector space. Thus M_1 is a manifold and $\dim(M_1) = 6$. We examine the structure of M_1 more closely in Chapter 4.

To obtain the equations of motion we perform the Legendre transform by setting

$$\alpha = \frac{\partial L}{\partial \Omega}(\Omega, t) = m(t) \cdot \Omega + F(t).$$

Since $m(t)$ is invertible we have $\Omega = m(t)^{-1} \cdot (\alpha - F(t))$. The Hamiltonian is then

$$\begin{aligned}\langle \alpha, \Omega \rangle - L(\Omega, t) &= \frac{1}{2}\langle \alpha - F(t), m(t)^{-1} \cdot (\alpha - F(t))\rangle. \\ &= \frac{1}{2}\langle \alpha, m(t)^{-1}\alpha\rangle - \langle \alpha, m(t)^{-1}F(t)\rangle + \frac{1}{2}\langle F(t), m(t)^{-1}F(t)\rangle.\end{aligned}$$

The last term is a function of t only, so dropping it will have no effect on the equations of motion. We take our Hamiltonian to be

$$\mathcal{H}(\alpha, t) = \frac{1}{2}\langle \alpha, m(t)^{-1}\alpha\rangle - \langle \alpha, m(t)^{-1}F(t)\rangle.$$

Therefore

$$\nabla\mathcal{H}(\alpha, t) = m(t)^{-1}(\alpha - F(t)) = \Omega,$$

and by §2.2(iii) the equations are

$$\begin{cases} \text{(a)} & \dot{\alpha} = \alpha \times m(t)^{-1}(\alpha - F(t)) \\ \text{(b)} & \dot{g} = g(\widehat{m(t)^{-1}(\alpha - F(t))}) \end{cases} \tag{2.12}$$

Note that for a rigid body $F = 0$, and m is constant, so that these reduce to the usual Euler (2.5), and group (2.6) equations.

In Koiller[16], these equations are derived using conservation of angular momentum alone. To see this, we first show

$$m(t)\Omega = \int_{\mathbb{R}^3} \rho(R,t)(R \times (\Omega \times R))\ d^3R. \tag{2.13}$$

Let $V \in \mathbb{R}^3$. Then by (2.11)

$$\begin{aligned} \langle V, m(t)\Omega\rangle &= \int_{\mathbb{R}^3} \rho(R,t)\langle V \times R, \Omega \times R\rangle\ d^3R \\ &= \int_{\mathbb{R}^3} \rho(R,t)\langle V, R \times (\Omega \times R))\rangle\ d^3R \\ &= \langle V, \int_{\mathbb{R}^3} \rho(R,t)(R \times (\Omega \times R))d^3R\rangle, \end{aligned}$$

proving (2.13). Now the classical expression for angular momentum is

$$\begin{aligned} \int_{\mathbb{R}^3} \rho(R,t)(r \times \dot{r})\ d^3R &= \int_{\mathbb{R}^3} \rho(R,t)[gR \times (\dot{g}R + gY(R,t))]\ d^3R \\ &= g \cdot \int_{\mathbb{R}^3} \rho(R,t)[R \times (g^{-1}\dot{g})R + R \times Y(R,t)]\ d^3R \\ &= g \cdot \left[\int_{\mathbb{R}^3} \rho(R,t)(R \times (\Omega \times R))\ d^3R + \int_{\mathbb{R}^3} \rho(R,t)(R \times Y(R,t))\ d^3R\right] \\ &= g(m(t)\Omega + F(t)) \\ &= g\alpha. \end{aligned}$$

As for a rigid body, the map $\mathbf{J} : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$, $(g, \alpha) \mapsto g\alpha$ gives the angular momentum vector in the inertial frame. Thus $\alpha \in \mathbb{R}^3$ is properly interpreted as the angular momentum as seen from the moving frame. Since $\mathbf{J}$ is conserved we have

$$0 = \frac{d}{dt} g\alpha = \dot{g}\alpha + g\dot{\alpha},$$

so that

$$0 = g^{-1}\dot{g}\alpha + \dot{\alpha} = \Omega \times \alpha + \dot{\alpha},$$

which is the same as (2.12.a). Also $\hat{\Omega} = g^{-1}\dot{g}$ gives $\dot{g} = g\hat{\Omega}$ which is identical to (2.12.b). The motion of the rotating frame is thus obtained by substituting the solution to (2.12.a) into (2.12.b) and solving. We remark that these equations are completely analogous to (2.5) and (2.6) respectively.

Thus far we have considered two systems: (A) the *rigid body* with Hamiltonian

$$H_m(\alpha) = \frac{1}{2}\langle \alpha, m^{-1}\alpha\rangle,$$

and equations

$$\begin{cases} \dot{\alpha} &= \alpha \times m^{-1}\alpha \\ \dot{g} &= g(\widehat{m^{-1}\alpha}); \end{cases}$$

and (B) the *deformable body* governed by (2.9), with time dependent Hamiltonian

$$\mathcal{H}(\alpha, t) = \frac{1}{2}\langle \alpha, m(t)^{-1}\alpha\rangle - \langle \alpha, m(t)^{-1}F(t)\rangle,$$

and equations

$$\begin{cases} \dot{\alpha} &= \alpha \times m(t)^{-1}\alpha - \alpha \times m(t)^{-1}F(t) \\ \dot{g} &= g(\widehat{m(t)^{-1}\alpha}) - g(\widehat{m(t)^{-1}F}(t)). \end{cases}$$

A third system related to both of these is (C) the *rigid body with variable inertia tensor* with Hamiltonian

$$H_{m(t)}(\alpha) = \frac{1}{2}\langle \alpha, m(t)^{-1}\alpha\rangle,$$

and equations

$$\begin{cases} \dot{\alpha} &= \alpha \times m(t)^{-1}\alpha \\ \dot{g} &= g(\widehat{m(t)^{-1}\alpha}). \end{cases}$$

Here $t \mapsto m(t)$ is to be a path in the space of rigid body inertia tensors as in (B). We briefly examine the relationship between (B) and (C).

First observe that if the motion of the system in the moving frame is entirely radial, which means $\dot{R}(t) = Y(R(t), t) := \phi(t)R(t)$ for some scalar function ϕ, then (B) and (C) are identical. Indeed from (2.10) $F(t) \equiv 0$, so the second terms in (B) vanish.

Consider the corresponding slow systems (B_ϵ) and (C_ϵ) defined as follows. Replace (2.10) by

$$\dot{R}(t) = \epsilon Y(R(t), \epsilon t) \qquad\qquad 0 \leq t \leq \frac{c}{\epsilon}$$

where $\epsilon > 0$ is called the slowness parameter. Introduce the slow time $\tau = \epsilon t$, then by (2.10) $F(t)$ is replaced by $\epsilon F(\tau)$. System (B_ϵ) is

$$\begin{cases} \dot{\alpha} &= \alpha \times m(\tau)^{-1}\alpha - \epsilon(\alpha \times m(\tau)^{-1}F(\tau)) \\ \dot{g} &= g(\widehat{m(\tau)^{-1}\alpha}) - \epsilon g(\widehat{m(\tau)^{-1}F}(\tau)) \\ \dot{\tau} &= \epsilon, \end{cases}$$

and (C_ϵ) is defined by

$$\begin{cases} \dot{\alpha} &= \alpha \times m(\tau)^{-1}\alpha \\ \dot{g} &= g(\widehat{m(\tau)^{-1}}\alpha) \\ \dot{\tau} &= \epsilon. \end{cases}$$

A standard theorem of perturbation theory (e.g. Lemma 2.3.2 in Sanders, Verhulst[26]) says the solutions to (B_ϵ) and (C_ϵ) differ by $\mathcal{O}(\epsilon)$ on a time scale of order 1. (This means the estimate is valid on a bounded interval which is independent of ϵ.) We conjecture that there exists a wide class of motions (i.e. vector fields $Y(R,t)$) for which such an estimate can be made on a longer time scale (such as $1/\epsilon$.) The above example (radial motion) in which (B_ϵ) and (C_ϵ) are identical lends some credence to this. We proceed in Chapter 4 to study system (C) for its own sake.

3. The Hannay-Berry Connection

In this chapter we summarize the work of Montgomery[24] in which the classical Hannay angles are generalized to non-integrable Hamiltonians. Even though our primary interest is in the rigid body, which is completely integrable, the results here are indispensable for the geometric insight and computational ease which they provide. Our setting is somewhat more general than in [24], in which only trivial bundles are considered, but still less general than Marsden, Montgomery, and Ratiu[21]. We also refer the reader to Weinstein[28] for another point of view.

3.1 Families of Hamiltonian Group Actions

Let (P, ω) be a symplectic manifold, G a Lie group with Lie algebra $\mathfrak{g}$, and M a manifold which we call the parameter space. Let π_1 and π_2 denote projections of $M \times P$ onto M and P respectively, and suppose $2n = \dim(P)$, and $k = \dim(M)$. Let $E \subset M \times P$ be a smooth submanifold for which $\pi_1|E : E \to M$ is a smooth (not necessarily trivial) subbundle, and each fiber $E_m = \pi_1^{-1}(m) \cap E$ is an open submanifold of $\pi_1^{-1}(m) = \{m\} \times P$. Thus E_m is a symplectic manifold with symplectic form $\pi_2^* \omega | E_m$.

Definition 3.1.1: *A smooth action of G on E is called a family of Hamiltonian G actions if the following are satisfied:*

1. *G preserves the fibers E_m of $\pi_1|E$.*
2. *The action restricted to each fiber is symplectic.*
3. *The action admits a parametrized momentum map $I : E \to \mathfrak{g}^*$.*

Some explanation of (3) is required. For $\xi \in \mathfrak{g}$ the infinitesimal generator is:

$$\xi_E(m,x) = \left.\frac{d}{dt}\right|_{t=0} (\exp t\xi) \cdot (m,x).$$

Here $g \cdot (m,x)$ denotes the action of $g \in G$ on $(m,x) \in M \times P$. Note that ξ_E is vertical with respect to the bundle $\pi_1|E$, by (1). Let d_P and d_M denote the exterior derivatives in the P and M directions respectively. That is, for $f \in C^\infty(M \times P)$

$$d_P f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} dq^i + \frac{\partial f}{\partial p_i} dp_i \right),$$

and

$$d_M f = \sum_{j=1}^{k} \frac{\partial f}{\partial m_j} dm_j,$$

where $\{q^i, p_i\}_{i=1}^n$ and $\{m_j\}_{j=1}^k$ are local coordinates on P and M respectively. The Hamiltonian vector field of $f \in C^\infty(E)$ is defined as the unique vector field $X_f \in \mathfrak{X}(E)$ which is vertical with respect to $\pi_1|E$ and satisfies

$$\pi_2^*\omega(X_f, \cdot) = d_P f.$$

For $\xi \in \mathfrak{g}$, let I^ξ denote the function on E given by $I^\xi(m,x) = \langle I(m,x), \xi \rangle$, where $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the natural paring. Then (3) says that for each $\xi \in \mathfrak{g}$,

$$X_{I^\xi} = \xi_E.$$

A trivial example is given by a Hamiltonian action of G on P with momentum map $\mathbf{J} : P \to \mathfrak{g}^*$. Let G act trivially on M and take the diagonal action on $E := M \times P$. In this case $I(m,x) = \mathbf{J}(x)$ and there is no dependence on the "parameters" M. We will be concerned primarily with integrable systems which depend nontrivially on a parameter.

Example 1. Let $G = S^1$ and suppose $SO(3)$ acts on (P, ω) in a Hamiltonian manner with equivariant momentum map $\mathbf{J} : P \to so(3)^* \cong \mathbb{R}^3$. Here we identify $so(3)^* \cong so(3) \cong \mathbb{R}^3$ as in §2.2. Let $M = S^2$, the unit sphere in $\mathbb{R}^3$, and take $E = M \times P$. Then for $x \in P$ and $\eta \in S^2$, $X_{\langle \mathbf{J}, \eta \rangle} = \eta_P(x)$ has flow $\Phi_t(x) = (\exp t\eta) \cdot x$, where $g \cdot x$ denotes the action of $g \in SO(3)$ on $x \in P$. We see that $\Phi_t(x)$ is 2π-periodic since $\|\eta\| = 1$. Thus for each $\eta \in S^2$ we have an action $x \mapsto \Phi_\theta(x)$, of $\theta \in S^1$ on P. Let S^1 act trivially on S^2 and take the diagonal action on $S^2 \times P$. This forms a family of Hamiltonian S^1 actions. The action clearly preserves fibers and is symplectic when restricted to fibers. For the parametrized momentum map take $I : S^2 \times P \to \mathbb{R} = \mathfrak{Lie}(S^1)$ to be $I(\eta, x) = \langle \mathbf{J}(\eta), x\rangle$. This example is used in Montgomery[24] to compute phases for the Foucault pendulum.

Example 2. Take $E = M \times P$, let $H \in C^\infty(M \times P)$ and consider the dynamical system given by X_H. Since X_H is vertical, the fibers of π_1 are invariant under the flow. We may consider X_H to be a Hamiltonian system on $P \cong \pi_1^{-1}(m)$ which depends on the parameter $m \in M$. We say that H defines a *parameter dependent integrable system* if for each $m \in M$, $H(m, \cdot) \in C^\infty(P)$ is integrable in the usual sense. This means that there are functions $f_1, \ldots, f_n = H \in C^\infty(M \times P)$ such that for each $m \in M$, $\{f_i(m, \cdot), f_j(m, \cdot)\} = 0$ (using the induced Poisson structure on $\pi_1^{-1}(m)$), and $d_P f_1(m, x) \wedge \ldots \wedge d_P f_n(m, x) \neq 0$ for almost every $x \in P$. By the Arnold-Liouville Theorem 2.1.1 the regular compact connected level sets of $f(m, \cdot) = (f_1(m, \cdot), \ldots, f_n(m, \cdot))$ are diffeomorphic to $\mathbb{T}^n$ and about such a torus there exist local action-angle coordinates. This means we have functions $I = (I_1, \ldots, I_n)$ locally defined in (m, x), whose Hamiltonian flows are 2π-periodic. Also we have a local diffeomorphism ϕ, on $\mathbb{R}^n$, depending on $m \in M$, such that $f(m, x) = \phi(m, I(m, x))$. In particular H is a function of m and I. For simplicity we assume that I is globally defined. Then by flowing along the trajectories of $X_{I_1}, \ldots, X_{I_n}$ (or equivalently by advancing the angle coordinates conjugate to $I_1, \ldots, I_n$) we have an action of $\mathbb{T}^n$ on $M \times P$. We see readily that this is a family of Hamiltonian $\mathbb{T}^n$ actions. Indeed, (1) in the definition is satisfied since X_{I_i} $(1 \leq i \leq n)$ is vertical, and (2) holds since Hamiltonian flows are necessarily symplectic. By definition of the $\mathbb{T}^n$ action we have for $\xi \in \mathbb{R}^n = \mathfrak{Lie}(\mathbb{T}^n)$,

$$\xi_{M \times P} = \xi_1 X_{I_1} + \cdots + \xi_n X_{I_n} = X_{\xi_1 I_1 + \cdots + \xi_n I_n} = X_{I^\xi},$$

showing that (3) holds.

Example 3. Let $f_1, \ldots, f_n = H \in C^\infty(M \times P)$ form a parametrized integrable system as in the previous example. We examine the case where action-angle coordinates are *not* globally defined, which is the more common occurrence. The discussion preceding Theorem 2.1.3 shows that for each parameter value $m \in M$, there is an open submanifold $P(m) \subset P$ on which $f(m, \cdot) = (f_1(m, \cdot), \ldots, f_n(m, \cdot))$ is a proper submersion. Typically $P(m)$ is obtained from P by removing certain submanifolds of codimension at least one, so that $P(m)$ is dense in P. As $m \in M$ is allowed to vary, the connected components of $P(m)$ may migrate through P, change topology, or even disappear. Let us assume that the number of components remains constant for all $m \in M$; say

$$P(m) = \bigcup_{i=1}^{k} P_i(m),$$

where each $P_i(m)$ is connected and open. Define

$$E = \{(m, x) \in M \times P \mid x \in P(m)\}$$

and consider the subbundle $\pi_1|E : E \to M$, with fibers $E_m = \{m\} \times P(m)$. If we follow the i^{th} component as m executes a loop in M, it may be that $P_i(m)$ does not return to itself, but say to $P_j(m)$. In other words the labeling map

$$\{\text{components of } P(m)\} \to \{1, \ldots, k\}$$

may be defined only locally on M. Put another way, the bundle $\pi_1|E : E \to M$ may not be trivial. Now assume that each component of $P(m)$ admits a single action-angle chart. While not occurring generally, this situation is exactly what happens for the rigid body (see §2.3). Just as in Example 2, $\mathbb{T}^n$ acts in a Hamiltonian manner on each component $P_i(m)$, $1 \leq i \leq k$, and hence on the fiber E_m, thus forming a family of Hamiltonian $\mathbb{T}^n$ actions on E.

In [21], Definition 3.1.1 is generalized further to the case when $\pi_1 : E \to M$ is a Poisson fiber bundle, not necessarily contained in a trivial bundle.

3.2 Definition of the HB Connection

Let $E \subset M \times P$ be equipped with a family of Hamiltonian G actions, and assume G is compact and connected. We will denote the action by $\Phi_g(m,x)$ for $g \in G$, $(m,x) \in E$. Let dg denote Haar measure on G, and let σ be a tensor field defined along (not necessarily on) a G invariant submanifold of E.

Definition 3.2.1: *The average of σ over G is the tensor field of the same type given by*

$$\langle \sigma \rangle = \frac{1}{|G|} \int_G \Phi_g^* \sigma \, dg,$$

where $|G|$ denotes the volume of G with respect to dg.

Observe that if σ is G-invariant, which means that $\Phi_g^* \sigma = \sigma$ for $g \in G$, then $\langle \sigma \rangle = \sigma$. Conversely $\langle \sigma \rangle = \sigma$ implies $\Phi_g^* \sigma = \sigma$ by the translation invariance of Haar measure. Thus $\langle \sigma \rangle$ is itself G-invariant. Note that the map $\sigma \mapsto \langle \sigma \rangle$ is $\mathbb{R}$ linear, and in fact linear over the ring of G-invariant functions.

Let $v \in T_m M$. The horizontal lift of v with respect to the trivial connection on $\pi_1 : E \to M$ is simply $(v, 0)$. Let $v \oplus 0$ denote the vector field on E_m whose value at (m,x) is $(v,0) \in T_{(m,x)}(E)$.

Definition 3.2.2: *The Hannay-Berry (HB) connection is the Ehresman connection on $\pi_1|E : E \to M$ whose horizontal lift is given by*

$$\mathrm{Hor}_{(m,x)}(v) = \langle v \oplus 0 \rangle (m,x).$$

Note that if the G action is independent of m, then $\Phi_g^*(v \oplus 0) = v \oplus 0$ whence $\langle v \oplus 0 \rangle = v \oplus 0$, and the HB connection is trivial in this case.

To motivate this definition recall Example 2 of the previous section. We consider a parameter dependent integrable system with global action angle variables $(\mathbf{I}, \boldsymbol{\theta})$. Let $\{m_j\}_{j=1}^k$ be local coordinates on M for which $(\mathbf{I}, \boldsymbol{\theta})$ are defined. Then

$$(m_1, \ldots, m_k, I_1, \ldots, I_n, \theta_1, \ldots, \theta_n)$$

are local parameter dependent coordinates on $E := M \times P$. We take $G = \mathbb{T}^n$ in the above definition, and denote the induced $\mathbb{T}^n$ action by

$$\Phi_{\boldsymbol{\phi}}(m, \mathbf{I}, \boldsymbol{\theta}) := (m, \mathbf{I}, \boldsymbol{\theta} + \boldsymbol{\phi})$$

for $\boldsymbol{\phi} \in \mathbb{T}^n$. Note that this action depends on m since the coordinates $(\mathbf{I}, \boldsymbol{\theta})$ do. To determine the holonomy of the corresponding HB connection we first compute $\text{Hor} \cdot \frac{\partial}{\partial m_j} \in \mathfrak{X}(M \times P)$. Observe

$$\begin{aligned}
(\text{Hor} \cdot \frac{\partial}{\partial m_j})[I_i] &= dI_i \cdot \left\langle \frac{\partial}{\partial m_j} \oplus 0 \right\rangle \\
&= (2\pi)^{-n} \int_{\mathbb{T}^n} dI_i \cdot \Phi^*_{\boldsymbol{\theta}}(\frac{\partial}{\partial m_j} \oplus 0)\, d\boldsymbol{\theta} \\
&= (2\pi)^{-n} \int_{\mathbb{T}^n} (\frac{\partial}{\partial m_j} \oplus 0)[I_i \circ \Phi_{-\boldsymbol{\theta}}] \circ \Phi_{\boldsymbol{\theta}}\, d\boldsymbol{\theta} \\
&= (2\pi)^{-n} \int_{\mathbb{T}^n} \Phi^*_{\boldsymbol{\theta}} \left((\frac{\partial}{\partial m_j} \oplus 0)[I_i] \right) d\boldsymbol{\theta} \\
&= \left\langle \frac{\partial I_i}{\partial m_j} \right\rangle .
\end{aligned}$$

In a similar manner we get

$$(\text{Hor} \cdot \frac{\partial}{\partial m_j})[\theta_i] = \left\langle \frac{\partial \theta_i}{\partial m_j} \right\rangle ,$$

and

$$(\text{Hor} \cdot \frac{\partial}{\partial m_j})[m_i] = \delta_{ij},$$

whence

$$\text{Hor} \cdot \frac{\partial}{\partial m_j} = \frac{\partial}{\partial m_j} \oplus \sum_{i=i}^{n} \left(\left\langle \frac{\partial I_i}{\partial m_j} \right\rangle \frac{\partial}{\partial I_i} + \left\langle \frac{\partial \theta_i}{\partial m_j} \right\rangle \frac{\partial}{\partial \theta_i} \right) .$$

Recall that *standard actions* were those obtained via Theorem 2.1.2. The following result is one of the basic facts concerning slowly varying completely integrable systems. A proof is implicit in the work of Weinstein[28].

Lemma 3.2.1: *If the actions are standard then* $\langle d_M \mathbf{I} \rangle = 0$ *i.e* $\left\langle \frac{\partial I_i}{\partial m_j} \right\rangle = 0$, *for* $1 \leq i \leq n,\ 1 \leq j \leq k$.

Now assume that the actions are standard, or at least that $\langle d_M \mathbf{I} \rangle = 0$. Then the horizontal lift becomes

$$\mathrm{Hor} \cdot \frac{\partial}{\partial m_j} = \frac{\partial}{\partial m_j} \oplus \sum_{i=i}^{n} \left\langle \frac{\partial \theta_i}{\partial m_j} \right\rangle \frac{\partial}{\partial \theta_i}.$$

Let $m : [0,1] \to M$ be a smooth path whose image C, lies in the domain of the coordinates $\{m_j\}_{j=1}^{k}$. Then

$$\begin{aligned}
\mathrm{Hor} \cdot m' &= \mathrm{Hor} \cdot \left(\sum_{j=1}^{k} m_j' \frac{\partial}{\partial m_j} \right) \\
&= \sum_{j=1}^{k} m_j' \left(\frac{\partial}{\partial m_j} \oplus \sum_{i=1}^{n} \left\langle \frac{\partial \theta_i}{\partial m_j} \right\rangle \frac{\partial}{\partial \theta_i} \right) \\
&= \sum_{j=1}^{k} m_j' \frac{\partial}{\partial m_j} \oplus \sum_{i=1}^{n} \left(\sum_{j=1}^{k} \left\langle \frac{\partial \theta_i}{\partial m_j} \right\rangle m_j' \right) \frac{\partial}{\partial \theta_i} \\
&= m' \oplus \sum_{i=1}^{n} \langle d_M \theta_i \rangle \cdot m',
\end{aligned}$$

and the parallel transport equations are

$$\begin{cases} \dot{I}_i &= 0 \\ \dot{\theta}_i &= \langle d_M \theta_i \rangle \cdot m'. \end{cases}$$

The parallel transport along $m(t)$ is then

$$(m(0), \mathbf{I}(0), \boldsymbol{\theta}(0)) \mapsto \left(m(1), \mathbf{I}(0), \boldsymbol{\theta}(0) + \int_C \langle d_M \boldsymbol{\theta} \rangle \right).$$

If C is a loop then the holonomy is

$$\boldsymbol{\theta}(1) - \boldsymbol{\theta}(0) = \int_C \langle d_M \boldsymbol{\theta} \rangle \,,$$

which is precisely the Hannay angles mentioned in the introduction. We have proved

Theorem 3.2.1: *If $\langle d_M \mathbf{I} \rangle = 0$, then for sufficiently small loops, the holonomy of the HB connection is the Hannay angles.*

Remark. Sufficiently small in this case means that the loop must lie in a region in M over which the m dependent action-angle coordinates can be defined. We see from this theorem that the HB connection serves to generalize the Hannay angles to non-integrable systems.

3.3 Geometric Properties of the HB Connection

In this section we introduce the notion of a Hamiltonian connection for a given family of Hamiltonian G actions. We quote a theorem of Montgomery regarding the existence and uniqueness of such connections, and discuss a method for calculating the Hamiltonian one-form (defined below.) This method will be used in the next chapter to construct the HB connection for the rigid body. For $m \in M$ define $P(m) := \{x \in P \mid (m, x) \in E_m\} = \pi_2(E_m)$.

Definition 3.3.1: *Let the bundle $\pi_1 : E \subset M \times P \to M$ be equipped with a family of Hamiltonian G actions, with parametrized momentum map $\mathbf{I} : E \to \mathfrak{g}^*$. A Hamiltonian connection for this family is an Ehresman connection on $\pi_1|E$ satisfying:*

1. *$D\mathbf{I} = 0$, where D denotes the covariant differentiation operator.*
2. *For each $v \in T_m M$, there is a function $\mathbf{K} \cdot v \in C^\infty(P(m))$ such the horizontal lift is given by*

$$\mathrm{Hor}_{(m,x)}(v) = v \oplus X_{\mathbf{K}\cdot v}(x),$$

for $x \in P(m)$.

3. *$\langle \mathbf{K} \cdot v \rangle$ is a constant, which can be taken to be zero.*

Property (1) says that $\mathbf{I}$ is constant along the horizontal lift $c(t) = (m(t), x(t))$ of a curve $m(t) \in M$. By (2), the parallel transport equations are

$$\begin{aligned} c'(t) &= \mathrm{Hor}_{c(t)} \cdot m'(t) \\ &= m'(t) \oplus X_{\mathbf{K}\cdot m'(t)}(x(t)), \end{aligned}$$

so that

$$x'(t) = X_{\mathbf{K}\cdot m'(t)}(x(t)).$$

Thus $x(t)$ is the flow of a time dependent Hamiltonian vector field on $P(m(t))$. One says that *parallel transport is Hamiltonian.* Regarding (3), note that $\mathbf{K} \cdot v$ is defined only up to an additive constant. Thus we may replace $\mathbf{K} \cdot v$ by $\mathbf{K} \cdot v - \langle \mathbf{K} \cdot v \rangle$ which has average zero.

If $Z \in \mathfrak{X}(M)$ we can regard $\mathbf{K} \cdot Z$ as a function on E which is defined up to addition of a smooth function on M. The map $Z \mapsto \mathbf{K} \cdot Z$ can clearly be taken to be linear. The operator $\mathbf{K}$ is thus a one-form on M taking values in the ring $C^\infty(E)/C^\infty(M)$, and determines the connection. $\mathbf{K}$ is called the *Hamiltonian one-form* for the connection. The following is proved in [21,24].

Theorem 3.3.1: *A family of Hamiltonian G actions admits a Hamiltonian connection if and only if the adiabatic condition $\langle d_M \mathbf{I} \rangle = 0$ holds. Furthermore, if such a connection exists it is unique and equals the Hannay-Berry connection. In particular the HB connection is Hamiltonian.*

The condition $\langle d_M \mathbf{I} \rangle = 0$ is not prohibitive, since it is always possible to replace $\mathbf{I}$ with $\mathbf{I}'$ satisfying $\langle d_M \mathbf{I}' \rangle = 0$, without changing the G action. The new momentum map is defined globally in P but only locally in M. See [21,24] for details. This subtlety doesn't arise for the rigid body since the actions are standard and hence $\langle d_M \mathbf{I} \rangle = 0$ by Lemma 3.2.1.

Let $v \in T_m M$ and $\xi \in \mathfrak{g}$. Using (2) of the definition, we have

$$\begin{aligned} D\mathbf{I}^\xi \cdot v &= d\mathbf{I}^\xi \cdot \mathrm{Hor} \cdot v \\ &= d\mathbf{I}^\xi \cdot (v \oplus X_{\mathbf{K}\cdot v}) \end{aligned}$$

$$\begin{aligned} &= d_M \mathbf{I}^\xi \cdot v + d_P \mathbf{I}^\xi \cdot X_{\mathbf{K}\cdot v} \\ &= d_M \mathbf{I}^\xi \cdot v + \{\mathbf{I}^\xi, \mathbf{K}\cdot v\}, \end{aligned}$$

so by (1), $\mathbf{K} \cdot v$ necessarily satisfies

$$d_M \mathbf{I}^\xi \cdot v + \{\mathbf{I}^\xi, \mathbf{K}\cdot v\} = 0$$

for each $\xi \in \mathfrak{g}$. This PDE is not sufficient however, to determine the function $\mathbf{K}\cdot v$. Instead we have

Proposition 3.3.1: *Suppose one can find a function $\widetilde{\mathbf{K}} \cdot v \in C^\infty(P(m))$ such that for all $\xi \in \mathfrak{g}$*

$$d_M \mathbf{I}^\xi \cdot v + \{\mathbf{I}^\xi, \widetilde{\mathbf{K}}\cdot v\} = 0. \tag{3.1}$$

Then $\mathbf{K} \cdot v = \widetilde{\mathbf{K}} \cdot v \ - \langle \widetilde{\mathbf{K}} \cdot v \ \rangle$ is the Hamiltonian 1-form for the HB connection.

Proof: Properties (2) and (3) of the definition are automatically satisfied. By the above computation

$$\begin{aligned} D\mathbf{I}^\xi \cdot v &= d_M \mathbf{I}^\xi \cdot v + \{\mathbf{I}^\xi, \mathbf{K}\cdot v\} \\ &= d_M \mathbf{I}^\xi \cdot v + \{\mathbf{I}^\xi, \widetilde{\mathbf{K}}\cdot v\} - \{\mathbf{I}^\xi, \langle \widetilde{\mathbf{K}}\cdot v\rangle\} \\ &= \xi_P[\langle \widetilde{\mathbf{K}}\cdot v\rangle] \\ &= 0, \end{aligned}$$

since $\langle \widetilde{\mathbf{K}} \cdot v\rangle$ is G invariant. This proves (1), and by uniqueness, $\mathbf{K} \cdot v$ determines the HB connection. ∎

We say that $\widetilde{\mathbf{K}} \cdot v$ *almost generates parallel translation.* This proposition provides a procedure for calculating the horizontal lift operator, which is sometimes more feasible than computing $\mathrm{Hor} \cdot v$ directly from its definition. This is because it is easier to average a function than a vector field. (For the latter, one must differentiate the group action $\Phi_g : E \to E,\ g \in G$ with respect to $x \in P(m)$.) Of course we have the added step of solving (3.1), but this PDE is sometimes quite simple.

If the family of Hamiltonian G actions admits additional symmetries the above procedure simplifies considerably.

Proposition 3.3.2: *Suppose another Lie group H, with Lie algebra $\mathfrak{h}$, acts on P in a Hamiltonian manner with equivariant momentum map $\mathbf{J} : P \to \mathfrak{h}^*$. Suppose also that H acts on M in such a way that the corresponding diagonal action $h\cdot(m, x) = (h\cdot m, h\cdot x)$, $h \in H$, preserves $\mathbf{I} : E \to \mathfrak{g}^*$. Then the action of $\mathbf{K}$ on vectors tangent to the H orbits in M is given by*

$$\mathbf{K} \cdot \eta_M = \mathbf{J}^\eta - \langle \mathbf{J}^\eta \rangle.$$

Here $\eta \in \mathfrak{h}$, and η_M denotes the infinitesimal generator of the H action on M.

Proof: Let $\xi \in \mathfrak{g}$. By hypothesis we have

$$\mathbf{I}^\xi(h \cdot m, h \cdot x) = \mathbf{I}^\xi(m, x) \tag{3.2}$$

for $h \in H$, $(m, x) \in E$. Let $\eta \in \mathfrak{h}$, put $h = \exp t\eta$ in (3.2), and differentiate both sides with respect to t at $t = 0$. We get

$$\begin{aligned} 0 &= d_M \mathbf{I}^\xi \cdot \eta_M(m) + d_P \mathbf{I}^\xi \cdot \eta_P(x) \\ &= d_M \mathbf{I}^\xi \cdot \eta_M(m) + \{\mathbf{I}^\xi, \mathbf{J}^\eta\}, \end{aligned}$$

showing that $\mathbf{J}^\eta$ is a solution to (3.1). The result follows from the previous proposition. ■

3.4 Curvature of the HB Connection

Recall that the curvature of an Ehresman connection is the vertical bundle valued two form given by the covariant derivative of the connection one form. The curvature induces a two form on the base (also called the curvature) by composition with the horizontal lift operator. Maintaining the notation of previous sections, let $\operatorname{curv}(V_1, V_2)(m, x)$ denote the curvature of an Ehresman connection on $\pi_1 : E \subset M \times P \to M$ applied to $V_1, V_2 \in T_{(m,x)}E$, $(m, x) \in E$. The induced form is then

$$\overline{\Omega}(v_1, v_2)(m, x) = \operatorname{curv}(\mathrm{Hor} \cdot v_1, \mathrm{Hor} \cdot v_2)(m, x),$$

where $v_1, v_2 \in T_m M$.

For the HB connection the form $\overline{\Omega}$ is Hamiltonian in the following sense.

Theorem 3.4.1: *Let $m \in M$ and $v_1, v_2 \in T_m M$. Then there is a smooth function $\Omega(v_1, v_2) : E_m \to \mathbb{R}$ such that*

$$\overline{\Omega}(v_1, v_2)(m, x) = X_{\Omega(v_1, v_2)}(m, x).$$

Ω is given by

$$\Omega(v_1, v_2) = \langle \{ \mathbf{K} \cdot v_1, \mathbf{K} \cdot v_2 \} \rangle.$$

The proof can be found in [21,24]. We will abuse the terminology slightly and call Ω the curvature of the HB connection.

Remark. As for the connection one-form, $\Omega(v_1, v_2)$ is defined only up to addition of a constant, so for $Z_1, Z_2 \in \mathfrak{X}(M)$, $\Omega(Z_1, Z_2)$ is a smooth function of E defined up to addition of a smooth function of M. Thus Ω is a two-form on M taking values in the ring $C^\infty(E)/C^\infty(M)$.

4. The HB Connection for the Rigid Body

In this chapter we study the rigid body with variable inertia tensor mentioned in §2.5. Recall this is the time dependent Hamiltonian system on $SO(3) \times \mathbb{R}^3$ with energy $H_{m(t)}(\alpha) = \langle \alpha, m^{-1}\alpha \rangle$, where $t \mapsto m(t)$ is a (piecewise) smooth path in the space of inertia tensors. We are particularly interested in the case where $m(t)$ is a loop.

As usual $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ denotes an inertial frame in $\mathbb{R}^3$ and $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ denotes the "body" frame in the sense of §2.5 (i.e. the motion of the system is known relative to $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$.) In the Euler description of the motion, one regards the body frame as fixed and the inertial frame as moving. Accordingly we consider $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ as a fixed basis for $\mathbb{R}^3$ throughout this chapter. It will be necessary to consider a third frame, denoted by $\{\mathcal{E}_1(m), \mathcal{E}_2(m), \mathcal{E}_3(m)\}$, which diagonalizes the inertia tensor m. Note that this basis is not unique and cannot be defined in a consistent manner along certain non-contractible loops in the space of inertia tensors. We shall see in §4.1 that for m with distinct eigenvalues, $\{\mathcal{E}_1(m), \mathcal{E}_2(m), \mathcal{E}_3(m)\}$ is defined only up to rotations by a discrete subgroup of $SO(3)$. When considering loops $m(t)$, we will usually take $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and $\{\mathcal{E}_1(m(t)), \mathcal{E}_2(m(t)), \mathcal{E}_3(m(t))\}$ to coincide initially.

Define the open sets $U(m),\ U_i(m),\ U_i^\nu(m) \subset \mathbb{R}^3$ and $P(m),\ P_i(m),\ P_i^\nu(m) \subset SO(3) \times \mathbb{R}^3$ for $i = 1, 3;\ \nu = +, -$, as in §2.3. In these definitions the coordinates α_j of $\alpha \in \mathbb{R}^3$ are taken with respect to $\{\mathcal{E}_1(m), \mathcal{E}_2(m), \mathcal{E}_3(m)\}$, so that these sets depend on m, as the notation suggests. Recall from §2.3 that we have the disjoint union

$$P(m) = P_1^+(m) \cup P_1^-(m) \cup P_3^+(m) \cup P_3^-(m),$$

where each $P_i^\nu(m)$ supports a single action-angle chart. As m executes a loop in the space of inertia tensors, one can see that the ellipsoids $H_m^{-1}(h)$ $(h > 0)$, the sets $U_i^\nu(m),\ P_i^\nu(m)$, and the basis $\{\mathcal{E}_1(m), \mathcal{E}_2(m), \mathcal{E}_3(m)\}$ all move together relative to $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$. Set

$$E = \{(m, g, \alpha) \mid (g, \alpha) \in P(m)\},$$

where M denotes the inertia tensors with distinct eigenvalues. Then $\pi_1|E : E \to M$ is a nontrivial subbundle of $\pi_1 : M \times (SO(3) \times \mathbb{R}^3) \to M,\ (m, g, \alpha) \mapsto m$. The action angle coordinates on $P_i^\nu(m)$ induce a family of Hamiltonian $\mathbb{T}^3$ actions on $\pi_1|E$, just as in §3.1 Example 3. The goal of this chapter is then to study the corresponding HB connection.

We begin in §4.1 by examining the space of inertia tensors in detail, showing that M has the structure of a smooth fiber bundle with contractible base and topologically non-trivial fiber. In §4.2 we compute the Hamiltonian one-form for the HB connection on vectors tangent to fiber and base respectively. In §4.3 we compute all but three terms in the curvature form, and in §4.4 we calculate the holonomy of various loops in M. Finally in §4.5 we consider whether the HB connection can be extended to the inertia tensors with multiple eigenvalues, and discuss averaging over a natural $\mathbb{T}^2$ action.

4.1 The Space of Inertia Tensors

The space of rigid body inertia tensors is the set M_1, of real, positive definite, symmetric matrices, whose eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfy $\lambda_i + \lambda_j > \lambda_k$ (i, j, k cyclic.) Recall from §2.5 that if we allow all the mass of the body to be concentrated in a plane then we have weak inequality above. We assume this is not the case, so that M_1 is an open subset of the six dimensional vector space of 3×3 symmetric matrices, and hence is a manifold. Each $m \in M_1$ induces a completely integrable system on the phase space $SO(3) \times \mathbb{R}^3 \cong T^*SO(3)$ via the Hamiltonian energy function $H_m(\alpha) = \frac{1}{2}\langle \alpha, m^{-1}\alpha \rangle$.

Define $M \subset M_1$ to be the set of $m \in M_1$ with distinct eigenvalues. We claim that M is an open submanifold of M_1. In fact the set Σ, consisting of $m \in M_1$ with exactly two eigenvalues equal, is a submanifold of codimension two in M_1. To see this observe that the elements of M_1 are in one to one correspondence with the ellipsoids in $\mathbb{R}^3$ which are the level sets of the corresponding quadratic form (i.e. kinetic energy function.) The ellipsoids corresponding to $m \in \Sigma$ are parametrized by the lengths of the distinct axes (giving two parameters since two of the three axes are equal) and by the direction of the one axis which is unequal to the others (giving two parameters.) Thus $\dim(\Sigma) = 4$. Similarly, the set Σ', of $m \in M_1$ with three equal eigenvalues forms a submanifold of dimension one, being parametrized by the radius of the corresponding level set (which is a sphere in this case.) Thus $M = M_1 \setminus (\Sigma \cup \Sigma')$, showing that M is an open submanifold of M_1 as claimed. One might have initially guessed that the codimension of Σ in M_1 is equal to one, since it is defined by a single equality. This is explained by noting that the functions which give the eigenvalues of m (defined locally on M_1), cease to be independent at points of Σ and hence cannot be considered as coordinates there. (See Arnold[2] Appendix 10.)

We take M as the parameter space for the HB connection. Note that the $\mathbb{T}^3$ action on $\pi_1|E : E \to M$ given by the parameter dependent integrable system H_m forms a family of Hamiltonian group actions by §3.1 Example 3. In §4.5 we will see that the HB connection cannot be extended to M_1, except in a certain restricted sense. The goal of this section is to show that M has the structure of a trivial fiber bundle over an open set in $\mathbb{R}^3$.

Let $SO(3)$ act on M_1 by conjugation: $g \cdot m = gmg^{-1}$ for $m \in M_1,\ g \in SO(3)$. Note that this action preserves eigenvalues, and therefore restricts to an action on M. It is an elementary fact of linear algebra that two elements of M_1 are in the same orbit if and only if they have the same eigenvalues, and furthermore that given $m \in M_1$, there is a $g \in SO(3)$ such that gmg^{-1} is diagonal. The infinitesimal generator of this action is

$$\begin{aligned} \xi_M(m) &= \left.\frac{d}{dt}\right|_{t=0} (\exp t\hat{\xi}) m (\exp t\hat{\xi})^{-1} \\ &= \hat{\xi} m - m\hat{\xi} = [\hat{\xi}, m], \end{aligned}$$

for $\xi \in \mathbb{R}^3$. If m is diagonal, say $m = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$, one computes

$$\xi_M(m) = \begin{pmatrix} 0 & (\lambda_1 - \lambda_2)\xi_3 & (\lambda_3 - \lambda_1)\xi_2 \\ (\lambda_1 - \lambda_2)\xi_3 & 0 & (\lambda_2 - \lambda_3)\xi_1 \\ (\lambda_3 - \lambda_1)\xi_2 & (\lambda_2 - \lambda_3)\xi_1 & 0 \end{pmatrix}. \tag{4.1}$$

The isotropy algebra $so(3)_m = \{\xi \in \mathbb{R}^3 \mid \xi_M(m) = 0\}$ is the Lie algebra of the isotropy subgroup $SO(3)_m = \{g \in SO(3) \mid gmg^{-1} = m\}$. If $m \in M$, then by (4.1), $\xi_M(m) = 0$ if and only if $\xi = 0$. This is also true even if m is not diagonal, as one verifies. Thus

$so(3)_m = 0$, and $SO(3)_m$ is discrete, showing that the action is locally free when restricted to M. From now on we consider the action on M only.

We now compute the isotropy on M. Again assume $m = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. We claim that

$$\begin{aligned} SO(3)_m &= \{id, \exp \pi\hat{\mathbf{E}}_1, \exp \pi\hat{\mathbf{E}}_2, \exp \pi\hat{\mathbf{E}}_3\} \\ &= \{id, \text{diag}(1,-1,-1), \text{diag}(-1,1,-1), \text{diag}(-1,-1,1)\}, \end{aligned}$$

which is isomorphic to the Klein four group. Indeed, direct computation shows the above matrices belong to $SO(3)_m$, and if one writes out the condition $gmg^{-1} = m$ in terms of the entries of g, one finds that g is itself diagonal since the λ_i are distinct. Since the rows of g are orthonormal, $g = \text{diag}(\pm 1, \pm 1, \pm 1)$, and since $\det(g) = 1$, we see that the above matrices exhaust $SO(3)_m$, as claimed. In general we have that gmg^{-1} is diagonal for some g, and one checks easily that $SO(3)_{gmg^{-1}} = g \cdot SO(3)_m \cdot g^{-1}$. Thus *all isotropy groups are conjugate.* The following result is a special case of Corollary (2.5) (p.309) of Bredon[7].

Theorem 4.1.1: *Let G be a compact Lie group acting smoothly on a manifold X. If all the isotropy groups are conjugate, then X/G is a manifold, and the projection $X \to X/G$ is a locally trivial fiber bundle with typical fiber G/G_x (for any $x \in X$) and structure group $N(G_x)/G_x$. Here $N(G_x)$ denotes the normalizer of G_x in G.*

Let $\pi : M \to M/SO(3)$ be the canonical projection and put

$$G_0 = \{\text{id}, \exp \pi\hat{\mathbf{E}}_1, \exp \pi\hat{\mathbf{E}}_2, \exp \pi\hat{\mathbf{E}}_3\}.$$

Then Theorem (4.1.1), together with the preceding discussion proves:

Corollary 4.1.1: *$M/SO(3)$ is a smooth manifold and $\pi : M \to M/SO(3)$ is a locally trivial fiber bundle with typical fiber $SO(3)/G_0$.*

In fact, as we'll see, $M/SO(3)$ is diffeomorphic to an open ball in $\mathbb{R}^3$, and hence π is globally trivial. Put

$$B = \{\lambda \in \mathbb{R}^3 \mid \lambda_1 > \lambda_2 > \lambda_3 > 0 \text{ and } \lambda_i + \lambda_j > \lambda_k \ (i,j,k \text{ cyclic})\}.$$

Define $\pi_B : M \to B$ by $m \mapsto \lambda = (\lambda_1, \lambda_2 \lambda_3)$ where λ_i $(1 \leq i \leq 3)$ are the distinct eigenvalues of m taken in descending order. Note that π_B is smooth since the eigenvalues are smooth functions of m. (Proof: By the implicit function theorem, the roots of a polynomial are smooth functions of the coefficients as long as the roots are distinct. Apply this to the characteristic polynomial of $m \in M$.) π_B is also clearly onto. Since the action preserves eigenvalues, π_B drops to a smooth function on the quotient. That is, we have a smooth function $\phi : M/SO(3) \to B$ satisfying $\phi \circ \pi = \pi_B$. We claim that ϕ is a diffeomorphism. To see this observe that π_B admits a smooth section $\sigma : B \to M,\ \lambda = (\lambda_1, \lambda_2, \lambda_3) \mapsto \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. Write $[m] = \pi(m)$ for the equivalence class of $m \in M$, so that $\phi([m]) = \pi_B(m)$. Then

$$(\pi \circ \sigma) \circ \phi([m]) = \pi(\sigma \circ \pi_B(m)) = \pi(m) = [m].$$

Also $\phi \circ (\pi \circ \sigma) = \pi_B \circ \sigma = \mathrm{identity}_B$, whence $\pi \circ \sigma : B \to M/SO(3)$ is a smooth inverse to ϕ. This shows that $M/SO(3) \cong B$, and in fact that π and π_B are isomorphic as fiber bundles.

Lemma 4.1.1: *B is contractible.*

Proof: Fix $a = (a_1, a_2, a_3) \in B$. Define $G : [0,1] \times B \to B$ by $G(t, \lambda) = t\lambda + (1-t)a$. Then $G(0, \cdot)$ is the constant map at a, $G(1, \cdot) = \mathrm{identity}_B$, and G is clearly continuous. We need only check $G(t, \lambda) \in B$ for all $\lambda \in B,\ t \in [0,1]$. Observe that

$$t\lambda_1 + (1-t)a_1 > t\lambda_2 + (1-t)a_2 > t\lambda_3 + (1-t)a_3,$$

since the analogous inequalities hold for $\lambda,\ a \in B$. Similarly

$$\begin{aligned}(t\lambda_i + (1-t)a_i) + (t\lambda_j + (1-t)a_j) &= t(\lambda_i + \lambda_j) + (1-t)(a_i + a_j) \\ &> (t\lambda_k + (1-t)a_k)\end{aligned}$$

(cycle on $i,\ j,\ k$.) Thus $G(t, \lambda) \in B$ as required. ■

Theorem 4.1.2: *$\pi : M \to M/SO(3)$ is a trivial fiber bundle isomorphic to $\pi_B : B \times F \to B$, with $F := SO(3)/G_0$. The fiber has homotopy group $\pi_1(F) = Q$, where Q is the quaternion group.*

Proof: The first statement follows from Corollary (4.1.1), Lemma (4.1.1), and the preceding discussion. For the second statement, let $\mathbb{H}$ denote the quaternions and identify $S^3 \subset \mathbb{R}^4$ with the quaternions of unit length. The quaternion group is $Q = \{\pm 1, \pm i, \pm j, \pm k\}$. If $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$, its conjugate is $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$, and its squared length is $|q|^2 = q\overline{q}$. The covering projection $\rho : S^3 \to SO(3)$ is defined as follows. The map $\mathbb{H} \to \mathbb{H},\ x \mapsto qx\overline{q},\ x \in \mathbb{H},\ q \in S^3$ is orthogonal since it preserves lengths. It also preserves the purely real quaternions, and so also their orthogonal complement, $\text{span}\{i, j, k\} \cong \mathbb{R}^3$. Let $\rho(q) \in O(3)$ be the restriction to $\mathbb{R}^3$:

$$\rho(q) \cdot v = qv\overline{q},$$

for $v = v_1 i + v_2 j + v_3 k \in \mathbb{R}^3$. The matrix of $\rho(q)$ is computed to be

$$\rho(q) = \begin{pmatrix} 2(q_0^2 + q_1^2) - 1 & 2(q_1 q_2 - q_0 q_3) & 2(q_1 q_3 + q_0 q_2) \\ 2(q_1 q_2 + q_0 q_3) & 2(q_0^2 + q_2^2) - 1 & 2(q_2 q_3 - q_1 q_0) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_2 q_3 + q_0 q_1) & 2(q_0^2 + q_3^2) - 1 \end{pmatrix}. \tag{4.2}$$

We see that $\det(\rho(q)) = 1$, whence $\rho(q) \in SO(3)$, and ρ is clearly smooth. Since $\dim(S^3) = \dim(SO(3)) = 3$, invariance of domain implies ρ is onto. For $q,\ p \in S^3$ and $v \in \mathbb{R}^3$ we have $\rho(qp) \cdot v = qpv\overline{qp} = q(pv\overline{p})\overline{q} = \rho(q) \circ \rho(p) \cdot v$, showing that ρ is a homomorphism. From (4.2) we obtain $\ker(\rho) = \{1, -1\} \subset Q$, and $\rho(Q) = G_0$. Thus the surjective homomorphism $\rho : S^3 \to SO(3)$ induces a diffeomorphism $\bar{\rho} : S^3/Q \to SO(3)/G_0$ of quotient manifolds, showing that $F \cong S^3/Q$. Now the natural projection $S^3 \to S^3/Q$ is a bundle with discrete fiber Q. The long exact sequence of homotopy groups arising from this fibering (see Gray[13]) is

$$\cdots \to \pi_{n+1}(S^3/Q) \to \pi_n(Q) \to \pi_n(S^3) \to \pi_n(S^3/Q) \to \pi_{n-1}(Q) \to \cdots,$$

which leads to

$$0 \to \pi_1(S^3/Q) \to \pi_0(Q) \to 0,$$

since $\pi_1(S^3) = \pi_0(S^3) = 0$. Thus $\pi_1(S^3/Q) \cong \pi_0(Q)$ and since Q is a discrete group, $\pi_0(Q) \cong Q$. Therefore $\pi_1(F) = Q$ and the proof is complete. ■

Under the identification $M \cong B \times F$, the orbits $F_\lambda := SO(3) \cdot \sigma(\lambda)$, $\lambda \in B$, correspond to the fibers $\{\lambda\} \times F \cong S^3/Q$ which are compact. Note that since B is contractible, M contracts onto a single fiber F_λ, and hence $\pi_1(M) = \pi_1(F_\lambda) = Q$. We can realize the isomorphism $Q \to \pi_1(F_\lambda)$ explicitly as follows. Define γ_1 to be the constant curve at $\sigma(\lambda) \in F_\lambda$ and

$$
\begin{aligned}
\gamma_{-1}(t) &= \exp t\widehat{\xi} \cdot \sigma(\lambda) && (0 \leq t \leq 2\pi), \\
\gamma_{\pm i}(t) &= \exp \pm t\widehat{\mathbf{E}_1} \cdot \sigma(\lambda) && (0 \leq t \leq \pi), \\
\gamma_{\pm j}(t) &= \exp \pm t\widehat{\mathbf{E}_2} \cdot \sigma(\lambda) && (0 \leq t \leq \pi), \\
\gamma_{\pm k}(t) &= \exp \pm t\widehat{\mathbf{E}_3} \cdot \sigma(\lambda) && (0 \leq t \leq \pi).
\end{aligned}
$$

In the definition of γ_{-1} we may take ξ to be any unit vector in $\mathbb{R}^3$. (If we choose some other unit vector $\eta \in \mathbb{R}^3$, then since $t \mapsto \exp t\hat{\xi}$ and $t \mapsto \exp t\hat{\eta}$ $(0 \leq t \leq 2\pi)$ are homotopic in $SO(3)$, the loops $t \mapsto \exp t\hat{\xi} \cdot \sigma(\lambda)$ and $t \mapsto \exp t\hat{\eta} \cdot \sigma(\lambda)$ $(0 \leq t \leq 2\pi)$ are homotopic in F_λ.) The isomorphism $Q \cong \pi_1(F_\lambda)$ is then given by $a \in Q \mapsto [\gamma_a]$, where $[\gamma_a]$ denotes the homotopy class of γ_a. We shall be interested in calculating the holonomy of the HB connection on loops belonging to these classes.

4.2 Hamiltonian One-Forms

Using $M \cong B \times F$ we calculate the Hamiltonian one-form $\mathbf{K} \cdot v$, for the HB connection on vectors $v \in T_m M$ tangent to F and B respectively. First we set some notation for what follows. Fix $m \in M$ and let $(\alpha_1, \alpha_2, \alpha_3)$ denote coordinates on $\mathbb{R}^3$ relative to $\{\mathcal{E}_1(m), \mathcal{E}_2(m), \mathcal{E}_3(m)\}$. Recall from §2.3 the definitions

$$\begin{aligned} U_1(m) &= \{\alpha \in \mathbb{R}^3 \mid \alpha_3^2 < \eta\alpha_1^2,\ \alpha_2^2 + \alpha_3^2 \neq 0\} \\ U_3(m) &= \{\alpha \in \mathbb{R}^3 \mid \alpha_3^2 > \eta\alpha_1^2,\ \alpha_1^2 + \alpha_2^2 \neq 0\} \\ U(m) &= U_1(m) \cup U_3(m). \end{aligned}$$

Here $m = \sigma(\lambda) = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3),\ \lambda \in B$, and

$$\eta = \frac{\lambda_3(\lambda_1 - \lambda_2)}{\lambda_1(\lambda_2 - \lambda_3)}.$$

Note that the definitions of $U_i(m),\ (i = 1, 3)$ don't depend on which diagonalizing basis we pick. (By §4.1, the frame $\{\mathcal{E}_j(m) \mid 1 \leq j \leq 3\}$ is defined only up to rotations by $G_0 = \{\mathrm{id}, \exp \pi\widehat{\mathcal{E}_1(m)}, \exp \pi\widehat{\mathcal{E}_2(m)}, \exp \pi\widehat{\mathcal{E}_3(m)}\}$.) Indeed, since the coordinates are squared in the above definitions, it is immaterial which direction on each principal axis is considered positive. Also from §2.3, we set

$$\begin{aligned} U_i^+(m) &= \{\alpha \in U_i(m) \mid \alpha_i > 0\} \\ U_i^-(m) &= \{\alpha \in U_i(m) \mid \alpha_i < 0\}, \end{aligned}$$

for $i = 1, 3$. If we transport these sets along the loop $m(t) = (\exp t\widehat{\mathbf{E}_2})m(\exp t\widehat{\mathbf{E}_2})^{-1},\ 0 \leq t \leq \pi$, we see that U_i^+ and U_i^- switch places, showing that these definitions are *not* intrinsic. Also set

$$\begin{aligned} P_i^\nu(m) &= \pi^{-1}(U_i^\nu(m)) \cap W \\ P_i(m) &= \pi^{-1}(U_i(m)) \cap W \\ P(m) &= \pi^{-1}(U(m)) \cap W, \end{aligned}$$

for $i = 1, 3;\ \nu = +, -$. Here $\pi : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3,\ (g, \alpha) \mapsto \alpha$ and $W \subset SO(3) \times \mathbb{R}^3$ is defined just as in §2.3. Finally put

$$E = \{(m, g, \alpha) \mid (g, \alpha) \in P(m)\},$$

and consider the HB connection on $\pi_1 : E \to M,\ (m, g, \alpha) \mapsto m$.

4.2.1 Case I: $v \in F_\lambda$

In this section we compute the Hamiltonian one-form for the HB connection on $\pi_1 : E \to M$ in the direction of vectors tangent to the orbits F_λ, of the $SO(3)$ action on M described in §4.1. To do this we appeal to Proposition (3.3.2) which gives $\mathbf{K} \cdot \xi_M(m)$ in terms of a momentum map for an $SO(3)$ action on $P(m)$. The correct action is not however, the usual lifted left action.

Consider the left $SO(3)$ action on $SO(3) \times \mathbb{R}^3$ given in body coordinates by

$$h \cdot (g, \alpha) = (gh^{-1}, h\alpha), \tag{4.3}$$

for $h \in SO(3)$. This is the cotangent lift of the left action on $SO(3)$ given by *right* multiplication by h^{-1}. We remark that this action commutes with the lifted left action §2.2(i). Also (4.3) admits an equivariant momentum map $\mathbf{L} : SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$, given by

$$\mathbf{L}(g, \alpha) = -\alpha.$$

To check this, we first compute the infinitesimal generator of (4.3). For $\xi \in \mathbb{R}^3$,

$$\begin{aligned} \xi_P(g,\alpha) &= \left.\frac{d}{dt}\right|_{t=0} \exp t\hat{\xi} \cdot (g,\alpha) \\ &= \left.\frac{d}{dt}\right|_{t=0} (g \exp(-t\hat{\xi}), (\exp t\hat{\xi})\alpha) \\ &= (-g\hat{\xi}, \xi \times \alpha) \\ &= -(g\hat{\xi}, \alpha \times \xi). \end{aligned}$$

Now $\mathbf{L}^\xi(g,\alpha) = -\langle \xi, \alpha \rangle$ so that $\nabla_\alpha \mathbf{L}^\xi = -\xi$, and hence by §2.2(iii)

$$\begin{aligned} X_{\mathbf{L}^\xi}(g,\alpha) &= (g \cdot \widehat{\nabla_\alpha \mathbf{L}^\xi}, \alpha \times \nabla_\alpha \mathbf{L}^\xi) \\ &= -(g \cdot \hat{\xi}, \alpha \times \xi) \\ &= \xi_P(g,\alpha). \end{aligned}$$

We have used the fact that $\mathbf{L}^\xi$ is independent of $g \in SO(3)$. One checks easily that $\mathbf{L}$ is equivariant with respect to (4.3) and the usual action of $SO(3)$ on $\mathbb{R}^3$.

To utilize Proposition (3.3.2) we must first check that the actions $\mathbf{I} = (I_1, I_2, I_3)$ for the rigid body are invariant under the corresponding diagonal action on $E \subset M \times (SO(3) \times \mathbb{R}^3)$:

$$h \cdot (m, g, \alpha) = (hmh^{-1}, gh^{-1}, h\alpha), \tag{4.4}$$

for $h \in SO(3),\ m \in M,\ (g, \alpha) \in P(m)$. Recall that $I_1 = \langle \mathbf{J}, \mathbf{e}_1 \rangle$, and $I_2 = \|\mathbf{J}\|$ do not depend on the inertia tensor m. Here $\mathbf{J}$ denotes the momentum map §2.2(v) for the usual lifted left action §2.2(i). Thus $I_1(gh^{-1}, h\alpha) = \langle ghh^{-1}\alpha, \mathbf{e}_1 \rangle = \langle g\alpha, \mathbf{e}_1 \rangle = I_1(g, \alpha)$, and $I_1(gh^{-1}, h\alpha) = \|ghh^{-1}\alpha\| = \|\alpha\| = I_2(g, \alpha)$, showing that I_1 and I_2 are invariant.

To check the invariance of I_3 we first note that the energy $H(m, \alpha) = \frac{1}{2}\langle \alpha, m^{-1}\alpha \rangle$ is invariant:

$$\begin{aligned} H(hmh^{-1}, h\alpha) &= \frac{1}{2}\langle h\alpha, (hmh^{-1})^{-1}h\alpha \rangle \\ &= \frac{1}{2}\langle h\alpha, hm^{-1}\alpha \rangle \\ &= H(m, \alpha). \end{aligned}$$

Thus the original integrals in involution $f = (I_1, I_2, H_m)$, are invariant under the diagonal action (4.4). Recall from §2.3 that

$$I_3(m, g, \alpha) = \int_{\gamma_3(f(m,g,\alpha),m)} \beta, \tag{4.5}$$

where β is the canonical 1-form on phase space and $\gamma_i(f(m, g, \alpha), m)$, $(1 \le i \le 3)$ denote closed curves which generate the $\mathbb{Z}$ homology of the Liouville torus passing through $(g, \alpha) \in P(m)$. Each γ_i, $(1 \le i \le 3)$ depends smoothly on the value $f(m, g, \alpha) \in \mathbb{R}^3$, as well as the parameter $m \in M$. Since f is invariant we have

$$I_3(h \cdot (m, g, \alpha)) = \int_{\gamma_3(f(m,g,\alpha),hmh^{-1})} \beta. \tag{4.6}$$

We showed in §2.3 that the value of the integral in (4.5) is $-r^{-1}$ times the oriented area of the spherical cap enclosed by the curve $S_r^2 \cap H_m^{-1}(c)$. Here $r, c > 0$ are constants, and S_r^2 is the sphere of radius r. Note that replacing m by hmh^{-1}, $h \in SO(3)$ has the effect of rigidly rotating the level sets of H about the origin, i.e.

$$H^{-1}_{hmh^{-1}}(c) = h \cdot H_m^{-1}(c).$$

Observe that this rotation does not alter the above area, and therefore the righthand sides of (4.5) and (4.6) are identical, showing that $I_3(hmh^{-1}, gh^{-1}, h\alpha) = I_3(m, g, \alpha)$ as required.

The invariance of $\mathbf{I} = (I_1, I_2, I_3)$ under the diagonal action together with Proposition (3.3.2) yields

Lemma 4.2.1: *The Hamiltonian* 1*-form acting on vectors tangent to the* $SO(3)$ *orbits in* M *is given by*

$$\mathbf{K} \cdot \xi_M(m) = \mathbf{L}^\xi - \langle \mathbf{L}^\xi \rangle,$$

for $m \in M,\ \xi \in \mathbb{R}^3$.

We remark that $\mathbf{K} \cdot \xi_M(m)$ is a left invariant function on $P(m)$ since $\mathbf{L}^\xi$ is. The next two lemmas will facilitate the computation of the above average. We see that under certain conditions, the averaging operation (over the $\mathbb{T}^3$ action) can be replaced by the time average over the rigid body dynamics.

Lemma 4.2.2: *Fix* $m \in M$ *and suppose* $F : P(m) \to \mathbb{R}$ *is continuous and invariant under the Hamiltonian flow of* I_1*. Then*

$$\langle F \rangle = \langle F \rangle_{H_m}.$$

Here $\langle F \rangle_{H_m}$ *denotes the time average of* F *along the flow of* H_m*:*

$$\langle F \rangle_{H_m}(g, \alpha) := \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau F(\Phi_t^{H_m}(g, \alpha))dt.$$

Remark. Any function invariant under the lifted left action $h \cdot (g, \alpha) = (hg, \alpha)$, is invariant under the flow $\Phi_t^{I_1}(g, \alpha) = ((\exp t\hat{\mathbf{e}}_1)g, \alpha)$, so that $\mathbf{L}^\xi$ satisfies the above hypothesis.

Proof: Let $\langle \cdot \rangle_{\mathbb{T}^2}$ denote averaging over the $\mathbb{T}^2$ action generated by the flows of I_2 and I_3. By Fubini's Theorem and the invariance of F we have immediately that $\langle F \rangle = \langle F \rangle_{\mathbb{T}^2}$, so we must show that $\langle F \rangle_{\mathbb{T}^2} = \langle F \rangle_{H_m}$.

Recall from §2.4 that the frequencies of rigid body motion are given by $\omega_1 = 0,\ \omega_2 = \Delta\Theta/T$, and $\omega_3 = 2\pi/T$. (Here T denotes the period of the reduced trajectory and $\Delta\Theta$ is the angle of rotation of the rigid body about the space angular momentum vector over time T.) If the initial point (g, α) is such that $\omega_2/\omega_3 = \Delta\Theta/2\pi$ is irrational then the flow of H_m is a dense winding on the 2-torus parametrized by the angles $\theta_2,\ \theta_3$, conjugate to $I_2,\ I_3$. Since in this case we know that the time average equals the space average (See Arnold[2]) we have $\langle F\rangle_{\mathbb{T}^2}(g, \alpha) = \langle F\rangle_{H_m}(g, \alpha)$. Its clear that $\langle F\rangle_{\mathbb{T}^2}$ and $\langle F\rangle_{H_m}$ are continuous functions so to show they are equal its sufficient to show they coincide on a dense subset of $P(m)$. We have reduced the problem to showing that for a dense set of initial conditions, $\Delta\Theta$ is an irrational multiple of 2π.

From the formulas for $\Delta\Theta$ in Appendix B, we see that $\Delta\Theta$ is a real analytic function on $P(m)$, being given by a combination of algebraic operations and complete elliptic integrals. Thus the critical points of $\Delta\Theta$ are isolated and therefore any neighborhood of $(g, \alpha) \in P(m)$ contains a regular point of $\Delta\Theta$, and hence also a point at which $\Delta\Theta/2\pi$ is irrational. This completes the proof. ■

The next result shows that the operator $\mathbf{K}$ is invariant under a certain $SO(3)$ action.

Lemma 4.2.3: *Let* $\xi \in \mathbb{R}^3,\ m \in M,\ \alpha \in U(m)$, *and* $h \in SO(3)$. *Then*

$$\left[\mathbf{K} \cdot (h\xi)_M(hmh^{-1})\right](h\alpha) = [\mathbf{K} \cdot \xi_M(m)]\,(\alpha).$$

Proof: Since $\mathbf{L}^{h\xi}(h\alpha) = -\langle h\xi, h\alpha\rangle = -\langle h, \alpha\rangle = \mathbf{L}^{\xi}(\alpha)$, we need only show

$$\langle \mathbf{L}^{h\xi}\rangle(hmh^{-1}, h\alpha) = \langle \mathbf{L}^{\xi}\rangle(m, \alpha).$$

Note that even though $\mathbf{L}^{\xi}$ does not depend on m, its average does since the $\mathbb{T}^3$ action, over which we average, does. By Lemma (4.2.2) its sufficient to show

$$\langle \mathbf{L}^{h\xi}\rangle_{H_{hmh^{-1}}}(h\alpha) = \langle \mathbf{L}^{\xi}\rangle_{H_m}(\alpha).$$

With a slight abuse of notation let $\Phi_t^{H_m}(\alpha)$ denote the flow of the Euler equations (2.5) in $\mathbb{R}^3$ with energy H_m and initial point $\alpha \in U$. We assert that

$$\Phi_t^{H_{hmh^{-1}}}(h\alpha) = h \cdot \Phi_t^{H_m}(\alpha).$$

This says nothing more than the fact that by rotating a given trajectory by $h \in SO(3)$, we obtain a trajectory for the system with rotated inertia tensor and rotated initial point, which is obvious. Thus

$$\begin{aligned}
\mathbf{L}^{h\xi}(\Phi_t^{H_{hmh^{-1}}}(h\alpha)) &= -\langle h\xi, \Phi_t^{H_{hmh^{-1}}}(h\alpha)\rangle \\
&= -\langle h\xi, h \cdot \Phi_t^{H_m}(\alpha)\rangle \\
&= -\langle \xi, \Phi_t^{H_m}(\alpha)\rangle \\
&= \mathbf{L}^{\xi}(\Phi_t^{H_m}(\alpha)),
\end{aligned}$$

and therefore

$$\begin{aligned}
\langle \mathbf{L}^{h\xi}\rangle_{H_{hmh^{-1}}}(h\alpha) &= \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \mathbf{L}^{h\xi}(\Phi_t^{H_{hmh^{-1}}}(h\alpha))dt \\
&= \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \mathbf{L}^{\xi}(\Phi_t^{H_m}(\alpha))dt \\
&= \langle \mathbf{L}^{\xi}\rangle_{H_m}(\alpha),
\end{aligned}$$

as required. ■

As a consequence of Lemma (4.2.3) we need only calculate $\mathbf{K} \cdot \xi_M(m)$ for m which are diagonal with respect to the fixed basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$. Indeed taking $m = \sigma(\lambda)$, $\lambda \in B$, then

$$\left[\mathbf{K} \cdot \xi_M(h\sigma(\lambda)h^{-1})\right](\alpha) = \left[\mathbf{K} \cdot (h^{-1}\xi)_M(\sigma(\lambda))\right](h^{-1}\alpha) \tag{4.7}$$

for any $h \in SO(3)$. (Replace ξ by $h^{-1}\xi$ and α by $h^{-1}\alpha$ in the statement of Lemma (4.2.3).) Note that $\alpha \in U(h\sigma(\lambda)h^{-1}) = h \cdot U(\sigma(\lambda))$ implies $h^{-1}\alpha \in U(\sigma(\lambda))$ so the right hand side is defined whenever the left is.

Proposition 4.2.1: *Let* $m = \sigma(\lambda)$, $\lambda \in B$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in U(m)$, *and* $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ *with all coordinates relative to* $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$. *Then*

$$[\mathbf{K} \cdot \xi_M(m)](\alpha) = \begin{cases} G_1(\alpha)\xi_1 - \langle \xi, \alpha \rangle & \text{for } \alpha \in U_1(m) \\ G_3(\alpha)\xi_3 - \langle \xi, \alpha \rangle & \text{for } \alpha \in U_3(m), \end{cases}$$

where

$$\begin{aligned} G_1(\alpha) &= \frac{\pi\sqrt{\alpha_1^2 + \mu\alpha_2^2}}{2K(k_1(\alpha))} \\ G_3(\alpha) &= \frac{\pi\sqrt{\nu\alpha_2^2 + \alpha_3^2}}{2K(k_3(\alpha))} \\ \mu &= \frac{\lambda_1(\lambda_2 - \lambda_3)}{\lambda_2(\lambda_1 - \lambda_3)} \\ \nu &= \frac{\lambda_3(\lambda_1 - \lambda_2)}{\lambda_2(\lambda_1 - \lambda_3)} \\ k_1(\alpha) &= k_3(\alpha)^{-1} = \sqrt{\frac{c_2\alpha_2^2 + c_3\alpha_3^2}{c_1\alpha_1^2 + c_2\alpha_2^2}} \\ c_1 &= \lambda_2\lambda_3(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \\ c_2 &= \lambda_1\lambda_3(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3) \\ c_3 &= \lambda_1\lambda_2(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3). \end{aligned}$$

Here $K(k)$ *denotes the complete elliptic integral of the first kind (see Appendix B.)*

Proof: By Lemmas (4.2.1) and (4.2.2) we have

$$\begin{aligned} [\mathbf{K} \cdot \xi_M(m)](\alpha) &= \mathbf{L}^\xi(\alpha) - \langle \mathbf{L}^\xi \rangle(m, \alpha) \\ &= -\langle \xi, \alpha \rangle - \langle \mathbf{L}^\xi \rangle_{H_m}(\alpha), \end{aligned}$$

so we must show that for $\alpha \in U_i(m)$, $i = 1, 3$:

$$\langle \mathbf{L}^\xi \rangle_{H_m}(\alpha) = -\xi_i G_i(\alpha). \tag{4.8}$$

As in the previous proof let $\Phi_t^{H_m}(\alpha)$ denote the solution to the Euler equations (2.5) with initial point α. For $\alpha \in U(m)$, $\Phi_t^{H_m}(\alpha)$ is periodic with period T, whence

$$\begin{aligned} \langle \mathbf{L}^\xi \rangle_{H_m}(\alpha) &= \frac{1}{T}\int_0^T \mathbf{L}^\xi(\Phi_t^{H_m}(\alpha))\, dt \\ &= -\frac{1}{T}\int_0^T \left\langle \xi, \Phi_t^{H_m}(\alpha) \right\rangle\, dt \\ &= -\left\langle \xi\ ,\ \frac{1}{T}\int_0^T \Phi_t^{H_m}(\alpha)\, dt \right\rangle. \end{aligned}$$

Comparing this to (4.8), we must show

$$\frac{1}{T}\int_0^T \Phi_t^{H_m}(\alpha)\,dt = G_i(\alpha)\mathbf{E}_i \tag{4.9}$$

for $\alpha \in U_i(m)$. Now

$$\Phi_t^{H_m}(\alpha) = \alpha_1(t)\mathbf{E}_1 + \alpha_2(t)\mathbf{E}_2 + \alpha_3(t)\mathbf{E}_3$$

is given by §2.2 cases (4e), (4d) for the initial point $\alpha(0) = \alpha \in U_1(m)$, $U_3(m)$ respectively.

Let $\alpha \in U_3(m)$. Examining §2.2(4d) we have

$$\int_0^T \alpha_1(t)\,dt = \int_0^T \alpha_2(t)\,dt = 0,$$

since $\operatorname{cn}(\cdot,k)$ and $\operatorname{sn}(\cdot,k)$ have average zero over one period. Thus in this case

$$\frac{1}{T}\int_0^T \Phi_t^{H_m}(\alpha)\,dt = \left(\frac{1}{T}\int_0^T \alpha_3(t)\,dt\right)\mathbf{E}_3.$$

Let R, s, k be as in §2.2(4d), then

$$\begin{aligned}
\frac{1}{T}\int_0^T \alpha_3(t)\,dt &= \frac{R}{T}\int_0^T \operatorname{dn}(s(t-t_0),k)\,dt \\
&= \frac{R}{4K(k)}\int_0^{4K(k)} \operatorname{dn}(u,k)\,du \\
&= \frac{R}{4K(k)}\operatorname{am}(u,k)\Big|_0^{4K(k)} \\
&= \frac{R}{4K(k)}(2\pi - 0) = \frac{\pi R}{2K(k)}.
\end{aligned}$$

In the second line above we have put $u = s(t-t_0)$ and used $T = 4s^{-1}K(k)$. Here $\operatorname{am}(\cdot,k)$ denotes the Jacobi amplitude function (see Byrd and Friedman[8].) By §2.2(4d) we have

$$\begin{aligned}
R^2 &= \frac{\lambda_3(2\lambda_1 h - r^2)}{\lambda_1 - \lambda_3} \\
&= \frac{\lambda_3}{\lambda_1 - \lambda_3}\left(\frac{\lambda_1 - \lambda_2}{\lambda_2}\alpha_2^2 + \frac{\lambda_1 - \lambda_3}{\lambda_3}\alpha_3^2\right) \\
&= \nu\alpha_2^2 + \alpha_3^2,
\end{aligned}$$

and

$$k^2 = \frac{(\lambda_1 - \lambda_2)(r^2 - 2\lambda_3 h)}{(\lambda_2 - \lambda_3)(2\lambda_1 h - r^2)}$$

$$
\begin{aligned}
&= \frac{(\lambda_1-\lambda_2)\left(\frac{\lambda_1-\lambda_3}{\lambda_1}\alpha_1^2+\frac{\lambda_2-\lambda_3}{\lambda_2}\alpha_2^2\right)}{(\lambda_2-\lambda_3)\left(\frac{\lambda_1-\lambda_2}{\lambda_2}\alpha_2^2+\frac{\lambda_1-\lambda_3}{\lambda_3}\alpha_3^2\right)} \\
&= \frac{c_1\alpha_1^2+c_2\alpha_2^2}{c_2\alpha_2^2+c_3\alpha_3^2} \\
&= k_3(\alpha)^2.
\end{aligned}
$$

We have used $r^2 = \alpha_1^2+\alpha_2^2+\alpha_3^2$ and $2h = \alpha_1^2/\lambda_1+\alpha_2^2/\lambda_2+\alpha_3^2/\lambda_3$. Therefore

$$
\frac{1}{T}\int_0^T \alpha_3(t)\,dt = \frac{\pi\sqrt{\nu\alpha_2^2+\alpha_3^2}}{2K(k_3(\alpha))} = G_3(\alpha),
$$

proving (4.9) and hence (4.8) for $\alpha \in U_3(m)$.

Similarly if $\alpha \in U_1(m)$ then §2.2(4e) yields

$$
\int_0^T \alpha_2(t)\,dt = \int_0^T \alpha_3(t)\,dt = 0,
$$

so that

$$
\frac{1}{T}\int_0^T \Phi_t^{H_m}(\alpha)\,dt = \left(\frac{1}{T}\int_0^T \alpha_1(t)\,dt\right)\mathbf{E}_1,
$$

and

$$
\frac{1}{T}\int_0^T \alpha_1(t)\,dt = \frac{\pi P}{4K(k)},
$$

where

$$
\begin{aligned}
P^2 &= \alpha_1^2+\mu\alpha_2^2 \\
k^2 &= \frac{c_2\alpha_2^2+c_3\alpha_3^2}{c_1\alpha_1^2+c_2\alpha_2^2} = k_1(\alpha)^2.
\end{aligned}
$$

Thus

$$
\frac{1}{T}\int_0^T \Phi_t^{H_m}(\alpha)\,dt = G_1(\alpha)\mathbf{E}_1,
$$

and the proof is complete. ∎

Using Proposition 4.2.1 and Equation (4.7) we have

$$
\left[\mathbf{K}\cdot\xi_M(h\sigma(\lambda)h^{-1})\right](\alpha) = G_i(h^{-1}\alpha)\langle\xi, h\mathbf{E}_i\rangle - \langle\xi,\alpha\rangle \tag{4.10}
$$

for any $h \in SO(3)$ and $\alpha \in U_i(h\sigma(\lambda)h^{-1}),\ i = 1, 3$. Equation (4.10) now gives the expression for $\mathbf{K} \cdot \xi_M(m)$ for any $m \in M$.

4.2.2 Case II: $v \in TB$

Our goal in this section is to compute $\mathbf{K} \cdot v$ for vectors $v \in TM$ tangent to B under the identification $M \cong B \times F$ in §4.1. These are vectors tangent to curves in M along which the principle axes of inertia remain fixed and the moments of inertia are allowed to vary. Throughout we take m to be diagonal with respect to the fixed frame $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, i.e $m = \sigma(\lambda),\ \lambda \in B$.

By Proposition 3.3.1 we must solve the system

$$d_M I_j \cdot v + \{I_j, \widetilde{\mathbf{K}} \cdot v\} = 0 \qquad (1 \le j \le 3) \tag{4.11}$$

for the unknown function $\widetilde{\mathbf{K}} \cdot v$, then take

$$\mathbf{K} \cdot v = \widetilde{\mathbf{K}} \cdot v - \langle \widetilde{\mathbf{K}} \cdot v \rangle.$$

Recall that $I_1 = \langle \mathbf{J}, \mathbf{e}_1 \rangle$, and $I_2 = \|\mathbf{J}\|$ do not depend on the parameter m so that $d_M I_j \cdot v = 0$ for $j = 1, 2$. Thus (4.11) becomes

$$\left\{ \begin{array}{rcl} \{I_1, \widetilde{\mathbf{K}} \cdot v\} & = & 0 \\ \{I_2, \widetilde{\mathbf{K}} \cdot v\} & = & 0 \\ d_m I_3 \cdot v + \{I_3, \widetilde{\mathbf{K}} \cdot v\} & = & 0 \end{array} \right.$$

The first two equations say that $\widetilde{\mathbf{K}} \cdot v$ is constant along the flows of I_1 and I_2. From §2.2, $\Phi_t^{I_1}(g, \alpha) = ((\exp t\hat{\mathbf{e}}_1)g, \alpha)$ and $\Phi_t^{I_2}(g, \alpha) = \left(g \exp t\frac{\hat{\alpha}}{\|\alpha\|}, \alpha\right)$, so if we assume that $\left[\widetilde{\mathbf{K}} \cdot v\right](g, \alpha)$ is independent of $g \in SO(3)$, then the first two equations are automatically satisfied. Therefore it is sufficient to find a smooth function $\widetilde{\mathbf{K}} \cdot v$, depending only on $\alpha \in U(m)$, which satisfies the single equation

$$\{\widetilde{\mathbf{K}} \cdot v, I_3\} = d_M I_3 \cdot v \tag{4.12}$$

Now recall that $I_3 = A/2\pi\|\mathbf{J}\|$ where A is the spherical area enclosed by one of the periodic trajectories of (2.5). We find in Appendix A that A is a function of $r = \|\alpha\| = \|\mathbf{J}\|$ and $h = H_m(\alpha)$. If $\alpha \in U_3(m)$ we have by (A.3)

$$A = -|A| = -\int\int_{\bar{R}} \frac{abr\,dxdy}{\sqrt{r^2 - a^2x^2 - b^2y^2}},$$

where $\bar{R} \subset \mathbb{R}^2$ is the unit disc, and

$$a^2 = \frac{\lambda_1(r^2 - 2\lambda_3 h)}{\lambda_1 - \lambda_3}, \qquad b^2 = \frac{\lambda_2(r^2 - 2\lambda_3 h)}{\lambda_2 - \lambda_3}.$$

Using $r^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$ and $2h = \alpha_1^2/\lambda_1 + \alpha_2^2/\lambda_2 + \alpha_3^2/\lambda_3$ we have

$$r^2 - 2\lambda_3 h = \frac{\lambda_1 - \lambda_3}{\lambda_1}\alpha_1^2 + \frac{\lambda_2 - \lambda_3}{\lambda_2}\alpha_2^2,$$

so that

$$a^2 = \alpha_1^2 + \mu\alpha_2^2, \qquad b^2 = \mu^{-1}\alpha_1^2 + \alpha_2^2,$$

where, as in Proposition 4.2.1, we have set

$$\mu = \frac{\lambda_1(\lambda_2 - \lambda_3)}{\lambda_2(\lambda_1 - \lambda_3)}.$$

Thus for $\alpha \in U_3(m)$

$$I_3(\alpha) = -\frac{1}{2\pi}\int\int_{\overline{R}} \frac{ab\,dxdy}{\sqrt{r^2 - a^2x^2 - b^2y^2}}$$

where a, b, r are the above functions of α and μ. Observe that I_3 depends on the parameter m only through the principle moments of inertia, and on these only through the quantity μ.

Similar calculations show that for $\alpha \in U_1(m)$, $I_3(\alpha)$ depends on the the inertia tensor only through the quantity

$$\nu = \frac{\lambda_3(\lambda_1 - \lambda_2)}{\lambda_2(\lambda_1 - \lambda_3)}.$$

A short computation shows that $\nabla\mu \neq 0 \neq \nabla\nu$ and that $\nabla\mu = -\nabla\nu$ for all $\lambda \in B$. Since their gradients are parallel at each point, the level sets of the functions $\mu(\lambda)$, $\nu(\lambda)$ coincide. Recall that

$$\eta = \frac{\lambda_3(\lambda_1 - \lambda_2)}{\lambda_1(\lambda_2 - \lambda_3)}$$

gives the slope of the separatrix planes which divide $\mathbb{R}^3$ into the regions $U_1(m)$, $U_3(m)$, and that $\eta = \nu/\mu$, so that $\eta(\lambda)$ has the same level sets as μ and ν. Thus if we let $\lambda \in B$ vary along such a level set, $U_1(m)$ and $U_3(m)$ are unchanged, and for fixed $\alpha \in U(m)$, $I_3(\alpha)$ remains constant.

One also checks that $\nabla\eta \neq 0$ on B. Since B is contractible (Lemma 4.1.1), it follows that η can serve as a coordinate function on B. Put $\eta_1 = \eta$, then there exist smooth functions η_2, η_3 on B such that the gradients $\nabla\eta_j$ are linearly independent at each $\lambda \in B$. The coordinate vector fields $\partial/\partial\eta_j$ are convenient directions along which to compute $\mathbf{K}\cdot v$.

Proposition 4.2.2: *For* $j = 2, 3$

$$\mathbf{K} \cdot \frac{\partial}{\partial \eta_j} = 0.$$

Proof: From the preceding discussion we have

$$d_M I_3 \cdot \frac{\partial}{\partial \eta_2} = d_M I_3 \cdot \frac{\partial}{\partial \eta_3} = 0,$$

so that for $v = \partial/\partial\eta_j$, $j = 2, 3$, (4.12) becomes $\{\widetilde{\mathbf{K}} \cdot v, I_3\} = 0$, which has the simple solution $\widetilde{\mathbf{K}} \cdot v = 0$. Therefore

$$\mathbf{K} \cdot \frac{\partial}{\partial \eta_j} = \widetilde{\mathbf{K}} \cdot \frac{\partial}{\partial \eta_j} - \left\langle \widetilde{\mathbf{K}} \cdot \frac{\partial}{\partial \eta_j} \right\rangle = 0$$

for $j = 2, 3$, as required. ■

For $v = \partial/\partial\eta_1$, (4.12) becomes

$$\{\widetilde{\mathbf{K}} \cdot v, I_3\} = \frac{\partial I_3}{\partial \eta_1}. \tag{4.13}$$

Using §2.2(iii), we have

$$\begin{aligned} \{\widetilde{\mathbf{K}} \cdot v, I_3\}(\alpha) &= -\left\langle \alpha \,,\, \nabla_\alpha \left(\widetilde{\mathbf{K}} \cdot v\right) \times \nabla_\alpha I_3 \right\rangle \\ &= \left\langle \alpha \times \nabla_\alpha I_3 \,,\, \nabla_\alpha \left(\widetilde{\mathbf{K}} \cdot v\right) \right\rangle \\ &= \mathcal{D}\left(\widetilde{\mathbf{K}} \cdot v\right)(\alpha), \end{aligned}$$

where $\mathcal{D}$ is the linear differential operator

$$\begin{aligned}\mathcal{D} &= \langle \alpha \times \nabla_\alpha I_3, \nabla_\alpha \rangle \\ &= \left(\alpha_2 \frac{\partial I_3}{\partial \alpha_3} - \alpha_3 \frac{\partial I_3}{\partial \alpha_2}\right) \frac{\partial}{\partial \alpha_1} + \left(\alpha_3 \frac{\partial I_3}{\partial \alpha_1} - \alpha_1 \frac{\partial I_3}{\partial \alpha_3}\right) \frac{\partial}{\partial \alpha_2} + \left(\alpha_1 \frac{\partial I_3}{\partial \alpha_2} - \alpha_2 \frac{\partial I_3}{\partial \alpha_1}\right) \frac{\partial}{\partial \alpha_3}.\end{aligned}$$

Thus (4.13) is

$$\mathcal{D}\left(\widetilde{\mathbf{K}} \cdot v\right) = \partial I_3 / \partial \eta_1,$$

a linear PDE whose coefficients and right hand side are elliptic integrals. Fortunately it is not necessary to solve this equation in order to determine the holonomy of loops in B, as we shall see in §4.3.

4.3 Curvature

We now turn our attention to calculation of the curvature form on M, described in §3.4. Recall that for $m \in M,\ v_1, v_2 \in T_m M$,

$$\Omega(v_1, v_2) = \langle \{\mathbf{K} \cdot v_1, \mathbf{K} \cdot v_2\} \rangle \tag{4.14}$$

gives the smooth function on E_m whose Hamiltonian vector field is the curvature applied to $\mathrm{Hor} \cdot v_1, \mathrm{Hor} \cdot v_2$. Note Ω is a two-form on M with values in $C^\infty(E)/C^\infty(M)$, and is also called the curvature for obvious reasons (see §3.4). Throughout we take $m = \sigma(\lambda),\ \lambda \in B$, diagonal with respect to $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, and identify $M \cong B \times F_\lambda$ as in §4.1. We will calculate Ω on a conveniently chosen basis for $T_m M$.

Note that since the $SO(3)$ action on M is locally free, the map $\mathbb{R}^3 \to T_m F_\lambda$ given by $\xi \mapsto \xi_M(m)$ is an isomorphism, so that the infinitesimal generators $\{(\mathbf{E}_j)_M(m) \mid 1 \leq j \leq 3\}$ form a basis of $T_m F_\lambda$. We will use the coordinates (η_1, η_2, η_3), described in §4.2.2, on B, and the coordinate vector fields $\{\partial/\partial\eta_1, \partial/\partial\eta_1, \partial/\partial\eta_1\}$ as a basis on $T_m B$.

Define

$$\begin{aligned} \mathcal{K}^i &= \mathbf{K} \cdot \frac{\partial}{\partial \eta_i} && 1 \le i \le 3 \\ \mathcal{K}_j &= \mathbf{K} \cdot (\mathbf{E}_j)_M(m) && 1 \le j \le 3, \end{aligned}$$

and

$$\begin{aligned} \Omega^{ij} &= \Omega\left(\frac{\partial}{\partial \eta_i}, \frac{\partial}{\partial \eta_j}\right) \\ &= \langle\{\mathcal{K}^i, \mathcal{K}^j\}\rangle && 1 \le i < j \le 3 \\ \Omega_{ij} &= \Omega\left((\mathbf{E}_i)_M(m), (\mathbf{E}_j)_M(m)\right) \\ &= \langle\{\mathcal{K}_i, \mathcal{K}_j\}\rangle && 1 \le i < j \le 3 \\ \Omega^i_j &= \Omega\left(\frac{\partial}{\partial \eta_i}, (\mathbf{E}_j)_M(m)\right) \\ &= \langle\{\mathcal{K}^i, \mathcal{K}_j\}\rangle && 1 \le i, j \le 3. \end{aligned}$$

With these conventions we have

Theorem 4.3.1: *For $\alpha \in U(m)$*

$$\Omega^{12}(\alpha) = \Omega^{13}(\alpha) = \Omega^{23}(\alpha) = 0.$$

Proof: Proposition 4.2.2 yields $\mathcal{K}^2 = \mathcal{K}^3 = 0$ while $\mathcal{K}^1$ is unknown. Thus $\{\mathcal{K}^i, \mathcal{K}^j\} = 0,\ 1 \le i < j \le 3$ and the result follows. ■

Since B is contractible, an immediate consequence is

Corollary 4.3.1: *The holonomy of the HB connection about any loop lying in B is trivial.*

Thus the Hannay angles corresponding to any closed curve in M along which the principal moments of inertia vary while the principal axes remain fixed, are zero. The next result gives the curvature on F_λ.

Theorem 4.3.2: *Let $\alpha \in U(m)$, then*

$$\Omega_{23}(\alpha) = \begin{cases} -G_1(\alpha) & \alpha \in U_1(m) \\ 0 & \alpha \in U_3(m) \end{cases}$$

$$\Omega_{13}(\alpha) = 0$$

$$\Omega_{12}(\alpha) = \begin{cases} 0 & \alpha \in U_1(m) \\ -G_3(\alpha) & \alpha \in U_3(m), \end{cases}$$

where G_1 and G_3 are defined in Proposition 4.2.1.

To prove Theorem 4.3.2 we first establish two lemmas. An immediate consequence of Proposition 4.2.1 is

Lemma 4.3.1: *For $\alpha \in U(m)$*

$$\mathcal{K}_1(\alpha) = \begin{cases} G_1(\alpha) - \alpha_1 & \alpha \in U_1(m) \\ -\alpha_1 & \alpha \in U_3(m) \end{cases}$$

$$\mathcal{K}_2(\alpha) = -\alpha_2$$

$$\mathcal{K}_3(\alpha) = \begin{cases} -\alpha_3 & \alpha \in U_1(m) \\ G_3(\alpha) - \alpha_3 & \alpha \in U_3(m), \end{cases}$$

where G_1 and G_3 are defined in Proposition 4.2.1.

Lemma 4.3.2: *Let $\alpha \in U(m) = U_1(m) \cup U_3(m)$. If $\alpha \in U_1(m)$ then*

$$\begin{cases} \{\mathcal{K}_2, \mathcal{K}_3\}(\alpha) = -\alpha_1 \\ \{\mathcal{K}_1, \mathcal{K}_3\}(\alpha) = \alpha_2 + \alpha_1\alpha_2 f_1(\alpha_1^2, \alpha_2^2, \alpha_3^2) \\ \{\mathcal{K}_1, \mathcal{K}_2\}(\alpha) = -\alpha_3 + \alpha_1\alpha_3 f_2(\alpha_1^2, \alpha_2^2, \alpha_3^2), \end{cases} \tag{4.15}$$

and if $\alpha \in U_3(m)$ then

$$\begin{cases} \{\mathcal{K}_2, \mathcal{K}_3\}(\alpha) = -\alpha_1 + \alpha_1\alpha_3 f_3(\alpha_1^2, \alpha_2^2, \alpha_3^2) \\ \{\mathcal{K}_1, \mathcal{K}_3\}(\alpha) = \alpha_2 + \alpha_2\alpha_3 f_4(\alpha_1^2, \alpha_2^2, \alpha_3^2) \\ \{\mathcal{K}_1, \mathcal{K}_2\}(\alpha) = -\alpha_3, \end{cases} \tag{4.16}$$

where f_1, f_2, f_3, f_4 are certain smooth functions of $(\alpha_1^2, \alpha_2^2, \alpha_3^2)$.

Proof: The Poisson bracket for left invariant functions is given by §2.2 as

$$\{\mathcal{K}_i, \mathcal{K}_j\}(\alpha) = -\langle \alpha, \nabla\mathcal{K}_i \times \nabla\mathcal{K}_j \rangle.$$

Let $\alpha \in U_1(m)$, then by Lemma 4.3.1

$$\begin{aligned} \nabla\mathcal{K}_1 &= \nabla G_1 - \mathbf{E}_1 \\ \nabla\mathcal{K}_2 &= -\mathbf{E}_2 \\ \nabla\mathcal{K}_3 &= -\mathbf{E}_3. \end{aligned}$$

Now observe from the statement of Proposition 4.2.1 that $G_1(\alpha)$ is actually a function of $(\alpha_1^2, \alpha_2^2, \alpha_3^2)$. Hence by the chain rule

$$\nabla G_1 = (\alpha_1 g_1, \alpha_2 g_2, \alpha_3 g_3)$$

for some smooth functions g_1, g_2, g_3 of $(\alpha_1^2, \alpha_2^2, \alpha_3^2)$. (Namely $g_i = 2\partial G_1/\partial(\alpha_i^2),\ 1 \leq i \leq 3$.) We compute:

$$\begin{aligned} \nabla\mathcal{K}_2 \times \nabla\mathcal{K}_3 &= (-\mathbf{E}_2) \times (-\mathbf{E}_3) \\ &= \mathbf{E}_1 \\ \nabla\mathcal{K}_1 \times \nabla\mathcal{K}_3 &= (\nabla G_1 - \mathbf{E}_1) \times (-\mathbf{E}_3) \\ &= -\nabla G_1 \times \mathbf{E}_3 - \mathbf{E}_2 \\ &= (-\alpha_2 g_2 \mathbf{E}_1 + \alpha_1 g_1 \mathbf{E}_2) - \mathbf{E}_2 \\ &= -\alpha_2 g_2 \mathbf{E}_1 + (\alpha_1 g_1 - 1)\mathbf{E}_2 \\ \nabla\mathcal{K}_1 \times \nabla\mathcal{K}_2 &= (\nabla G_1 - \mathbf{E}_1) \times (-\mathbf{E}_2) \\ &= -\nabla G_1 \times \mathbf{E}_2 + \mathbf{E}_3 \\ &= (\alpha_3 g_3 \mathbf{E}_1 - \alpha_1 g_1 \mathbf{E}_3) + \mathbf{E}_3 \\ &= \alpha_3 g_3 \mathbf{E}_1 + (1 - \alpha_1 g_1)\mathbf{E}_3. \end{aligned}$$

Thus

$$\begin{aligned} \{\mathcal{K}_2, \mathcal{K}_3\}(\alpha) &= -\langle \alpha, \mathbf{E}_1 \rangle = -\alpha_1 \\ \{\mathcal{K}_1, \mathcal{K}_3\}(\alpha) &= -\langle \alpha, -\alpha_2 g_2 \mathbf{E}_1 + (\alpha_1 g_1 - 1)\mathbf{E}_2 \rangle \\ &= \alpha_1(\alpha_2 g_2) - \alpha_2(\alpha_1 g_1 - 1) \\ &= \alpha_2 + \alpha_1\alpha_2(g_2 - g_1) \end{aligned}$$

$$
\begin{aligned}
&= \alpha_2 + \alpha_1\alpha_2 f_1 \\
\{\mathcal{K}_1, \mathcal{K}_2\}(\alpha) &= -\langle \alpha, \alpha_3 g_3 \mathbf{E}_1 + (1-\alpha_1 g_1)\mathbf{E}_3 \rangle \\
&= -\alpha_1(\alpha_3 g_3) - \alpha_3(1-\alpha_1 g_1) \\
&= -\alpha_3 + \alpha_1\alpha_3(g_1 - g_3) \\
&= -\alpha_3 + \alpha_1\alpha_3 f_2.
\end{aligned}
$$

Note that $f_1 := g_2 - g_1$ and $f_2 := g_1 - g_3$ are smooth functions of $(\alpha_1^2, \alpha_2^2, \alpha_3^2)$ as required. This proves (4.15). Equation (4.16) is proved for the case $\alpha \in U_3(m)$ by a similar argument. ■

Proof of Theorem 4.3.2: By (4.14) we must average the expressions in Lemma 4.3.2 over the $\mathbb{T}^3$ action induced by $I_1,\ I_2,\ I_3$. Since in all cases $\{\mathcal{K}_i, \mathcal{K}_j\}$ is left invariant, we may instead take the time average over the rigid body dynamics. (See Lemma 4.2.2 and the remark immediately following.) Thus we substitute into the expressions (4.15) and (4.16) the appropriate solution to the Euler equations (2.5) (§2.2 cases (4e), (4d) for $\alpha \in U_1(m),\ U_3(m)$ respectively) and average over one period of the motion.

We review a few facts concerning the Jacobi elliptic functions. (See Byrd and Friedman[8] or Lawden[18] for additional details.) The functions $\mathrm{cn}(u,k),\ \mathrm{sn}(u,k),\ (k^2 < 1)$, are periodic in u of period $4K(k)$, while $\mathrm{dn}(u,k)$ has period $2K(k)$. Now $\mathrm{cn}(u,k)$ is an even function with respect to the point $u = 0$, and odd with respect to $u = K$; $\mathrm{sn}(u,k)$ is odd with respect to $u = 0$, and even with respect to $u = K$; finally $\mathrm{dn}(u,k)$ is even with respect to both $u = 0$ and $u = K$.

Let $\alpha \in U_3(m)$. Then combining the above information with §2.2(4d) we see that $\alpha(t)$ (with initial point $\alpha(0) = \alpha$) has period $T = 4s^{-1}K(k)$; $\alpha_1(t)$ is even with respect to $t = 0$, odd with respect to $t = T/4$; $\alpha_2(t)$ is odd with respect to $t = 0$, even with respect to $t = T/4$; and $\alpha_3(t)$ is even with respect to both $t = 0$ and $t = T/4$.

Now the integral of an odd periodic function over one of its periods is zero. We see from (4.16) that $\{\mathcal{K}_2, \mathcal{K}_3\}(\alpha(t))$ is an odd function with respect to $t = T/4$, while $\{\mathcal{K}_1, \mathcal{K}_3\}(\alpha(t))$

is odd with respect to $t = 0$. Hence

$$\langle\{\mathcal{K}_2, \mathcal{K}_3\}\rangle = \langle\{\mathcal{K}_1, \mathcal{K}_3\}\rangle = 0,$$

and

$$\begin{aligned} \langle\{\mathcal{K}_1, \mathcal{K}_2\}\rangle(\alpha) &= -\langle\alpha_3\rangle_{H_m} \\ &= -\frac{1}{T}\int_0^T \alpha_3(t)\, dt \\ &= -G_3(\alpha). \end{aligned}$$

The last calculation was done in the proof of Proposition 4.2.1. Thus for $\alpha \in U_3(m)$

$$\begin{cases} \Omega_{23}(\alpha) &= 0 \\ \Omega_{13}(\alpha) &= 0 \\ \Omega_{12}(\alpha) &= -G_3(\alpha). \end{cases}$$

In a similar manner we obtain from (4.15) that for $\alpha \in U_1(m)$

$$\begin{cases} \Omega_{23}(\alpha) &= -G_1(\alpha) \\ \Omega_{13}(\alpha) &= 0 \\ \Omega_{12}(\alpha) &= 0. \end{cases}$$

This completes the proof. ■

The fact that $\mathcal{K}^2 = \mathcal{K}^3 = 0$ implies that six of the nine cross terms are zero as well. Three of the cross terms remain unknown since $\mathcal{K}^1$ is undetermined. Summarizing the results of this section:

$$\text{on } B\text{:} \quad \left\{ \Omega^{23} = \Omega^{13} = \Omega^{12} = 0 \right.$$

$$\text{on } F_\lambda\text{:} \quad \begin{cases} \Omega_{23}(\alpha) = \begin{cases} -G_1(\alpha) & \alpha \in U_1(m) \\ 0 & \alpha \in U_3(m) \end{cases} \\ \Omega_{13}(\alpha) = 0 \\ \Omega_{12}(\alpha) = \begin{cases} 0 & \alpha \in U_1(m) \\ -G_3(\alpha) & \alpha \in U_3(m), \end{cases} \end{cases}$$

$$\text{cross terms:} \quad \begin{cases} \Omega^2_j = \Omega^3_j = 0 & 1 \le j \le 3 \\ \Omega^1_1,\ \Omega^1_2,\ \Omega^1_3 & \text{unknown.} \end{cases}$$

The presence of so many zero terms in the curvature suggests that there are many loops in M with trivial holonomy.

4.4 Holonomy

As in the last section we identify $M \cong B \times F_\lambda$. We saw in Corollary 4.3.1 that if $t \mapsto m(t)$ is a (piecewise) smooth loop lying in B, then its holonomy is trivial; equivalently the Hannay angles are zero. In this section we compute the holonomy of certain loops lying in F_λ, $\lambda \in B$. These are closed curves in M along which the principal moments of inertia remain fixed, while the principal axes rotate about a fixed vector in $\mathbb{R}^3$.

Throughout this section we consider curves of the form

$$m(t) = (\exp t\hat{\xi})\sigma(\lambda)(\exp t\hat{\xi})^{-1} \tag{4.17}$$

where $\sigma(\lambda) = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\lambda \in B$, and $\xi \in \mathbb{R}^3$ with $\|\xi\| = 1$. These are integral curves of the infinitesimal generator ξ_M, i.e. $m'(t) = \xi_M(m(t))$. The interval of t values for $m(t)$ will depend on our choice of ξ.

Let $\alpha \in U_i(m(t))$, $i = 1, 3$. Then by (4.10)

$$\begin{aligned}
[\mathbf{K} \cdot m'(t)]\,(\alpha) &= [\mathbf{K} \cdot \xi_M(m(t))]\,(\alpha) \\
&= G_i\left((\exp t\hat{\xi})^{-1}\alpha\right)\langle \xi, (\exp t\hat{\xi})\mathbf{E}_i\rangle - \langle \xi, \alpha\rangle.
\end{aligned}$$

Thus

$$[\mathbf{K} \cdot m'(t)]\,(\alpha) = G_i\left((\exp t\hat{\xi})^{-1}\alpha\right)\langle \xi, \mathbf{E}_i\rangle - \langle \xi, \alpha\rangle, \tag{4.18}$$

since $\exp t\hat{\xi} \in SO(3)$ is a rotation about ξ. Note that $U_i(m(t)) = \exp t\hat{\xi} \cdot U_i(\sigma(\lambda))$, so that $(\exp t\hat{\xi})^{-1}\alpha \in U_i(\sigma(\lambda))$ as required.

The Hamiltonian vector field of $\mathbf{K} \cdot m'(t)$ yields the parallel transport equations for the path $m(t)$, which we wish to solve. In general these equations are quite complicated, involving the derivatives of G_i, which contains an elliptic integral. A possible approach would be to attack these equations numerically, to determine the parallel transport operator. Our strategy will be to choose the vector ξ so that $\mathbf{K}\cdot m'(t)$ is a simple as possible. For $\xi \in \mathbf{E}_i^{\perp}$, $i = 1$ or 3 we have

$$[\mathbf{K}\cdot m'(t)]\,(\alpha) = -\langle \xi, \alpha\rangle.$$

Observe that this function does not depend on t. Its gradient is

$$\nabla\,[\mathbf{K}\cdot m'(t)]\,(\alpha) = -\xi,$$

and by §2.2(iii) its Hamiltonian vector field is

$$\begin{aligned} X_{\mathbf{K}\cdot m'(t)}(g,\alpha) &= \left(g(-\hat{\xi}), \alpha\times(-\xi)\right) \\ &= \left(-g\hat{\xi}, \hat{\xi}(\alpha)\right), \end{aligned}$$

which is autonomous. The parallel transport equations are then

$$\begin{cases} \dot{g} = -g\hat{\xi} & g(0) = g_0 \\ \dot{\alpha} = \hat{\xi}(\alpha) & \alpha(0) = \alpha_0, \end{cases}$$

with solution

$$\begin{cases} g(t) = g_0(\exp t\hat{\xi})^{-1} \\ \alpha(t) = (\exp t\hat{\xi})\alpha_0. \end{cases} \tag{4.19}$$

Note that this flow transports (g_0, α_0) along the orbits of the action (4.3). Also observe that if $\alpha_0 \in U_i(\sigma(\lambda))$ then $\alpha(t) \in (\exp t\hat{\xi})\cdot U_i(\sigma(\lambda))$ as required by (4.18).

Thus (4.19) gives the parallel transport of $(g_0, \alpha_0) \in P_i(\sigma(\lambda))$, $i = 1, 3$, along the path consisting of rotation of the inertia tensor about $\xi \in \mathbf{E}_i^{\perp}$. For definiteness we take $\xi \in \mathbf{E}_3^{\perp}$ and consider only initial points $(g_0, \alpha_0) \in P_3(\sigma(\lambda))$. A glance at the definition (4.17) shows that $m(0) = m(2\pi) = \sigma(\lambda)$ since $\|\xi\| = 1$, so by (4.19) the holonomy of the loop $m(t)$, $0 \le t \le 2\pi$, is trivial. Note from the discussion at the end of §4.1 that these loops belong to a single nontrivial homotopy class.

If we take $\xi = \pm\mathbf{E}_1$ or $\pm\mathbf{E}_2$, (4.17) shows that $m(0) = m(\pi) = \sigma(\lambda)$. Recall that these loops represent four distinct non-trivial homotopy classes in F_λ (see §4.1). By (4.19) the holonomy of these loops is given by

$$(g_0, \alpha_0) \mapsto \left(g_0 \exp(\pi\widehat{\mathbf{E}_i}), \exp(\pi\widehat{\mathbf{E}_i})\alpha_0\right),$$

for $i = 1, 3$. We remark that the above holonomy maps do not depend on the moments of inertia $\lambda \in B$.

4.5 Conclusions

Using the results obtained so far, we now draw several conclusions regarding the adiabatic invariance of the action integrals, and averaging over a natural $\mathbb{T}^2$ action. We also address the question as to whether the connection can be extended to $\Sigma \subset M_1$.

4.5.1 Adiabatic Invariance

An *adiabatic invariant* of a time dependent Hamiltonian system is a phase space function which is conserved to order ϵ on a time scale of order $1/\epsilon$. For single frequency systems in which the frequency is nowhere zero, the action integral is a well known adiabatic invariant. Arnold[2] proves this using the single frequency averaging theorem. In [3] Arnold introduces the notion of an *almost adiabatic invariant*, which is a quantity that is conserved to order (some power of) ϵ on a time scale of order $1/\epsilon$, excluding a set of initial points whose measure approaches zero with ϵ. A theorem of Neishtadt[25] states that for multifrequency systems, the action integrals are almost adiabatic invariants. A proof in which all constants are given explicitly appears in Golin, Knauf, and Marmi[11].

For the rigid body with time dependent inertia tensor however, the situation is somewhat better that for general three frequency systems. One checks easily that the angular momentum $\mathbf{J}$ is exactly conserved, and thus the actions $I_1 = \langle \mathbf{J}, \mathbf{e}_1 \rangle$ and $I_2 = \|\mathbf{J}\|$ are actually first integrals of this system. Now recall from §2.3 that the third action $I_3 = A/2\pi\|\mathbf{J}\|$ is also an action for the reduced system on S^2_r, $r = \|\mathbf{J}\|$, which has one degree of freedom. Thus I_3 is a true adiabatic invariant for the reduced, and hence also for the full system.

We conclude that for slow variations of the inertia tensor, I_1 and I_2 remain constant, while I_3 is adiabatically constant, regardless of the initial point.

4.5.2 $\mathbb{T}^2$ Averaging

The basis for the averaging principle is the fact that for multifrequency systems without resonances, the time average over a dynamic trajectory can be replaced by a space average (see Arnold[2] Chapter 10). We saw in §2.4 that the generic trajectories of the rigid body are dense windings on the 2-torus parametrized by the angles θ_2, θ_3 conjugate to I_2, I_3. Viewing this as a three frequency system, it has a proper resonance, while as a two frequency system it is generically non-resonant. It would therefore seem more reasonable to study the HB connection associated to the family of Hamiltonian $\mathbb{T}^2$ actions induced by the flows of I_2, I_3. In fact the results would be identical to those already obtained, as we now prove.

In calculating the Hamiltonian one-form in case I: $v \in TF_\lambda$, we find that we must replace the average appearing in the statement of Lemma 4.2.1 with the average over the $\mathbb{T}^2$ action. But we argue in the proof of Lemma 4.2.2 that since the momentum map L, is left invariant, and hence doesn't see the flow of I_1, the $\mathbb{T}^2$ and $\mathbb{T}^3$ averages of L coincide. Thus the Hamiltonian one-forms are identical in this case.

In case II: $v \in TB$, Proposition 3.3.1 indicates that we must simply remove the first equation from system (4.11). Proceeding as before we find it is sufficient to solve the single equation (4.12). The Hamiltonian one-forms are again identical. Since the Hamiltonian one-form uniquely determines the HB connection, it follows that the connections associated to the $\mathbb{T}^2$ and $\mathbb{T}^3$ actions are identical.

Observe however that the $\mathbb{T}^2$ action can be defined over a slightly larger region in $SO(3) \times \mathbb{R}^3$ than can the $\mathbb{T}^3$ action. In particular there is no need to assure that dI_1 be independent of dI_2 and dH_m, which means we can drop the requirement $g\alpha \times \mathbf{e}_1 \neq 0$ in the definition of $W \subset SO(3) \times \mathbb{R}^3$ (see Lemma 2.3.1). For $m \in M$, set

$$W' = \{(g, \alpha) \in SO(3) \times \mathbb{R}^3 \mid \alpha \times m^{-1}\alpha \neq 0\},$$

and define $P(m) \subset SO(3) \times \mathbb{R}^3$, $E \subset M \times (SO(3) \times \mathbb{R}^3)$ just as in §4.2 with W replaced by W'. Our formulas for the Hamiltonian one-form, curvature, and holonomy are then valid on this slightly larger bundle.

That these two actions give the same results ultimately derives from the arbitrariness with which we defined the action I_1. Recall we could have taken $I_1 = \langle \mathbf{J}, \mathbf{u} \rangle$ where $\mathbf{u} \in \mathbb{R}^3$ is any unit vector. Then the corresponding action-angle charts do not cover points (g, α) at which $g\alpha \times \mathbf{u} = 0$. The different $\mathbb{T}^3$ actions (one for each choice of $\mathbf{u}$) induce connections defined on bundles which exclude different codimension one submanifolds of $SO(3) \times \mathbb{R}^3$. The resulting formulas are the same in each case so we need not leave out any such submanifold.

4.5.3 Extending the Connection

Up until now we have not considered inertia tensors with multiple eigenvalues. As in §4.1 let Σ, Σ' denote those $m \in M_1$ with two and three eigenvalues equal, respectively. If $m \in \Sigma'$ then m is a multiple of the identity, so that every $\alpha \in \mathbb{R}^3$ is an eigenvector of m, and $\alpha \times m^{-1}\alpha = 0$. We saw in the proof of Lemma 2.3.1 that this implies dI_2 and dH_m are linearly dependent everywhere, so that not even the $\mathbb{T}^2$ action is defined (see §4.5.1). Therefore the connection cannot be extended to Σ'.

Neither can it be extended to Σ as we now show. Let $\lambda_i(t)$, $0 \leq t \leq 1$, for $1 \leq i \leq 3$, be smooth functions satisfying

$$\begin{array}{ccccccc} \lambda_1(t) & > & \lambda_2(t) & > & \lambda_3(t) & & t \in [0, \frac{1}{2}) \\ \lambda_1(\frac{1}{2}) & = & \lambda_2(\frac{1}{2}) & > & \lambda_3(\frac{1}{2}) & & \\ \lambda_2(t) & > & \lambda_1(t) & > & \lambda_3(t) & & t \in (\frac{1}{2}, 1], \end{array}$$

and set

$$m(t) = \operatorname{diag}(\lambda_1(t), \lambda_2(t), \lambda_3(t)),$$

for $0 \le t \le 1$. Then $m(1/2) \in \Sigma$, or equivalently for $h > 0$, $H^{-1}_{m(1/2)}(h)$ is an ellipsoid of revolution with $\mathbf{E}_3$ as symmetry axis. Suppose the HB connection can be extended somehow from $\pi_1 : E \to M$ to a bundle which includes $m(1/2)$ in its base. Let $(g(t), \alpha(t)),\ 0 \le t \le 1$ be the horizontal lift of $m(t)$, with initial point $(g(0), \alpha(0)) \in P_1(m(0))$. Define

$$\rho(t) = \frac{\|\alpha(t)\|^2}{2H_{m(t)}(\alpha(t))}$$

(this is just the quantity $r^2/2h$ from §2.3 evaluated along the horizontally lifted path.) Our choice of initial point forces

$$\lambda_1(0) > \rho(0) > \lambda_2(0) > \lambda_3(0)$$

(see §2.3.) Thus for some $t_0 \in (0, 1)$, $\rho(t_0) = \lambda_1(t_0)$ or $\rho(t_0) = \lambda_2(t_0)$ and therefore either $(g(t_0), \alpha(t_0))$ lies on a separatrix of the frozen system $H_{m(t_0)}$, or $\alpha(t_0) \times m(t_0)^{-1}\alpha(t_0) = 0$. In neither case can a torus action be defined at $(g(t_0), \alpha(t_0))$. We conclude that the horizontal lift of $m(t)$ cannot be continued through $t = t_0$, and hence the HB connection, as it is presently defined, cannot be extended to Σ.

If we choose instead $(g(0), \alpha(0)) \in P_3(m(0))$ the above problem does not occur, but a similar difficulty arises if we try to parallel transport along a path for which λ_2 and λ_3 cross. In order to accommodate parameter variations with eigenvalue collisions, we must change the bundle on which we define the HB connection.

We take as base for a new bundle

$$B_1 = \{\lambda \in \mathbb{R}^3 \mid \lambda_1 > \lambda_3,\ \lambda_2 > \lambda_3,\ \text{and } \lambda_i + \lambda_j > \lambda_k\ (i, j, k \text{ cyclic})\},$$

and identify B_1 with the matrices $m = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3),\ \lambda \in B_1$, which are diagonal with respect to $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$. As initial points for parallel transport we allow only those (g, α) for which

$$\lambda_1 > \frac{r^2}{2h} > \lambda_3 \qquad\qquad \lambda_2 > \frac{r^2}{2h} > \lambda_3,$$

where $r = \|\alpha\|$ and $h = H_m(\alpha)$. We thus allow $\lambda_1 = \lambda_2$, but not $\lambda_1 = \lambda_3$ or $\lambda_2 = \lambda_3$. Define for $\lambda \in B_1$

$$U(\lambda) = \{\alpha \in \mathbb{R}^3 \mid \lambda_1 > r^2/2h > \lambda_3,\ \lambda_2 > r^2/2h > \lambda_3\},$$

and set

$$P(\lambda) = \pi^{-1}(U(\lambda)) \cap W,$$

with π and W as in §2.3. Define

$$E = \{(\lambda, g, \alpha) \in B_1 \times (SO(3) \times \mathbb{R}^3) \mid \lambda \in B_1,\ (g, \alpha) \in P(\lambda)\}$$

and form the bundle $\pi_1 : E \to B_1,\ (\lambda, g, \alpha) \mapsto \lambda$, with fiber $E_\lambda = \pi_1^{-1}(\lambda) = \{\lambda\} \times P(\lambda)$. Note that $P(\lambda) \in SO(3) \times \mathbb{R}^3$ is the union of exactly two action-angle charts (formerly denoted $P_3^+(m),\ P_3^-(m)$.) Thus $\pi_1 : E \to B_1$ admits a family of Hamiltonian $\mathbb{T}^3$ actions (or $\mathbb{T}^2$ actions, see §4.5.1) and we can consider the corresponding HB connection.

We find that $I_3(\alpha),\ \alpha \in U(\lambda)$, for $\lambda \in B_1$, has the same expression as in §4.2.2. As before I_3 depends on λ only through the quantity $\eta(\lambda)$, and $\nabla\eta(\lambda) \neq 0$ for all $\lambda \in B_1$. Thus there is only one direction in B_1 along which parallel transport is non-trivial, showing as in §4.3, that the curvature is zero. One checks that B_1 is contractible, whence the holonomy about any loop in B_1 is trivial.

Appendix A. Calculation of the Spherical Area

Consider the trajectory of the Euler equations (2.5) whose initial point $\alpha(0)$, lies in one of the regions $U_1,\ U_3 \subset \mathbb{R}^3$ on which action angle coordinates are defined. The solution $\alpha(t)$, is then given by either Case (4d) or (4e) of §2.2. Our goal in this section is to compute the oriented surface area A, on the sphere $S_r^2 = \{\alpha \in \mathbb{R}^3 \mid \|\alpha\| = r\}$ enclosed by this trajectory, where $r = \|\alpha(t)\|$. For definiteness we assume throughout this section that $\alpha(0) \in U_3$ so (4d) applies. (The details would be similar if $\alpha(0) \in U_1$.) As noted in §2.2 this curve is one component of the intersection $S_r^2 \cap H_m^{-1}(h)$, where $h = H(\alpha(t))$. Our assumption (4d) says that $\lambda_1 > \lambda_2 > r^2/2h > \lambda_3$, and one verifies that in this case the two components of $S_r^2 \cap H_m^{-1}(h)$ lie in the two half spaces $\{\alpha_3 > 0\}$ and $\{\alpha_3 < 0\}$ respectively. We assume that $\alpha_3(t) > 0$. Let $|A|$ denote the *positive* area of the spherical region in question. The orientation of this region is that given by the direction of the curve $\alpha(t)$. Examination of the solutions in this case shows that $A = -|A|$. (In Case (4e) one has $A = |A|$.) Now $\alpha \in S_r^2 \cap H_m^{-1}(h)$ if and only if

$$\begin{cases} \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = r^2 \\ \frac{\alpha_1^2}{\lambda_1} + \frac{\alpha_2^2}{\lambda_2} + \frac{\alpha_3^2}{\lambda_3} = 2h. \end{cases} \tag{A.1}$$

We project this curve onto the α_1-α_2 plane by eliminating α_3 from (A.1), obtaining

$$\frac{\alpha_1^2}{a^2} + \frac{\alpha_2^2}{b^2} = 1$$

where

$$a^2 = \frac{\lambda_1(r^2 - 2\lambda_3 h)}{\lambda_1 - \lambda_3} \quad \text{and} \quad b^2 = \frac{\lambda_2(r^2 - 2\lambda_3 h)}{\lambda_2 - \lambda_3}. \tag{A.2}$$

Let $R = \{(\alpha_1, \alpha_2) \mid \frac{\alpha_1^2}{a^2} + \frac{\alpha_2^2}{b^2} \le 1\}$, and put $f(\alpha_1, \alpha_2) = \sqrt{r^2 - \alpha_1^2 - \alpha_2^2}$. We seek the positive surface area, $|A|$, on the graph of f which lies over the elliptical region $R \subset \mathbb{R}^2$. Thus

$$|A| = \int\int_R \sqrt{\left(\frac{\partial f}{\partial \alpha_1}\right)^2 + \left(\frac{\partial f}{\partial \alpha_2}\right)^2 + 1}\, d\alpha_1 d\alpha_2 = \int\int_R \frac{r\, d\alpha_1 d\alpha_2}{\sqrt{r^2 - \alpha_1^2 - \alpha_2^2}}.$$

To facilitate this integration we change coordinates as follows. Put $\bar{R} = \{(x,y)|\ x^2+y^2 \leq 1\}$, and apply the diffeomorphism $(x,y) \mapsto (ax, by)$ with Jacobian ab. We obtain

$$|A| = \int\int_{\bar{R}} \frac{abr\, dxdy}{\sqrt{r^2 - a^2x^2 - b^2y^2}}. \tag{A.3}$$

Converting this to polar coordinates ρ, ϕ yields

$$|A| = abr \int_0^{2\pi}\int_0^1 \frac{\rho\, d\rho d\phi}{\sqrt{r^2 - (a^2\cos^2\phi + b^2\sin^2\phi)\rho^2}} = 4abr\int_0^{\pi/2}\int_0^1 \frac{\rho\, d\rho d\phi}{\sqrt{r^2 - e^2\rho^2}}, \tag{A.4}$$

where

$$e^2 = a^2\cos^2\phi + b^2\sin^2\phi = a^2 + (b^2 - a^2)\sin^2\phi.$$

The inner integral (with respect to ρ) is readily computed as:

$$\begin{aligned}\int_0^1 \frac{\rho\, d\rho}{\sqrt{r^2 - e^2\rho^2}} &= \frac{r - \sqrt{r^2 - e^2}}{e^2} = \frac{r}{e^2} + \left[\frac{1}{\sqrt{r^2 - e^2}} - \frac{r^2}{e^2\sqrt{r^2 - e^2}}\right] \\ &= \frac{r}{a^2\cos^2\phi + b^2\sin^2\phi} + \frac{1}{a^2\sqrt{r^2 - a^2}}\left[\frac{a^2}{\Delta(\phi,k)} - \frac{r^2}{(1 + \eta\sin^2\phi)\Delta(\phi,k)}\right],\end{aligned}$$

where we have set

$$\eta = \frac{b^2 - a^2}{a^2} = \frac{\lambda_3(\lambda_1 - \lambda_2)}{\lambda_1(\lambda_2 - \lambda_3)},$$

$$k^2 = \frac{b^2 - a^2}{r^2 - a^2} = \frac{(\lambda_1 - \lambda_2)(r^2 - 2\lambda_3 h)}{(\lambda_2 - \lambda_3)(2\lambda_1 h - r^2)},$$

and

$$\Delta(\phi, k) = \sqrt{1 - k^2\sin^2\phi}.$$

Recall from Case (4c) in §2.2 that $\pm\sqrt{\eta}$ gives the slope of the intersection of the separatrix planes with the α_1-α_3 plane. Referring to (A.4) we then have that

$$\begin{aligned}|A| &= 4abr^2\int_0^{\pi/2}\frac{d\phi}{a^2\cos^2\phi + b^2\sin^2\phi} \\ &\quad + \frac{4br}{a\sqrt{r^2 - a^2}}\left[a^2\int_0^{\pi/2}\frac{d\phi}{\Delta(\phi,k)} - r^2\int_0^{\pi/2}\frac{d\phi}{(1 + \eta\sin^2\phi)\Delta(\phi,k)}\right].\end{aligned}$$

The first term becomes, upon integration, $2\pi r^2$ (by putting $y = \tan\phi$), while we recognize the terms in brackets as complete elliptic integrals of the first and third kinds respectively (see Byrd & Friedman[8].) We denote these by $K(k)$, and $\Pi(\eta, k)$, so that

$$|A| = 2\pi r^2 + \frac{4br}{a\sqrt{r^2 - a^2}}\left[a^2K(k) - r^2\Pi(\eta,k)\right].$$

Remark. The first term above is half the total surface area of S_r^2, and so the second term is the negative area of the complement of our enclosed spherical cap in the upper hemisphere. One checks that this term is indeed negative. The modulus k, appearing above is also the modulus of the elliptic functions which give the explicit solution to the Euler equations (2.5), as well as that of the complete elliptic integral which gives the period T. (see Appendix B.)

Using (A.2), we write $|A|$ in terms of r, h, and λ_i $(1 \leq i \leq 3)$,

$$|A| = 2\pi r^2 + 4r \left(\frac{\lambda_1 \lambda_2 \lambda_3}{(\lambda_2 - \lambda_3)(2\lambda_1 h - r^2)} \right)^{1/2} \left[\frac{(r^2 - 2\lambda_3 h)}{\lambda_3} K - \frac{r^2(\lambda_1 - \lambda_3)}{\lambda_1 \lambda_3} \Pi \right].$$

From §2.2 Case (4d) we have

$$s^2 = \frac{(\lambda_2 - \lambda_3)(2\lambda_1 h - r^2)}{\lambda_1 \lambda_2 \lambda_3},$$

thus

$$|A| = 2\pi r^2 + 4rs^{-1} \left[\frac{(r^2 - 2\lambda_3 h)}{\lambda_3} K - \frac{r^2(\lambda_1 - \lambda_3)}{\lambda_1 \lambda_3} \Pi \right].$$

Appendix B. Calculation of dI_3

In this section we show that I_3 defined by

$$2\pi I_3 = \frac{A}{r}$$

satisfies

$$d(2\pi I_3) = T\,dh - \Delta\Theta\,dr.$$

Here as in §2.2, T denotes the period of $\alpha(t)$, the solution to the Euler equations, and $\Delta\Theta$ is the angle in space by which the rigid body has rotated after time T. Recall also $h = H(\alpha(t))$ and $r = \|\alpha(t)\|$. We show by direct computation that

$$\frac{\partial}{\partial h}(2\pi I_3) = T, \tag{B.1}$$

and

$$\frac{\partial}{\partial r}(2\pi I_3) = -\Delta\Theta. \tag{B.2}$$

This calculation is performed specifically for $\alpha(t) \in U_3 \subset \mathbb{R}^3$ defined in §2.2. The case $\alpha(t) \in U_1$ is is entirely similar and we omit it. (The remaining cases represent trajectories in phase space not contained in the domain of any action-angle variables.)

Now the Jacobi elliptic functions of modulus k are periodic with period $4K(k)$, where

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\Delta(\phi, k)}$$

is the complete elliptic integral of the first kind. Here we have

$$\Delta(\phi, k) = \sqrt{1 - k^2 \sin^2 \phi},$$

and in our case

$$k^2 = \frac{(\lambda_1 - \lambda_2)(r^2 - 2\lambda_3 h)}{(\lambda_2 - \lambda_3)(2\lambda_1 h - r^2)}.$$

Examining §2.2 case (4d) we see the period of $\alpha(t)$ is

$$T = 4s^{-1}K(k), \tag{B.3}$$

where

$$s^2 = \frac{(\lambda_2 - \lambda_3)(2\lambda_1 h - r^2)}{\lambda_1 \lambda_2 \lambda_3}.$$

From Appendix A we have that the oriented surface area enclosed by $\alpha(t)$ in this case is

$$A = -2\pi r^2 - 4rs^{-1}\left[\frac{(r^2 - 2\lambda_3 h)}{\lambda_3}K - \frac{r^2(\lambda_1 - \lambda_3)}{\lambda_1\lambda_3}\Pi\right], \tag{B.4}$$

where

$$\Pi(\eta, k) = \int_0^{\pi/2} \frac{d\phi}{(1 + \eta \sin^2\phi)\Delta(\phi, k)}$$

is the complete elliptic integral of the third kind, and

$$\eta = \frac{\lambda_3(\lambda_1 - \lambda_2)}{\lambda_1(\lambda_2 - \lambda_3)}.$$

Combining Montgomery's formula (2.8) with (B.3) and (B.4) yields the expression for $\Delta\Theta$ in terms of elliptic integrals:

$$\begin{aligned} \Delta\Theta &= -\frac{A}{r^2} + \frac{2hT}{r} \\ &= 2\pi + 4rs^{-1}\left[\frac{1}{\lambda_3}K - \frac{\lambda_1 - \lambda_3}{\lambda_1\lambda_3}\Pi\right]. \end{aligned}$$

One could at this point calculate

$$\frac{\partial}{\partial r}T = -\frac{\partial}{\partial h}\Delta\Theta,$$

showing that $T\,dh - \Delta\Theta\,dr$ is closed. Instead we show directly that this form integrates to $2\pi I_3$.

To this end we first calculate the derivatives of $K(k)$ and $\Pi(\eta, k)$ with respect to r and h. Formulas (710.07) and (710.12) of Byrd and Friedman[8] yield

$$\frac{d}{dk}K(k) = \frac{E(k) - (1 - k^2)K(k)}{k(1 - k^2)},$$

and

$$\frac{d}{dk}\Pi(\eta, k) = \frac{k}{(1 - k^2)(\eta + k^2)}\left[E(k) - (1 - k^2)\Pi(\eta, k)\right],$$

where $E(k)$ is the complete elliptic integral of the second kind:

$$E(k) = \int_0^{\pi/2} \Delta(\phi, k)d\phi.$$

Using these formulas, the definitions of k and η, and diligent application of the chain rule, we obtain

$$\begin{aligned}
\frac{\partial}{\partial h}K &= \frac{-r^2(\lambda_2-\lambda_3)}{(2\lambda_2 h - r^2)(r^2-2\lambda_3 h)}E + \frac{r^2(\lambda_1-\lambda_3)}{(2\lambda_1 h - r^2)(r^2-2\lambda_3 h)}K \\
\frac{\partial}{\partial r}K &= \frac{2rh(\lambda_2-\lambda_3)}{(2\lambda_2 h - r^2)(r^2-2\lambda_3 h)}E - \frac{2rh(\lambda_1-\lambda_3)}{(2\lambda_1 h - r^2)(r^2-2\lambda_3 h)}K \\
\frac{\partial}{\partial h}\Pi &= \frac{-\lambda_1(\lambda_2-\lambda_3)}{(\lambda_1-\lambda_3)(2\lambda_2 h - r^2)}E + \frac{\lambda_1}{(2\lambda_1 h - r^2)}\Pi \\
\frac{\partial}{\partial r}\Pi &= \frac{2\lambda_1 h(\lambda_2-\lambda_3)}{r(\lambda_1-\lambda_3)(2\lambda_2 h - r^2)}E - \frac{2\lambda_1 h}{r(2\lambda_1 h - r^2)}\Pi.
\end{aligned}$$

These formulas were worked by hand then checked using Mathematica.

Now by (B.4)

$$\begin{aligned}
2\pi I_3 &= \frac{A}{r} \\
&= -2\pi r - 4s^{-1}\left[\frac{(r^2-2\lambda_3 h)}{\lambda_3}K - \frac{r^2(\lambda_1-\lambda_3)}{\lambda_1\lambda_3}\Pi\right].
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{\partial}{\partial h}(2\pi I_3) \\
&= 4s^{-2}\frac{\partial s}{\partial h}\left\{\frac{r^2-2\lambda_3 h}{\lambda_3}K - \frac{r^2(\lambda_1-\lambda_3)}{\lambda_1\lambda_3}\Pi\right\} \\
&\quad -4s^{-1}\left\{\frac{r^2-2\lambda_3 h}{\lambda_3}\frac{\partial K}{\partial h} - 2K - \frac{r^2(\lambda_1-\lambda_3)}{\lambda_1\lambda_3}\frac{\partial \Pi}{\partial h}\right\} \\
&= 4s^{-3}\left(\frac{\lambda_2-\lambda_3}{\lambda_2\lambda_3}\right)\left\{\frac{r^2-2\lambda_3 h}{\lambda_3}K - \frac{r^2(\lambda_1-\lambda_3)}{\lambda_1\lambda_3}\Pi\right\} \\
&\quad -4s^{-1}\left\{\frac{r^2-2\lambda_3 h}{\lambda_3}\left[\frac{-r^2(\lambda_2-\lambda_3)}{(2\lambda_2 h - r^2)(r^2-2\lambda_3 h)}E + \frac{r^2(\lambda_1-\lambda_3)}{(2\lambda_1 h - r^2)(r^2-2\lambda_3 h)}K\right]\right. \\
&\qquad \left. -2K - \frac{r^2(\lambda_1-\lambda_3)}{\lambda_1\lambda_3}\left[\frac{-\lambda_1(\lambda_2-\lambda_3)}{(\lambda_1-\lambda_3)(2\lambda_2 h - r^2)}E + \frac{\lambda_1}{2\lambda_1 h - r^2}\Pi\right]\right\} \\
&= 4s^{-1}\left\{\frac{\lambda_1(r^2-2\lambda_3 h)}{\lambda_3(2\lambda_1 h - r^2)}K - \frac{r^2(\lambda_1-\lambda_3)}{\lambda_3(2\lambda_1 h - r^2)}\Pi + \frac{r^2(\lambda_2-\lambda_3)}{\lambda_3(2\lambda_2 h - r^2)}E\right. \\
&\qquad \left. -\frac{r^2(\lambda_1-\lambda_3)}{\lambda_3(2\lambda_1 h - r^2)}K + 2K - \frac{r^2(\lambda_2-\lambda_3)}{\lambda_3(2\lambda_2 h - r^2)}E + \frac{r^2(\lambda_1-\lambda_3)}{\lambda_3(2\lambda_1 h - r^2)}\Pi\right\} \\
&= 4s^{-1}\left\{\frac{\lambda_1(r^2-2\lambda_3 h) - r^2(\lambda_1-\lambda_3) + 2\lambda_3(2\lambda_1 h - r^2)}{\lambda_3(2\lambda_1 h - r^2)}\right\}K \\
&= 4s^{-1}K = T,
\end{aligned}$$

which proves (B.1). Also

$$\begin{aligned}
&\frac{\partial}{\partial r}(2\pi I_3)\\
&= -2\pi + 4s^{-2}\frac{\partial s}{\partial r}\left\{\frac{r^2-2\lambda_3 h}{\lambda_3}K - \frac{r^2(\lambda_1-\lambda_3)}{\lambda_1\lambda_3}\Pi\right\}\\
&\quad -4s^{-1}\left\{\frac{2r}{\lambda_3}K + \frac{r^2-2\lambda_3 h}{\lambda_3}\frac{\partial K}{\partial r} - \frac{2r(\lambda_1-\lambda_3)}{\lambda_1\lambda_3}\Pi - \frac{r^2(\lambda_1-\lambda_3)}{\lambda_1\lambda_3}\frac{\partial \Pi}{\partial r}\right\}\\
&= -2\pi - 4s^{-3}\frac{r(\lambda_2-\lambda_3)}{\lambda_1\lambda_2\lambda_3}\left\{\frac{r^2-2\lambda_3 h}{\lambda_3}K - \frac{r^2(\lambda_1-\lambda_3)}{\lambda_1\lambda_3}\Pi\right\}\\
&\quad -4s^{-1}\left\{\frac{2r}{\lambda_3}K + \frac{r^2-2\lambda_3 h}{\lambda_3}\left[\frac{2rh(\lambda_2-\lambda_3)}{(2\lambda_2 h-r^2)(r^2-2\lambda_3 h)}E - \frac{2rh(\lambda_1-\lambda_3)}{(2\lambda_1 h-r^2)(r^2-2\lambda_3 h)}K\right]\right.\\
&\qquad \left. -\frac{2r(\lambda_1-\lambda_3)}{\lambda_1\lambda_3}\Pi - \frac{r^2(\lambda_1-\lambda_3)}{\lambda_1\lambda_3}\left[\frac{2\lambda_1 h(\lambda_2-\lambda_3)}{r(\lambda_1-\lambda_3)(2\lambda_2 h-r^2)}E - \frac{2\lambda_1 h}{r(2\lambda_1 h-r^2)}\Pi\right]\right\}\\
&= -2\pi - 4s^{-1}\left\{\frac{r(r^2-2\lambda_3 h)}{\lambda_3(2\lambda_1 h-r^2)}K - \frac{r^3(\lambda_1-\lambda_3)}{\lambda_1\lambda_3(2\lambda_1 h-r^2)}\Pi + \frac{2r}{\lambda_3}K + \frac{2rh(\lambda_2-\lambda_3)}{\lambda_3(2\lambda_2 h-r^2)}E\right.\\
&\qquad \left. -\frac{2rh(\lambda_1-\lambda_3)}{\lambda_3(2\lambda_1 h-r^2)}K - \frac{2r(\lambda_1-\lambda_3)}{\lambda_1\lambda_3}\Pi - \frac{2rh(\lambda_2-\lambda_3)}{\lambda_3(2\lambda_2 h-r^2)}E + \frac{2rh(\lambda_1-\lambda_3)}{\lambda_3(2\lambda_1 h-r^2)}\Pi\right\}\\
&= -2\pi - 4s^{-1}\left\{\left[\frac{r(r^2-2\lambda_3 h)+2r(2\lambda_1 h-r^2)-2rh(\lambda_1-\lambda_3)}{\lambda_3(2\lambda_1 h-r^2)}\right]K\right.\\
&\qquad \left. + \left[\frac{-r^3(\lambda_1-\lambda_3)-2r(\lambda_1-\lambda_3)(2\lambda_1 h-r^2)+2rh\lambda_1(\lambda_1-\lambda_3)}{\lambda_1\lambda_3(2\lambda_1 h-r^2)}\right]\Pi\right\}\\
&= -2\pi - 4s^{-1}\left\{\frac{1}{\lambda_3}K - \frac{\lambda_1-\lambda_3}{\lambda_1\lambda_3}\Pi\right\}\\
&= -\Delta\Theta,
\end{aligned}$$

proving (B.2).

References

1. Abraham, R., J.E. Marsden: *Foundations of Mechanics*. Addison-Wesley (1978).
2. Arnold, V.I.: *Mathematical Methods of Classical Mechanics (Second Edition)*. Springer-Verlag (1989).
3. Arnold, V.I.: *Geometrical Methods in the Theory of Ordinary Differential Equations*. Springer-Verlag (1983).
4. Berry, M.: Quantal Phase Factors Accompanying Adiabatic Changes. *Proc. R. Soc. Lond. A*, **392**, 45-57 (1984).
5. Berry, M.: Classical Adiabatic Angles and Quantal Adiabatic Phase. *J. Phys. A: Math. Gen.*, **18**, 15-27 (1985).
6. Berry, M., J.H. Hannay: Classical Non-Adiabatic Angles. *J. Phys. A: Math. Gen.*, **21**, L325-L331 (1988).
7. Bredon, G.E.: *Introduction to Compact Transformation Groups*. Academic Press (1972).
8. Byrd, P.F., M.D. Friedman: *Handbook of Elliptic Integrals for Engineers and Scientists (second edition)*. Springer-Verlag (1971).
9. Duistermaat, J.J.: On Global Action-Angle Coordinates. *Comm. on Pure and App. Math.*, **33**, 687-706 (1980).
10. Fomenko, A.T., V.V. Trofimov: *Integrable Systems on Lie Algebras and Symmetric Spaces*. Gordon and Breach (1988).
11. Golin, S., A. Knauf, S. Marmi: The Hannay angles: Geometry, Adiabaticity, and an Example. *Commun. Math. Phys.*, **123**, 95-122 (1989).
12. Golin, S., S. Marmi: Symmetries, Hannay Angles, and Precession of Orbits. *Europhys. Lett.*, **8**, 399-404 (1989).
13. Gray, B.: *Homotopy Theory; an Introduction to Algebraic Topology*. Academic Press (1975).

14. Hannay, J.H.: Angle Variable Holonomy in Adiabatic Excursion of an Integrable Hamiltonian. *J. Phys. A: Math. Gen.*, **18** 221-230 (1985).

15. Holm, D.D., J.E. Marsden: The Rotor and the Pendulum. *Symplectic Geometry and Mathematical Physics.* Edited by: P. Donato, C. Duval, J. Elhadad, and G. M. Tuynman. **99**, 189-203, Birkhauser (1991).

16. Koiller, J.: Classical Adiabatic Angles for Slowly Moving Mechanical Systems. *Contemporary Mathematics*, **97**, 159-185 (1989).

17. Landau, L.D., E.M. Lifshitz: *Mechanics (third edition).* Pergamon Press (1976).

18. Lawden, D.F.: *Elliptic Functions and Applications.* Springer-Verlag (1989).

19. Markus, L., K.R. Meyer: *Generic Hamiltonian Systems are neither Integrable nor Ergodic.* Memoirs AMS, **144** (1974).

20. Marsden J.E.: *Lectures on Mechanics.* Lond. Math. Soc. Lecture Notes **174**, Cambridge University Press (1992).

21. Marsden, J.E., R. Montgomery and T. Ratiu: *Reduction, Symmetry, and Phases in Mechanics.* Memoirs AMS, **436** (1990).

22. Marsden, J.E., T.S. Ratiu: *An Introduction to Mechanics and Symmetry.* In preparation.

23. Montgomery, R.: By How Much Does the Rigid Body Rotate? A Berry Phase from the 18^{th} Century. *American J. of Phys.*, **59**, n.5, 394-397, (1990).

24. Montgomery, R.: The Connection Whose Holonomy is the Classical Adiabatic Angles of Hannay and Berry and Its Generalization to the Non-Integrable Case. *Commun. Math. Phys.*, **120**, 269-294 (1988).

25. Neishtadt, A.I.: Averaging in Multifrequency Systems. *II. Dokl. Akad. Nauk SSSR.* **226**, 1295-1298, (1976). [English Translation: *Sov. Phys. Dokl.* **21**, 80-82, (1976).]

26. Sanders, J.A., F.Verhulst: *Averaging Methods in Non-linear Dynamical Systems.* Applied Mathematical Sciences, **59**, Springer-Verlag, (1985).

27. Simon, B.: Holonomy, the Quantum Adiabatic Theorem and Berry's Phase. *Phys. Rev. Lett.*, **51**, 2167-2170 (1983).

28. Weinstein, A.: Connections of Berry and Hannay Type for Moving Lagrangian Submanifolds. *Adv. in Math.*, **82**, 133-159, (1990).