UNIVERSITY OF CALIFORNIA
SANTA CRUZ

# A COMPARISON OF TWO METHODS OF RESOLUTION: BLOW UP AND PROLONGATION

A dissertation submitted in partial satisfaction of the
requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

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March 2009

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# Abstract


A Comparison of two methods of resolution: blow up and prolongation

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A standard method for resolving a plane curve singularity is method of blow up. We describe a less-known alternative method which we call prolongation, in honor of Cartan's work in the direction. This method is known to algebraic geometers as Nash blow up.

A standard method for resolving a plane curve singularity is the method of blow up. We describe a less-known alternative method which we call prolongation, in honor of Cartan's work in this direction. This method is known to algebraic geometers as Nash blow up. With each application of prolongation the dimension of the ambient space containing the new 'prolonged' singularity increases by one. The new singularity is tangent to a canonical plane field on the ambient space. Our main result, Theorem 3.3.1 asserts that the two methods, blow up and prolongation, yield the same resolution for unibranched singularities. The primary difficulties encountered are around understanding the prolongation analogues of the exceptional divisors from blow up. These analogues are called critical curves. Most of the critical curves are abnormal extremals in the sense of optimal control theory as it applies to rank 2 distributions (2 controls).

dedication To myself,

Perry H. Disdainful,

the only person worthy of my company.

# Acknowledgments

I want to “thank” my committee, without whose ridiculous demands, I would have graduated so, so, very much faster.

# Part I

# First Part

# Chapter 1

# Introduction

This paper introduces connections between two methods of resolving plane curve singularities: blow up and prolongation. The first method, blow up, is classical and well-known, [10], [4], [5] and will be reviewed later. The second method, prolongation, was introduced recently in [8], and is based on Cartan's notions of "prolongation" [2], [3]. Prolongation appeared earlier in a slightly different guise in algebraic geometry, where it is known as Nash blow up, [7]. In this thesis we explore examples that led us to believe that the two resolutions are "the same" for the class of analytic, singular plane curve germs. These examples led to the paper [9], where it is proved that the two resolutions are equal. Some care will be needed in explaining what we mean by the same.

Each method consists of iteratively applying a transformation – blow up or prolongation – to curves. Each application of the respective transformation, blow up or prolongation, adds a projective line to the transforms of the curves from the previous steps, so that after $k$ applications we have the transform of the original curve – known as the "proper transform" in the case of blow up — together with $k$ projective lines, these being called the "exceptional divisors" or "exceptional curves" in the case of blow up, and critical curves in the case of prolongation. We continue applying the transformation until the proper transform of the original curve is smooth, and all intersections amongst all curves are "normal" . The meaning of "normal" is well-known in the case of blow up: it means all intersections are transverse and there are no triple intersections. In the case of prolongation some thought is needed to make sense of the notion of normal. When the proper transform is smooth and all intersections are normal we stop, and declare the resulting union of curves to be the resolution (by whichever method) of the original plane curve singularity . The integer $k$ at which resolution is achieved will be called "the number of steps to resolution". This number could, a priori, be different for blow up and prolongation. (It is not!) In both cases the result of resolution can be encoded combinatorially by a graph. The verticies of the graph are the component curves: some number $k$ of projective lines and the proper transform. Two verticies are joined by an edge if the curves they represent are "incident". In the case of blow up "to be incident" means to intersect. In the case of prolongation the meaning of "to be incident" is more subtle, but roughly it means the curves which the edges represent intersect at some appropriate "level" in an appropriate way.

We will call the resulting graphs, the "resolution graphs". One of the main new results here is how to construct, in a natural way, a resolution graph for prolongation. In our recent paper [9], it is proved that the resolution graphs for blow up and for prolongation are isomorphic.

### 1.0.1 Outline of Paper

Section 1: Planar Curves.

Section 2: Blow up, a review.

Section 3: Prolongation. The Monster is introduced here.

Section 4:$(t^3, t^8)$, an example for which both prolongation and blow up resolve the curve singularity in the same number of steps.

Section 5: Critical Curves.

Section 6: Resolution by prolongation

Section 7: $(t^3, t^8)$, an example for which both prolongation and blow up have isomorphic resolution graphs. Main theorem is stated here.

Section 8: $A_{2k}$ singularity.$(t^2, t^7)$,and $(t^2, t^{2k+1})$.

Section 9: $E$ singularity $(t^3, t^7)$.

Section 10: Quasi-Homogeneous Case: $(t^m, t^n)$. (Case of Puiseux characteristic of length 2, mention Brieskorn and the sequence of Euclidean Algorithms)

Section 11: $(t^3, t^{10} + t^{11})$, a non-planar example and the failure of intermediate step equivalences. This example proves that a step-by-step diffeomorphism between the results of blow up and prolongation does not exist.

Section 12: Conclusion

# Part II

# Second Part

# Chapter 2

# Preliminaries

## 2.1 Plane curves

Curves in the plane can be represented by a set of points whose coordinates satisfy some defining equation $f(x, y) = 0$ or as the images of parametrizations $t \mapsto (x(t), y(t))$, where $x$ and $y$ may be taken as polynomials or power series. To pass from a defining equation to a parametrization near a regular point, use the implicit function theorem. Near a singular point $p$ use the Newton-Puiseux expansion, an algorithm for expressing the curve locally as the finite union of images of parameterized curves. These parametrized curves are called the branches of the singularity at $p$. A curve $c$ has a unique decomposition as a finite union of branches. Each branch has a unique tangent direction at the singular point, $p$. A parametrization $(x(t), y(t))$ determines a unique branch. See [4], or [10], for details on plane curves and the Newton-Puiseux expansion.

In this thesis we are concerned with germs of analytic unibranched singularities:

singular curves consisting of a single branch. Examples are $x^p - y^q = 0$, $p, q$ relatively prime integers. Such a curve is parameterized as $x = t^q, y = t^p$. *After a rotation of the xy plane, any unibranched analytic singularity can be parameterized as*

$$x = t^m, y = \Sigma_{r>m} a_r t^r. \tag{2.1.1}$$

It is worth pointing out that being unibranched for an analytic curve germ singularity $f(x, y) = 0$ is equivalent to $f$ being irreducible, in the usual sense of algebra, within the space of complex analytic function germs of two variables. See for example, ch. 2 of [10].

We have chosen to work over the field of complex rather than real numbers because $\mathbb{C}$ is the traditional field within which to blow up plane curves and so we have found it easier to state and compare results when working over $\mathbb{C}$. Also, we work over the complex numbers so that $x, y$ and the parameter $t$ all take values in $\mathbb{C}$. All of our definitions and constructions here carry through for real analytic plane curves but $\mathbb{C}$ is the traditional field over which to perform blow up so we will work there.

We consider a parametrization to be *good*, or a curve to be well-parametrized if a general point of the curve corresponds to just one value of the parameter $t$.

**Definition 1.** *A curve germ $t \mapsto c(t)$, $t \in \mathbb{C}$ defined near $t = 0$ is well-parameterized if there is a neighborhood of $t = 0$ such that the parameterization is one-to-one*

Example. The standard parameterization $x = t^2, y = t^3$ makes the the cusp $y^2 = x^3$ well -parameterized. But if we parameterize the cusp by $x = t^4, y = t^6$ then this parameterized curve is not-well parameterized.

**Remark.** If we work over the reals, a somewhat different definition of well-parameterized must be given. See [8].

**Remark.** We have the following number-theoretic way of establishing whether or not an analytic curve is well-parameterized. Suppose the curve to be given by equation (2.1.1). Let $supp(y) \subset \mathbb{N}$ be the set of exponents $r$ occurring in the expansion of $y(t)$ such that $a_r \neq 0$. This set $supp(y)$ will be called the support of $y(t)$. Then $c(t)$ is well-parameterized if and only if the greatest common divisor of $m \cap supp(y)$ is 1. See section 2.3, [10].

Henceforth, we will fix our attention primarily on singular well-parameterized planar curve germs.

## 2.2 Blow up

We review the method of blow up to resolve plane curve singularities. We set up notation to calculate examples that illustrate the main theorem. There are numerous good references for this section, including [4], [5], [10].

**Constructing blow ups.** The blow up of the plane at the origin can be realized as that subvariety $Bl_0(\mathbb{C}^2)$ of $P^1(\mathbb{C}^2)$ consisting of those pairs $(p, \ell) \in P^1(\mathbb{C}^2)$ for which $\ell$ passes through the origin 0 as well as through $p$. The restriction of the natural projection $P^1(\mathbb{C}^2) \to \mathbb{C}^2$ to $Bl_0\mathbb{C}^2$ is called the "blow-down map" and denoted $\beta : Bl_0\mathbb{C}^2 \to \mathbb{C}^2$: $\beta(p, \ell) = p$. We set $E = \beta^{-1}(0)$ and call $E$ the exceptional curve. $E$ is an embedded copy of $\mathbb{C}P^1$. It coincides with the vertical curve $V_1$ over 0. Away from $E$ the blow-down map is a diffeomorphism since if $\beta(p, \ell) = p$, and $p \neq 0$ then $\ell = span(p)$.

**Coordinates on the blow up.** Away from the exceptional fiber $E$ planar coordinates $(x, y)$ coordinatize the blow up since $\beta$ is a diffeomorphism off $E$. To cover points of $E$ we need two coordinate charts. Take $p_1 = ((0,0), \ell_0) \in E$ and suppose $\ell_0 \neq y-$axis. As a neighborhood of $p_1$ consider all points $((x, y), \ell)$ of the blow up for which $\ell \neq y-$axis. Use affine coordinate $w_1$ for these lines: $\ell = [1, w_1]$ and $p_1 = ((0, 0), [1, w_1])$. Then $(x, w_1)$ coordinatize our neighborhood . The condition defining $Bl_0(\mathbb{C}^2)$ is that $[x, y] = [1, w_1]$ for $(x, y) \neq (0, 0)$, which is to say that $y = xw_1$ and so the blow-down map is

$$\beta : (x, w_1) \mapsto (x, xw_1) = (x, y).$$

In the coordinates $(x, w_1)$ the exceptional curve $E$ is defined by $x = 0$.

**Notation.** By a slight abuse of notation we will say that $w_1$ is defined by the equation

$$w_1 = y/x \tag{2.2.1}$$

which is valid for $x \neq 0$. Even though equation (2.2.1) does not make sense on the exceptional curve $E$, there is no ambiguity in our original definition of $w_1$ and that equation uniquely picks out $w_1$ as an affine coordinate when restricted to the projective line $E$. The notational abuse of equation (2.2.1) will be useful later on when defining coordinates on iterated blow ups.

The coordinates $(x, w_1)$ miss one point, the point $((0, 0), [0, 1])$ corresponding to $\ell_0 = y$-axis (and so $w_1 = \infty$). To cover this missing point use $(y, v_1)$ for which $[v_1, 1]$

are the affine coordinates, and for which the the blow-down map is

$$\beta : (y, v_1) \mapsto (yv_1, y) = (x, y).$$

By the same abuse of notation, we write $v_1 = x/y$ in this case.

**Blowing up the curve.** If $c$ is a curve with singular point at the origin, its blow up is the curve $Bl^1(c) = \beta^{-1}(c) \subset Bl_0(\mathbb{C}^2)$, called the *total transform.* It consists of two components, the "proper transform" which is the closure of $\beta^{-1}(c \setminus \{0\})$, and the exceptional fiber $E$. If $c$ is algebraic, its blow up is algebraic. If $c$ is analytic its blow up is analytic.

**Iterated blow up.** To resolve $c$ we typically need to blow up more than once. In order to blow up a second time, we realize that $Bl_0(\mathbb{C}^2)$ is itself an analytic surface, and define the blow up operation works on any analytic surface. So, let us define the blow up $Bl_p(S)$ of an analytic surface $S$ at a point $p \in S$. Choose coordinates $(x, y)$ centered at $p$, coordinatizing a neighborhood $U$ of $p$ and so identifying $U$ with a neighborhood $V$ of 0 in $\mathbb{C}^2$. Using these coordinates, we identify $Bl_p(S)$ over $p$ with the open set $\beta^{-1}(V) \subset Bl_0(\mathbb{C}^2)$. In these coordinates the blow-down map $\beta$ takes the same form as it did in the plane. The exceptional curve is $E = \beta^{-1}(p)$ and is a $\mathbb{P}^1 \subset Bl_p(S)$. Away from $p$, we declare $Bl_p(S) \to S$ to be an analytic diffeomorphism, and this endows $Bl_p(S)$ with the structure of an analytic surface. If $c \subset S$ is an analytic curve with singularity $p$ then its blow up at $p$ is $\beta^{-1}(c)$, which again splits up into two parts, the proper transform, and the exceptional curve.

We are now able to iterate the blow up process. Suppose that the first blow up $B(c)$ of the unibranched curve singularity $c$ is still singular. Then $B(c)$ will have a single singular point $p_1$ which must be the intersection point of the proper transform $\tilde{c}$ with the exceptional curve $E = E_1$. We blow up $X_1 = Bl_0(\mathbb{C}^2)$ at $p_1$ so as to form a new surface $X_2 = Bl_{p_1}Bl_0(\mathbb{C}^2)$, and a new blown up curve $B^2(c) = B(B(c)) \subset X_2$ which consists of the new proper transform, still denoted $\tilde{c}$, and two exceptional curves, the new one, written $E_2$, and the proper transform of the old one, typically written $\tilde{E}_1$. If this configuration is still deemed singular, we keep going. At the kth iteration of the process we have a curve $Bl^k(c)$ in an analytic surface $X_k = Bl_{p_k}(Bl_{p_{k-1}} \ldots Bl_0(\mathbb{C}^2)\ldots)$, with $p_i \in E_i$. The kth blow up of the curve has $k+1$ components (in the Zariski sense): $B^k(c) = \tilde{c}^k \cup \tilde{E}_1 \cup \ldots \cup \tilde{E}_{k-1} \cup E_k$, with $\tilde{c}^k$ denoting the proper transform of the original curve at the kth step, and $\tilde{E}_j, j < k$ denoting the proper transform of the exceptional curve $E_j$ arising from the earlier level $j$. We stop the process when this collection of curves is normal in the following sense.

**Definition 2.** *A collection of curves in a smooth surface has normal crossing singularities if each curve is smooth, no three meet in a point, and any intersection of two of them is transverse.*

**Definition 3.** *The number of blow ups required to reach this normal crossing situation is called the resolution number by blow up.*

## 2.3 Prolongation

Let $c$ be a complex analytic curve in a smooth complex manifold $M^n$. Its singular points $\Sigma$ are discrete. At each non-singular point $p \in c \setminus \Sigma$ the tangent line $T_p c$ to $c$ is uniquely defined, and can be viewed as a point in the projectivized tangent bundle $\mathbb{P}TM$. The closure of the set of points $(p, T_p C), p \in c \setminus \Sigma$ is defined to be the first prolongation of $c$, and is denoted by $c^1$. Away from $\Sigma$, the projection $\mathbb{P}TM \to M$ maps $c^1$ diffeomorphically onto $c \setminus \Sigma$. In [8] we prove that $c^1$ is analytic.

It is worth pointing out that if $c = c(t)$ is parameterized by the parameter $t$, then at regular points $t$ (where $dc/dt(t) \neq 0$) we have $c^1(t) = (c(t), \text{span } \{dc(t)/dt\})$. At singular points $t_*$ (where $dc/dt(t_*) = 0$) we have $c^1(t_*) = \lim_{t \to t_*} c^1(t)$.

If $c$ is tangent to a rank 2 (complex) distribution $D \subset TM$ then its prolongation $c^1$ must lie in the projectivization $\mathbb{P}D \subset \mathbb{P}TM$ of $D$. The space $\mathbb{P}D$ is a bundle over $M$ with fiber the complex projective line. Now $\mathbb{P}D$, viewed as a complex manifold, is itself endowed with a canonical rank 2 (complex) distribution which we denote $D^1$, and call the prolongation of $D$. We may define $D^1$ by

$$D^1(p) = (d\pi_m)^{-1}(\ell), p = (m, \ell) \in \mathbb{P}D.$$

Here $\pi : \mathbb{P}D \to M$ is the projection sending $(m, \ell)$ to $s$. Alternatively, a smooth curve $\gamma$ in $\mathbb{P}D$ consists of a moving point and a moving line $\gamma(t) = (m(t), \ell(t))$ and we can define $D^1$ by declaring that

$$\text{a curve } \gamma(t) = (m(t), \ell(t)) \text{ is tangent to } D^1 \text{ iff } dm(t)/dt \in \ell(t).$$

Now $c^1$ is tangent to $D^1$ at every point over $c \setminus \Sigma$. It is in fact tangent to $D^1$ at all of its points, by continuity of $D$ and the analyticity of $c$. Thus, we can repeat the procedure, to achieve the second prolongation $c^2 = (c^1)^1$ tangent to a rank 2 distribution $D_2 = (D^1)^1$ on the manifold $M_2 = \mathbb{P}(D^1)$. Continuing in this manner we get a sequence of prolonged curves $c^k$ tangent to rank 2 distributions $D_k$ on manifolds $M_k$. The $M_k$ , $k = 1, 2, \ldots$ form a tower of $\mathbb{CP}^1$-bundles

$$\ldots \to M_{k+1} \to M_k \to M_{k-1} \to \ldots M_1 = \mathbb{P}(D) \to M.$$

We apply this construction to a planar curve $c \subset \mathbb{C}^2$, assumed analytic. The curve is trivially tangent to the tangent bundle $\Delta_0 := T\mathbb{C}^2$ of $\mathbb{C}^2$, a rank 2 distribution. The first prolongation of the triple $(c, \Delta_0, \mathbb{C}^2)$ consists of an integral curve $c^1$, a rank 2 distribution $\Delta_1 = (\Delta_0)^1$, and a (complex) 3-dimensional manifold $\mathbb{P}T\mathbb{C}^2$ which supports both $\Delta_1$ and $c$. The points of $\mathbb{P}T\mathbb{C}^2$ are the marked lines described in the introduction. $\Delta_1$ is a contact distribution. Iterating the prolongation construction we obtain $(c^j, \Delta_j, P^j(\mathbb{C}^2))$. The $P^j(\mathbb{C}^2)$ fit together to form a tower of $\mathbb{P}^1$-bundles

$$\ldots \to P^{j+1}(\mathbb{C}^2) \to P^j(\mathbb{C}^2) \to \ldots \to P^1(\mathbb{C}^2) = \mathbb{P}T\mathbb{C}^2 \to \mathbb{C}^2.$$

Each $P^j(\mathbb{C}^2)$ is endowed with its rank 2 distribution $\Delta_j$, and $P^{j+1}(\mathbb{C}^2)$ is the total space of the projectivized bundle $\mathbb{P}(\Delta_j)$. Note that a point $p_{j+1} \in P^{j+1}(\mathbb{C}^2)$ is to be viewed as a pair $(p_j, \ell)$ with $p_j \in P^j(\mathbb{C}^2)$ and $\ell \subset \Delta_j(p_j)$ a line.

*We call this tower of* $\mathbb{P}^1$ *bundles, endowed with their distributions "the Monster tower", see [8]. When we say we are at level* $j$ *we mean we are working within* $(P^j(\mathbb{C}^2), \Delta_j)$.

The curves $c^j \subset P^j(\mathbb{C}^2)$ are tangent to $\Delta_j$ and are defined iteratively by $c^{j+1} = (c^j)^1$. They are all analytic. (See [8]).

The real version of $P^1(\mathbb{C}^2)$ occurs frequently in books and papers of Arnol'd. The real version $P^2(\mathbb{C}^2)$ occurs infrequently and is the primary example of an "Engel manifold".

#### 2.3.0.1 K-R coordinates

Kumpera and Ruiz introduced a special system of coordinates designed to fit Goursat distributions, see [1]. We use them to coordinatize the Monster. We write a K-R coordinate system for $\mathbb{P}^k\mathbb{C}^2$ as $(x, y, u_1, \ldots, u_k)$. The construction goes as follows: fix coordinates $x, y$ on the plane $\mathbb{C}^2$. Arbitrarily designate the lines parallel to the $y$-axis as being "vertical". Then $\{dx, dy\}$ form a coframe for $\Delta^0 = T\mathbb{C}^2$ and the vertical line is anihilated by $dx$. So we have that $[dx, dy]$ form homogeneous coordinates on $\Delta^0(x, y)$. There are two corresponding fiber-affine coordinates, obtained by writing $[dx, dy] = [1, \frac{dy}{dx}]$ or $[dx, dy] = [\frac{dx}{dy}, 1]$. If the point $p^*$ is not vertical then $u_1 = \frac{dy}{dx}$, and if the point $p^*$ is vertical, then $u_1 = \frac{dx}{dy}$. Then $\mathbb{P}^1\mathbb{C}^2$ is covered by two charts of the form $(x, y, u_1)$, and the coordinate transformation between the charts is $(x, y, u_1) \mapsto (x, y, 1/u_1)$.

If $u_1 = \frac{dy}{dx}$, then $dy - u_1 dx = 0$ and this relation defines the contact form $\Delta^1$ on $\mathbb{P}^1\mathbb{C}^2$ within this chart. Similarly, when $u_1 = \frac{dx}{dy}$, then the contact form is given by $dx - u_1 dy = 0$. Now to go from $k$ to $k+1$, we proceed inductively. Suppose that systems of K-R coordinates $x, y, u_1, \ldots, u_k$ have been constructed such that:

(a) $\{du_i, du_j\}$ form a basis for $(\Delta^{k-1})^*$ for$i, j \leq k-1$;

(b) one of $i$ or $j$ is $k-1$;

(c) $u_k = \frac{du_i}{du_j}$ is the corresponding fiber-affine coordinate at level $k$

Taking $p^* \in \mathbb{P}^{1+k}\mathbb{C}^2$, let $i, j$ satisfy (a), (b), and (c). Then define $u_{k+1} = \frac{du_k}{du_j}$ if the point $p^*$ is not vertical; or define $u_{k+1} = \frac{du_j}{du_k}$ if the point $p^*$ is vertical.

In a K-R coordinate system for $\mathbb{P}^k\mathbb{C}^2$ the 2-distribution $\Delta^k$ is described by 1-forms $\alpha_1, \ldots, \alpha_k$, whose form corresponds to the structure of the coordinates $u_i$. In general at level $j$ if the coordinate $u_j$ has the form $u_j = \frac{du_a}{du_b}$ then $\alpha_j = du_a - u_j du_b$.

*From here on, all coordinates in the Monster are taken to be K-R coordinates.*

Let $u_1, \ldots, u_i$ be the coordinates at the point $\gamma^i(0) \in \mathbb{P}^i\mathbb{C}^2$, then the $i^{th}$ prolongation of $\gamma(t)$ is $\gamma^i(t) = (x(t), y(t), U_1(t), \ldots, U_i(t))$, where $U_k(t) = u_k(\gamma^k(t))$. We calculate the KR cooordinate functions using the following formulae.

Define $u_{-1}, u_0$ and $U_{-1}, U_0$ by

$$u_{-1} = x, u_0 = y,\ U_{-1} = x(t),\ U_0(t) = y(t) \text{ if } \operatorname{ord}(y'(t)) \geq \operatorname{ord}(x'(t))$$

$$u_{-1} = y, u_0 = x,\ U_{-1} = y(t),\ U_0(t) = x(t) \text{ if } \operatorname{ord}(y'(t)) < \operatorname{ord}(x'(t))$$

Then for $i \geq 1$

$$u_i = \frac{du_{\beta_i}}{du_{\alpha_i}},\quad U_i(t) = \frac{U'_{\beta_i}}{U'_{\alpha_i}}(t),\quad \alpha, \beta \in \{-1, 0, \ldots, i-1\},$$

with $\alpha_1 = -1,\ \beta_1 = 0$ and $\alpha_{\geq 2}, \beta_{\geq 2}$ given by

$$\alpha_i = \alpha_{i-1},\ \beta_i = i-1 \text{ if } \operatorname{ord}(U'_{i-1}(t)) \geq \operatorname{ord}(U'_{\alpha_i - 1}(t));$$

$$\alpha_i = i-1,\ \beta_i = \alpha_{i-1} \text{ if } \operatorname{ord}(U'_{i-1}(t)) < \operatorname{ord}(U'_{\alpha_i - 1}(t))$$

## 2.4 Example

In this section we compute the blow ups and prolongations necessary to transform the curve singularity $(t^3, t^8)$ into a smooth, immersed curve. We note that after three iterations of each method, the resulting curves are non-singular. We begin with blow up.

### 2.4.1 Blow up

**Example 1. Blow up:** $(t^3, t^8)$

The curve $(t^3, t^8)$ has a singularity at the origin. We will work with the defining equation $x^8 = y^3$. Let the coordinates for $\mathbb{C}^2$ be given by $x, y$ and the affine coordinates for $\mathbb{C}P^1$ be given by $u_1, v_1$. Recall that $\mathbb{C}P^1$ is covered by two affine charts corresponding to $u \neq 0$ and $v \neq 0$ for homogeneous coordinates $[u; v]$ on $\mathbb{C}P^1$. We take $u_1 = \frac{u}{v}$ and $v_1 = \frac{v}{u}$, in the respective charts. We coordinatize the blow up of $\mathbb{C}^2$ at the origin by $X_1 = \{([u, v], x, y) \in \mathbb{C}P^1 \times \mathbb{C}^2 | \; xv = yu\}$. In the affine chart $[1, v_1]$, we have $u_1 = 1$ and $v_1 = \frac{v}{u}$, so that $y = xv_1$. The defining equation $x^8 = y^3$ becomes $(xv_1)^3 - x^8 = 0$, or $x^3(x^5 - v_1^3) = 0$. In the other chart $[u_1, 1]$ the equation of the curve is $y^3(y^5 u_1^8 - 1) = 0$. These are the equations for the total transforms in their respective charts. The proper transform is the part of the curve defined by setting the expression in the parentheses equal to zero. The factor outside the parentheses represents the exceptional divisor. Since the singularity of the proper transform lies at the origin of the first chart with coordinates $(x, v_1)$, we blow up there.

Coordinates for the surface $X_1$ near the singular point of the blown-up curve are $x, v_1$. Let $u_2, v_2$ be affine coordinates for the new $\mathbb{CP}^1$ corresponding to the second exceptional divisor. The blow up of $X_1$ at the singular point is given by $X_2 = \{([u,v], x, v_1) \mid xv = v_1 u\}$, locally over the singular point in $X_1$. In the affine chart $[1, v_2]$, we have $u_2 = 1$ so that $v_1 = xv_2$. The defining equation for the total transform becomes $x^3(x^5 - (xv_2)^3) = x^6(x^2 - v_2^3)$. In the chart $[u_2, 1]$, the defining equation is $u_2^3 v_1^6 (u_2^5 v_1^2 - 1)$. The singularity of the proper transform lies in the first chart and we blow up again.

$X_3 = \{([u,v], x, v_2) \mid xv = uv_2\}$, locally over the singular point in $X_2$. In the affine chart $[1, v_3]$, the equation of the curve is $x^8(1 - xv_3^3) = 0$, and in the chart $[u_3, 1]$, the equation is $v_2^8 u_3^6 (u_3^2 - v_2) = 0$. Now, the curve is immersed. We record this information in the following table.

Table 2.1: The blow up of the curve $c : x(t) = t^3, y(t) = t^8$.

| Blow up | total transform | proper transform |
|---|---|---|
| $B^1(c)$ | $x^3(x^5 - v_1^3)$ | $(x^5 - v_1^3)$ |
| $B^2(c)$ | $x^6(x^2 - v_2^3)$ | $(x^2 - v_2^3)$ |
| $B^3(c)$ | $v_2^8 u_3^6 (u_3^2 - v_2)$ | $(u_3^2 - v_2)$ |

### 2.4.2 Prolongation

**Example 2. Prolongation:** $(t^3, t^8)$

Table 2.2: The prolongation of the curve $c: x(t) = t^3, y(t) = t^8$.

| Prol | KR coord | KR function |
|---|---|---|
| $c^1$ | $u_1 = \frac{dy}{dx}$ | $U_1(t) = c_1 t^5$ |
| $c^2$ | $u_2 = \frac{du_1}{dx}$ | $U_2(t) = c_2 t^2$ |
| $c^3$ | $u_3 = \frac{dx}{du_2}$ | $U_3(t) = c_3 t$ |

The curve $x^8 = y^3$ is parameterized as $c(t) = (t^3, t^8)$. Introduce the fiber coordinate $u_1$ on $P^1(\mathbb{C}^2)$ by setting $[dx, dy] = [1, u_1]$ which is to say $u_1 = dy/dx$ is the slope of the tangent curve. We compute with the coordinates $x, y, u_1 = dy/dx$ its first prolongation $c^1$ is $(t^3, t^8, c_1 t^5)$ which is singular. For the second prolongation $c^2$, introduce the fiber coordinate $u_2$ on $P^2(\mathbb{C}^2)$, near the point $c^2(0) = (0, 0, 0, \text{span}\,\{\partial/\partial x\})$ by setting $[dx, du_1] = [1, u_2]$ which is to say $u_2 = du_1/dx$. In the coordinates $(x, y, u_1, u_2)$ the second prolongation $c^2$ is given by $(t^3, t^8, c_1 t^5, c_2 t^2)$. Again this curve is singular. Its third prolongation $c^3$ is immersed. To see this introduce the fiber coordinate $u_3$ on $P^3(\mathbb{C}^2)$, near the point $c^3(0) = (0, 0, 0, 0, \text{span}\,\{\partial/\partial x\})$ by setting $[dx, du_2] = [u_3, 1]$ which is to say $u_3 = dx/du_2$. In the coordinates $(x, y, u_1, u_2, u_3)$ the third prolongation $c^3$ is given by $(t^3, t^8, c_1 t^5, c_2 t^2, c_3 t)$, where the $c_i$'s are nonzero constants.

This example illustrates:

**Proposition 2.4.1** (Nobile [7], see also [8])**.** *Let $c$ be an analytic plane curve germ. Then there is a finite number $j$ such that the jth prolongation $c^j \subset P^j(\mathbb{C}^2)$ is a nonsingular curve germ.*

# Part III

# Third Part

# Chapter 3

# Isomorphism of Resolution Graphs

## 3.1 Resolution by Prolongation

### 3.1.1 Critical Curves

Proposition 2.4.1 guarantees that the $j$th prolongation $c^j$ of an analytic plane curve germ $c$ is non-singular for large enough $j$. If we stopped at the first such $j$ and simply compared this prolonged curve $c^j(t)$ with the resolution of $c$ by blow up (its proper transform) our story would be uninteresting. We would have two immersed curves, albeit in spaces of different dimensions. Any two immersed curve germs are equivalent from the viewpoint of local analytic geometry: they look like one of the coordinate axes in some coordinate system. We would not have anything interesting to compare beyond how many steps are required to resolution in the two cases.

What makes blow up of a planar curve germ $c$ interesting is the exceptional curves. These are projective lines added to the curve with each blow up. Together with the proper transform of $c$ they form a "multi-curve" in a surface : a finite union of curves in a surface whose union is the resolved curve of blow up. The intersections among the components of this multi-curve define a graph intrinsically related to the curve germ. *It is this graph that we want to "see" in prolongation.* In order to see it we need prolongation analogues of the exceptional fibers. These analogues are called "critical curves".

**Definition 4.** *A critical curve in $P^j(\mathbb{C}^2)$, $j > 0$ is an embedded integral curve for $\Delta_j$ whose projection to the plane $\mathbb{C}^2$ is a constant curve.*

The simplest critical curves are the vertical curves.

**Definition 5.** *The vertical curves at level $j$ are the fibers of the projection $P^j(\mathbb{C}^2) \to P^{j-1}(\mathbb{C}^2)$. Such a curve will be denoted $V_j$.*

**Remark 1** (Warning)**.** The definitions of critical curve and of vertical curves which we have just given differ at level 1 from the definitions in [8]. In [8] we do not consider the vertical curves $V_1$ at level 1, or its prolongations, to be critical curves. All the other $V_j, j > 1$ and their prolongations comprise the critical curves of [8]. See section 1 for more on this difference.

We can view a vertical curve as the prolongation of the point over which it lies. For example, think of the origin in $\mathbb{C}^2$ as the image of the constant curve $t \mapsto 0$. Every line through 0 is tangent to this curve, so the prolongation of 0, viewed as a constant curve, is the vertical curve over 0, i.e. the fiber over 0 for the fibration $P(\mathbb{C}^2) \to \mathbb{C}^2$. It is a copy of $\mathbb{CP}^1$ in $P(\mathbb{C}^2)$.

**Definition 6.** *A tangency curve is the prolongation* $(V_i)^j$, $j > 0$ *of some vertical curve* $V_i$, $i \geq 1$.

For the rationale between the terminology "tangency" see [8].

**Proposition 3.1.1.** *Let* $\gamma$ *be a critical curve. Then* $\gamma$ *is either a vertical curve, or a tangency curve. Tangency curves are not vertical curves.*

*Proof.* The proof of the corresponding proposition in [8] holds here. (See the Remark 1, and section 1 concerning the difference between critical curves there and here).

## 3.2 Resolution Graphs

Start with our singular curve $c \subset \mathbb{C}^2$. Add to the prolongation $c^1$ of $c$ the prolongation of each of $c$'s singular points of $c$, in this way adding a finite collection of vertical curves to the old prolongation $c^1$. The resulting collection of curves is called the full (first) prolongation

Iterate this construction, forming $P(c), P^2(c), \ldots, P^j(c) \subset P^j(\mathbb{C}^2)$. The different branches of $P^j(c)$ consist of the old prolongation $c^j$ and a finite collection of critical curves.

Consider the case of a unibranched curve germ $c = c(t)$. Suppose that its $j-1$st prolongation $c^{j-1}$ is singular, with singular point $p_{j-1} = c^{j-1}(0)$. When we prolong again to form $c^j$ we must add the vertical curve $V_j = p_j^1$ at level $j$. We carry along with us the previously introduced critical curves, by prolonging them. Since we add exactly one new critical curve upon each prolongation, $P^j(c)$ consists of $j+1$ curves, these being $c^j$ and the $j$ critical curves, $V_j, V^1_{j-1}, V^2_{j-1}, \ldots, V^{j-1}_1$.

At what step $j$ do we declare the multi-curve $P^j(c)$ to be "resolved"?

**Definition 7.** *A finite collection of embedded integral curves for a rank 2 distribution $D$ is said to form a 'normal system" if, whenever two curves intersect at a point $p$ , their tangent lines intersect transversally within $D(p)$, and no three curves intersect at a single point.*

**Remark 2** (Normal Crossing is a Normal System)**.** A collection of curves with a normal crossing singularity forms a normal system, by taking the distribution $D$ of 7 to be the whole tangent bundle of $X_k$

If $c^j$ is tangent to $V_j$, or to a $V^i_{j-i}$ then we will count that tangency point as a critical point even if $c^j$ is immersed. Similarly if $c^j$ is immersed but forms a triple point with two of the critical curves, we count that triple point as a singular point and we continue the prolongation process as before. (If two critical curves intersect, then their intersection is transverse within $\Delta_j$, so that tangencies between critical curves cannot occur.)

**Definition 8.** *We will say that the unibranched singularity $c$ has been resolved by prolongation when $c^j$ is immersed and $P^j(c) = c^j \cup V_j \cup V^1_{j-1} \cup \ldots \cup V^{j-1}_1$ forms a normal system of curves for $\Delta_j$.*

**Theorem 3.2.1.** *Any well-parameterized curve germ can be resolved by prolongation in a finite number $r$ of steps.*

*Proof.* The theorem is almost completely proved in [8]. We prove there that for any such curve germ $c$, there is a finite number $k$ (depending only on $c$'s Puiseux characteristic ) such that after $k$ prolongations $c^k$ becomes immersed and regular, where "regular" means that $c^k$ is not tangent to a critical curve. For $j < k$ the $c^k$ are either not immersed, or are tangent to a critical direction. At step $k$ we are in a situation identical to the penultimate step in the cubic cusp example just presented: $c^k$ forms a triple intersection with the vertical curve and a tangency curve. One more prolongation yields the resolution in the sense of Definition 8. Thus the $r$ of Theorem 3.2.1 is $k + 1$ where $k$ is the regularization number of [8].

## 3.3 Main theorem

Definition 8 was made in analogy with the definitions in blow up where the $j$th blow up $B^j(c)$ of $c$ consists of the proper transform of $c$, and $j$ exceptional curves $E_1, \ldots, E_j$, each one an embedded $\mathbb{CP}^1$. These curves all lie on a surface $X_j$. We declare the curve to be resolved by blow up when the component curves of $B^j(c)$ form a normal system for the tangent bundle $D = T(X_j)$, in the sense of Definition 8.

There is a standard way to draw a graph, associated to blow up, sometimes called the "dual graph", which encodes the combinatorial relationships among component curves in resolution by blow up. (See for example [5]. The dual graph is dual to the 'diagram' there.) The vertices of the dual graph are the exceptional curves. Two vertices are connected by an edge if and only if they intersect. Finally, there is an arrow representing the proper transform and this arrow connects to the exceptional curve to which it intersects, this curve always being the last occurring exceptional curve.

If we associate a graph to the normal system in resolution by prolongation in this same way, we will obtain at a dual graph having little relation to the graph for blow up. What one finds is that most of the critical curves intersect no other critical curves within the prolongation, and hence most of the critical curves correspond to isolated vertices. On the other hand, in blow up, every exceptional curve intersects some other exceptional curve, and one finds that the blow up graph is connected. The discrepancy between the two graphs is a direct consequence of the difference between what happens to a pair of transversally intersecting curves when we prolong, versus when we blow up. When we prolong two integral curves which intersect transversally within $\Delta_j$, the resulting curves do not intersect at all. On the other hand, when we blow up two curves which intersect transversally at *any point besides the point which is the center of the blow up operation*, then the resulting curves continue to intersect transversally.

To get the correct diagram for prolongation we must alter our definition of what it means for two component curves to "intersect". We call the new relation of intersection "incidence". To present the definition of incidence, we first introduce a notational labelling conventions for the critical curves of $P^j(c)$.

**Notational convention.** Let $V_r^{j-r}$ be one of the critical curves comprising $P^j(c)$. We will use the symbol $V_r$ to denote this curve, viewed at any level of the monster. Since the curve first arises at level $r$, it is declared to be the empty curve when viewed at levels $k < r$, that is , we declare, for incidence counting purposes, that $V_r$, viewed in $P^k(\mathbb{C}^2), k < r$ is the empty curve. When we view $V_r$ at level $k > r$ we mean the $k - r$-fold prolongation $V_r^{k-r}$ of $V_r$. Similarly, when we say we are viewing $c^j$ at level $i < j$ we are speaking of the $i$th prolongation $c^i$.

Using this notation, we have that $P^j(c) = c^j \cup V_j \cup V_{j-1} \cup \ldots \cup V_1$.

**Definition 9.** *We declare that two component curves $A, B$ of $P^j(c)$ to be incident if, for some $i \leq j$ they intersect normally within $P^i(c)$. In other words, when viewed at level $i \leq j$ the curves $A$ and $B$ intersect transversally at some point $q \in P^i(\mathbb{C}^2)$, and no other component of $P^i(c)$ passes through $q$.*

### 3.3.1 Main Result

**Theorem 3.3.1** (Main Theorem)**.** *Let $C$ be a plane curve singularity consisting of a single branch. Consider two graphs associated to $C$, one for $C$'s resolution by blow up, the other graph for $C$'s resolution by prolongation. Use definitions for 7 and 9 for resolution by prolongation. Then these two labelled graphs are isomorphic. In particular the number of steps to resolution by blow up is equal to the number of steps to resolution by prolongation, these numbers being the number of verticies for each graph.*

This theorem is proved in our paper [9].

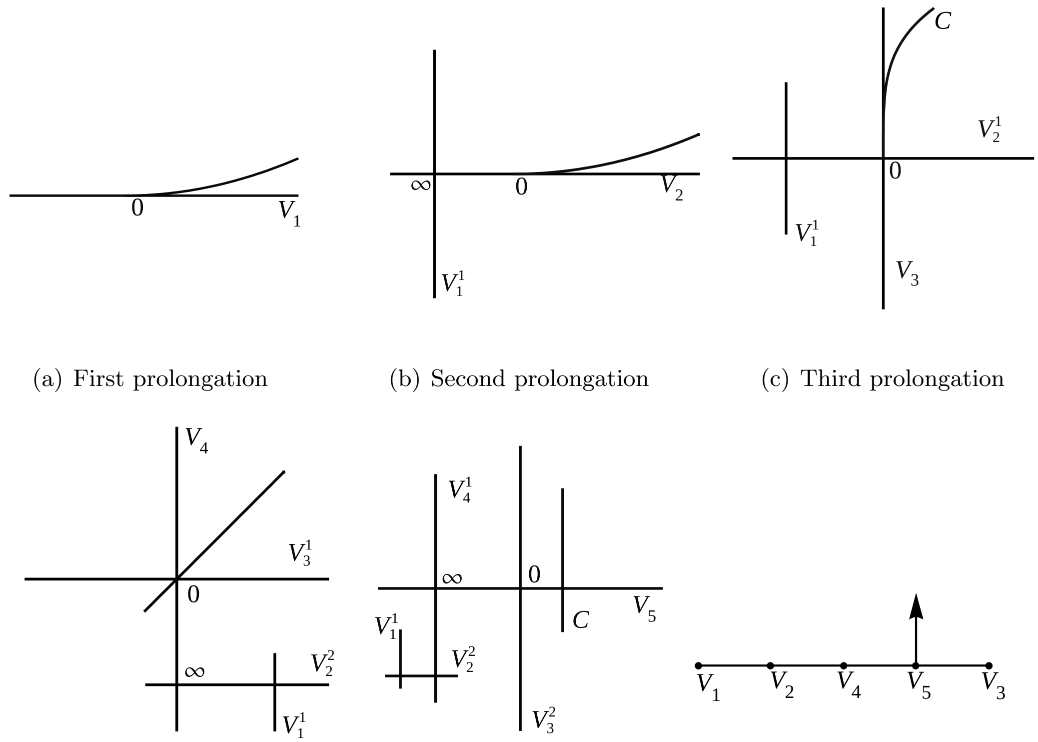
### 3.3.2 $(t^3, t^8)$ an example for which both prolongation and blow up have isomorphic resolution graphs

#### 3.3.2.1 Prolongation

**Example 3** (Resolution by prolongation)**.** We return to the curve $c(t) = (x(t), y(t)) = (t^3, t^8)$. We saw (Example 2) that its first prolongation is coordinatized as $(t^3, t^8, c_1t^5)$ with the last coordinate representing $u_1 = dy/dx$. In these coordinates the vertical curve $V_1$ is $(0, 0, t)$. We see that $c^1$ is tangent to the vertical curve and still singular. See Figure 3.1(a). So we must prolong again. This is done by introducing the new coordinate $u_2 = du_1/dx$ which represents a fiber affine coordinate on $P^2(\mathbb{C}^2) \to P^1(\mathbb{C}^2)$. In the $x, y, u_1, u_2$ coordinates we find that $c^2 = (t^3, t^8, c_1t^5, c_2t^2)$ while $V_1^1 = (0, 0, t, \infty)$, and $V_2$, the new vertical curve is given by $(0, 0, 0, t)$. The vertical curve $V_2$ is incident to the tangency curve $V_1^1$ by Definition 9, since both the curves intersect normally and since $c^2$ does not pass through their point of intersection. See Figure 3.1(b). We carry along the tangency curve $V_1^1$ through further prolongations by recording the intersection of $V_1^1$ with $V_2$ in the diagrams to come. The distribution $\Delta_2$ at level 2 is given in these coordinates by $dy - u_1dx = 0$ and $du_1 - u_2dx = 0$. The second prolongation $c^2$ is still singular and at $t = 0$ is tangent to the vertical curve $V_2$, since $u_2$ is the lower order coordinate of $u_1, u_2$. So we prolong again. We introduce the new coordinate $u_3 = \frac{dx}{du_2}$. In the $x, y, u_1, u_2, u_3$ coordinates we find that $c^3 = (t^3, t^8, c_1t^5, c_2t^2, c_3t)$ while $V_2^1 = (0, 0, 0, t, 0)$, and $V_3$, the new vertical curve is given by $(0, 0, 0, 0, t)$. See Figure 3.1(c). Although $V_3$ and $V_2^1$ intersect normally at a point $q$, the prolonged curve

$c_3$ also passes through $q$, so by Definition 9, $V_3$ and $V_2^1$ are not incident. We see that $c^3$, though immersed, is tangent to the vertical curve. So we must prolong again.

This is done by introducing the new coordinate $u_4 = du_2/du_3$ which represents a fiber affine coordinate on $P^4(\mathbb{C}^2) \to P^3(\mathbb{C}^2)$. In the $x, y, u_1, u_2, u_3, u_4$ coordinates we find that $c^4 = (t^3, t^8, c_1t^5, c_2t^2, c_3t, c_4t)$ while $V_3^1 = (0, 0, 0, 0, t, 0)$, the tangency curve $V_2^2 = (0, 0, 0, t, 0, \infty)$ and $V_4$, the new vertical curve is given by $(0, 0, 0, 0, 0, t)$. See Figure 3.1(d). The vertical curve $V_4$ is incident to $V_2^2$. We carry along $V_2^2$ through further prolongations by recording its intersection with $V_4$. The vertical curve $V_4$ is not incident to $V_3^1$, since $V_4$, $V_3^1$, and $c^4$ intersect in a point. All three curves pass through the coordinate origin, and their tangents form three distinct lines, $du_3 = 0, du_4 = 0$ and $du_3 = du_4$ within $\Delta_2(0, 0, 0, 0, 0, 0)$. We have a triple intersection. One more prolongation is required to resolve the singularity according to the definition. We find that $P^5(c) = c^5 \cup V_3^2 \cup V_4^1 \cup V_5$. At level 5, we have that $c^5$ and $V_5$ intersect transversally, and $c^5$ intersects none of the other curves $V_3^2$ and $V_4^1$. Thus the component curves form a normal system and the singularity is resolved. The configuration of the component curves of $P^5(c)$ is depicted with a resolution diagram in Figure 3.1(e). To obtain the dual graph, we represent each of the component curves with a vertex, and an edge joining two verticies when the component curves are incident. An arrow stemming from a vertex indicates that the prolonged curve intersects the component curve represented by the vertex. The dual graph is depicted in Figure 3.1(f).




(a) First prolongation (b) Second prolongation (c) Third prolongation

(d) Fourth prolongation (e) Fifth prolongation (f) Dual graph

Figure 3.1: The Prolongations of $(t^3, t^8)$

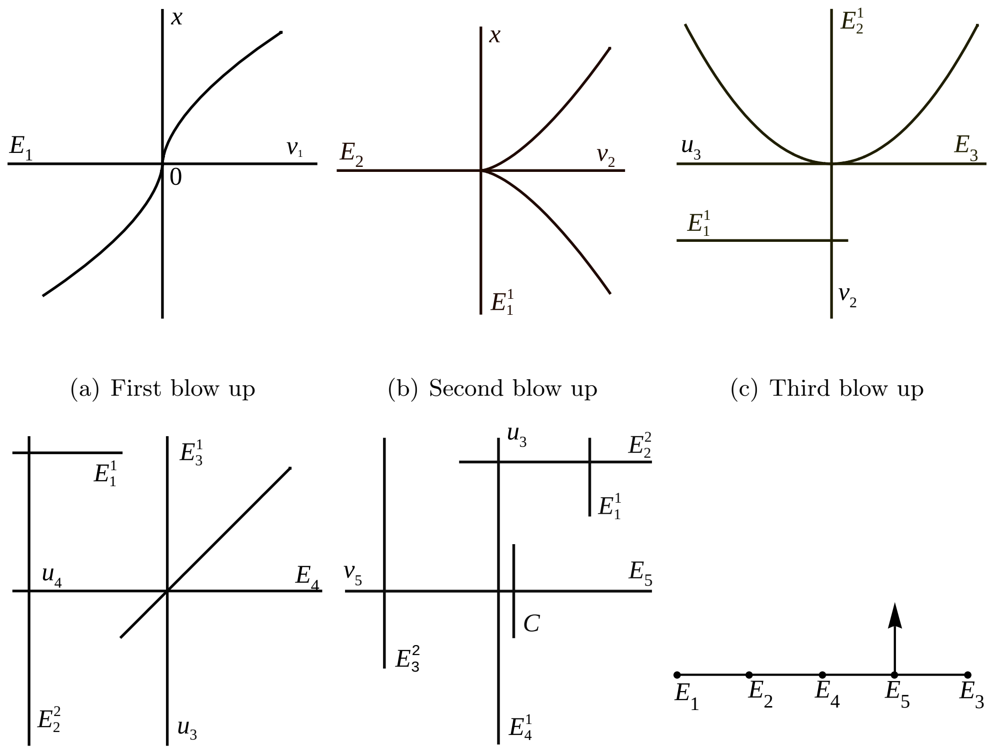
#### 3.3.2.2 Blow up

In this section, we repeatedly blow up the singular point of the curve singularity $(t^3, t^8)$ until we reach an immersed curve with normal crossings with the exceptional curves. We demonstrate the construction of the resolution diagrams and resolution graphs for blow up, see [5], [10] for more details.

**Example 4** (Resolution by blow up)**.** We return to the curve $(t^3, t^8)$ with defining equation $x^8 = y^3$. We saw in (Example 1) that after the first blow up, the total transform in the chart containing the singular point is given by the equation $x^3(x^5 - v_1^3) = 0$. The factor outside the parentheses represents the exceptional curve. Each exceptional curve has a multiplicity. The multiplicity of the exceptional curve is the exponent on the factor representing the exceptional curve. The multiplicity helps to keep track of which exceptional curves appear locally near the singular point. After the first blow up, we have an exceptional curve with multiplicity 3, denoted by $E_1$. The coordinates for the blown up surface near the singular point are $(x, v_1)$. The exceptional curve is given by the equation $x = 0$, which is the $v_1$-axis. See Figure 3.2(a). We blow up again at the point $(x, v_1) = (0, 0)$. The singular point of the resulting curve lies in the chart $[1, v_2]$, where the total transform has defining equation $x^6(x^2 - v_2^3)$. Since the multiplicity of $x$ is 6, the exceptional fiber $E_2$, given by $x = 0$, has multiplicity equal to 6. The exceptional divisor is marked by its multiplicity. In the other affine chart, the equation $u_2 = 0$ with multiplicity 3 gives us the blow up of $E_1$, which we now call $E_1^1$, still with multiplicity equal to 3. The singular point of the proper transform lies in the first chart and we blow up again. See Figure 3.2(b).

In the chart $[u_3, 1]$, the equation is $v_2^8 u_3^6 (u_3^2 - v_2) = 0$. Our new exceptional divisor $E_3$ has multiplicity equal to 8, and $E_2^1$, the blow up of $E_2$ still with multiplicity equal to 6. Now, the curve is immersed, but we continue until our proper transform is transverse to all of the exceptional divisors, and we have no triple intersection points. See Figure 3.2(c). We blow up again. In the chart $[u_4, 1]$, we have $u_3^{15} u_4^8 (u_3 - u_4) = 0$. Our new exceptional divisor $E_4$ has multiplicity equal to 15. The second chart contains both of the exceptional curves $E_4$ and $E_2^2$. In the chart containing the origin, all three curves intersect at the origin, we call this a triple intersection point and we must blow up there one last time. See Figure 3.2(d).

In the affine chart $[1, v_5]$, the equation of the curve is $u_3^{24} v_5^8 (1 - v_5) = 0$. In the chart $[u_5, 1]$ the equation is $u_4^{24} u_5^{15} (u_5 - 1) = 0$. This chart contains the exceptional curve $E_5$ with multiplicity 24, as well as $E_4^1$ with multiplicity 15. In either chart the proper transforms are smooth curves and intersect the corresponding exceptional curves transversally. In both charts the total transforms have only have normal crossing singularities, so we have reached a good resolution in five steps. The resolution diagram is depicted in Figure 3.2(e). We can also describe the configuration of curves with the dual graph: we have five verticies, one for each exceptional curve, and four edges, one for each transverse intersection between the exceptional curves. We have an arrow stemming from the vertex which represents $E_5$, to indicate that the proper transform intersects the exceptional curve $E_5$. The number of edges of the dual graph represents the number of transverse intersections of the exceptional curves. See Figure 3.2(f).




(a) First blow up (b) Second blow up (c) Third blow up

(d) Fourth blow up (e) Fifth blow up (f) Dual graph

Figure 3.2: The Blow ups of $(t^3, t^8)$

# Chapter 4

# Examples and Figures

## 4.1 $A_{2k}$ singularity $(t^2, t^7)$,and $(t^2, t^{2k+1})$.

### 4.1.1 Blow up of $(t^2, t^7)$

**Example 5.** The curve $(t^2, t^7)$ has a singularity at the origin. We will work with the defining equation $x^7 = y^2$. Let the coordinates for $\mathbb{C}^2$ be given by $x, y$ and the affine coordinates for $\mathbb{C}P^1$ be given by $u_1, v_1$. We take $u_1 = \frac{u}{v}$ and $v_1 = \frac{v}{u}$ for homogeneous coordinates $[u; v]$ on $\mathbb{C}P^1$. We coordinatize the blow up of $\mathbb{C}^2$ at the origin by $X_1 = \{([u, v], x, y) \in \mathbb{C}P^1 \times \mathbb{C}^2 | \ xv = yu\}$. In the affine chart $[1, v_1]$, we have $u_1 = 1$ and $v_1 = \frac{v}{u}$, so that $y = xv_1$. The defining equation $x^7 = y^2$ becomes $x^2(x^5 - v_1^2) = 0$. In the other chart $[u_1, 1]$ the equation of the curve is $y^2(y^5 u_1^7 - 1) = 0$. These are the equations for the total transforms in their respective charts. Recall that the proper transform is the part of the curve defined by setting the expression in the parentheses equal to zero and the factor outside the parentheses represents the exceptional curve.

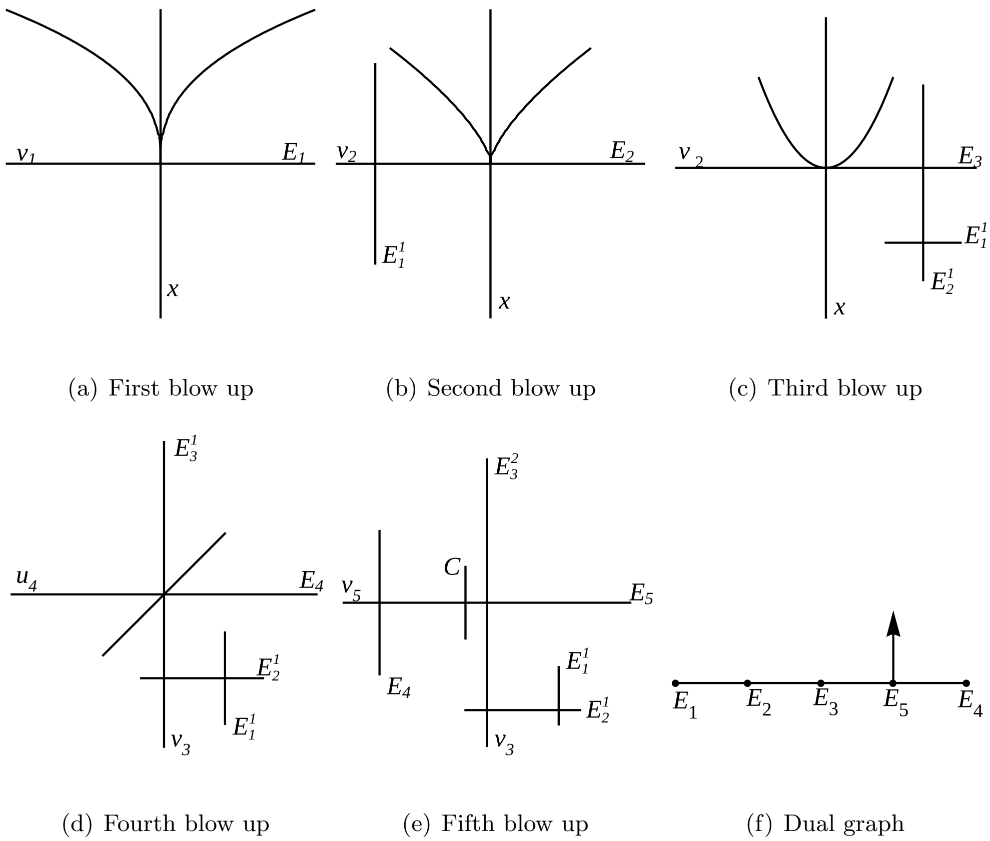
The multiplicity of the exceptional curve is the exponent on the factor representing the exceptional curve. After the first blow up, we have an exceptional curve with multiplicity 2, denoted by $E_1$. The coordinates for the blown up surface near the singular point are $(x, v_1)$. The exceptional curve is given by the equation $x = 0$, which is the $v_1$-axis. Since the singularity of the proper transform lies at the origin of the first chart, we blow up there. See 4.1(a).

Coordinates for the surface $X_1$ near the singular point of the blown-up curve are $x, v_1$. Let $u_2, v_2$ be affine coordinates for the new $\mathbb{CP}^1$ corresponding to the second exceptional divisor. The blow up of $X_1$ at the singular point is given by $X_2 = \{([u, v], x, v_1) \; |xv = v_1 u\}$, locally over the singular point in $X_1$. In the affine chart $[1, v_2]$, we have $u_2 = 1$ so that $v_1 = xv_2$. The defining equation for the total transform becomes $x^4(x^3 - v_2^2) = 0$. In the chart $[u_2, 1]$, the defining equation is $u_2^2 v_1^4 (u_2^5 v_1^3 - 1) = 0$. The singular point of the resulting curve lies in the chart $[1, v_2]$. Since the multiplicity of $x$ is 4, the exceptional fiber $E_2$, given by $x = 0$, has multiplicity equal to 4. In the other affine chart, the equation $u_2 = 0$ with multiplicity 2 gives us the blow up of $E_1$, which we now call $E_1^1$, with multiplicity equal to 2. The singular point of the proper transform lies in the first chart and we blow up again. See Figure 4.1(b).

$X_3 = \{([u, v], x, v_2) \; |xv = uv_2\}$, locally over the singular point in $X_2$. In the affine chart $[1, v_3]$, the equation of the curve is $x^6(x - v_3^2) = 0$, and in the chart $[u_3, 1]$, the equation is $v_2^6 u_3^4 (v_2 u_3^3 - 1) = 0$. Our new exceptional divisor $E_3$ has multiplicity equal to 6, and $E_2^1$, the blow up of $E_2$ still with multiplicity equal to 4. Now, the curve is immersed, but we continue until our proper transform is transverse to all of the exceptional curves, and we have no triple intersection points. See Figure 4.1(c).

We blow up again. In the chart $[u_4, 1]$, we have $v_3^7 u_4^6 (u_4 - v_3) = 0$. This chart contains both our new exceptional divisor $E_4$ with multiplicity equal to 7, as well as $E_3^1$ with multiplicity 6. In the chart $[1, v_4]$, we have $x^7(1 - v_4^2 x) = 0$. In the first chart, which contains the origin, all three curves intersect at the origin, and we call this a triple intersection point. We must blow up there one last time. See Figure 4.1(d).

In the affine chart $[1, v_5]$, the equation of the curve is $v_3^{14} v_5^6 (v_5 - 1) = 0$. In the chart $[u_5, 1]$ the equation is $u_4^{14} u_5^7 (1 - u_5) = 0$. The affine chart $[1, v_5]$ contains the exceptional curve $E_5$ with multiplicity 14, as well as $E_3^2$ with multiplicity 6. The affine chart $[u_5, 1]$ contains the new exceptional curve $E_5$ and the old $E_4^1$ with multiplicity 7. In either chart the proper transforms are smooth curves and intersect the corresponding exceptional curves transversally. In both charts the total transforms have only have normal crossing singularities, so we have resolved the curve singularity in five steps. The resolution diagram is depicted in Figure 4.1(e). The dual graph has five verticies, one for each exceptional curve, and four edges, one for each transverse intersection between the exceptional curves. We have an arrow stemming from the vertex which represents $E_5$, to indicate that the proper transform intersects the exceptional curve $E_5$. See Figure 4.1(f). We record the proper and total transforms in a table, see Table 4.1.




(a) First blow up

(b) Second blow up

(c) Third blow up

(d) Fourth blow up

(e) Fifth blow up

(f) Dual graph

Figure 4.1: The Blow ups of $(t^2, t^7)$

Table 4.1: The blow up of the curve $c : x(t) = t^2, y(t) = t^7$.

| Blow up | total transform | proper transform |
|---|---|---|
| $B^1(c)$ | $x^2(x^5 - v_1^2)$ | $(x^5 - v_1^2)$ |
| $B^2(c)$ | $x^4(x^3 - v_2^2)$ | $(x^3 - v_2^2)$ |
| $B^3(c)$ | $x^6(x - v_3^2)$ | $(x - v_3^2)$ |
| $B^4(c)$ | $v_3^7 u_4^6(u_4 - v_3)$ | $(u_4 - v_3)$ |
| $B^5(c)$ | $v_3^{14} v_5^6(v_5 - 1)$ | $(v_5 - 1)$ |

### 4.1.2 Prolongation of $(t^2, t^7)$

**Example 6.** The curve $x^7 = y^2$ is parameterized as $c(t) = (t^2, t^7)$. We introduce the fiber coordinate $u_1$ on $P^1(\mathbb{C}^2)$ by setting $[dx, dy] = [1, u_1]$ where $u_1 = dy/dx$ is the slope of the tangent curve. We compute with the coordinates $x, y, u_1 = dy/dx$ its first prolongation $c^1$ is $(t^2, t^7, c_1 t^5)$. In these coordinates the vertical curve $V_1$ is $(0, 0, t)$. The prolonged curve $c^1$ is tangent to the vertical curve and still singular. See Figure 4.2(a). We must prolong again.

We introduce the new coordinate $u_2 = du_1/dx$ which represents a fiber affine coordinate on $P^2(\mathbb{C}^2) \to P^1(\mathbb{C}^2)$. In the $x, y, u_1, u_2$ coordinates we find that $c^2 = (t^2, t^7, c_1 t^5, c_2 t^3)$ while $V_1^1 = (0, 0, t, \infty)$, and $V_2$, the new vertical curve is given by $(0, 0, 0, t)$. The vertical curve $V_2$ is incident to the tangency curve $V_1^1$ by Definition 9, since both the curves intersect normally and since $c^2$ does not pass through their point of intersection. See Figure 4.2(b). We carry along the tangency curve $V_1^1$ through further prolongations by recording the intersection of $V_1^1$ with $V_2$ in the diagrams to come. The distribution $\Delta_2$ at level 2 is given in these coordinates by $dy - u_1 dx = 0$ and $du_1 - u_2 dx = 0$. The second prolongation $c^2$ is still singular. At $t = 0$, $c^2$ is tangent to the vertical curve $V_2$, since $u_2$ is the lower order coordinate of $u_1, u_2$. So we prolong again. We introduce the fiber coordinate $u_3$ on $P^3(\mathbb{C}^2)$, near the point $c^3(0)$ by setting $[dx, du_2] = [1, u_3]$ which is to say $u_3 = du_2/dx$. In the coordinates $(x, y, u_1, u_2, u_3)$ the third prolongation $c^3$ is given by $(t^2, t^7, c_1 t^5, c_2 t^3, c_3 t)$, where the $c_i$'s are nonzero constants. The tangency curve $V_2^1$ is given by $V_2^1 = (0, 0, 0, t, \infty)$, and $V_3$, the new vertical curve is given by $(0, 0, 0, 0, t)$. See Figure 4.2(c). The vertical curve $V_3$ and the tangency curve $V_2^1$ are incident and we carry along $V_2^1$ through further prolongations by recording its last seen intersection with $V_3$. We see that $c^3$, though immersed, is tangent to the vertical curve. So we must prolong again.

This is done by introducing the new coordinate $u_4 = dx/du_3$ which represents a fiber affine coordinate on $P^4(\mathbb{C}^2) \to P^3(\mathbb{C}^2)$. In the $x, y, u_1, u_2, u_3, u_4$ coordinates we find that $c^4 = (t^2, t^7, c_1 t^5, c_2 t^3, c_3 t, c_4 t)$ while $V_3^1 = (0, 0, 0, 0, t, 0)$, the tangency curve $V_2^2 = (0, 0, 0, t, 0, \infty)$ and $V_4$, the new vertical curve is given by $(0, 0, 0, 0, 0, t)$. See Figure 4.2(d). The vertical curve $V_4$ intersects $V_3^1$ and $c^4$ in a point. All three curves pass through the coordinate origin, and their tangents form three distinct lines, $du_3 = 0, du_4 = 0$ and $du_3 = du_4$ within $\Delta_2(0, 0, 0, 0, 0, 0)$. We have a triple intersection. One more prolongation is required to resolve the singularity according to the definition. We find that $P^5(c) = c^5 \cup V_3^2 \cup V_4^1 \cup V_5$. At level 5, we have that $c^5$ and $V_5$ intersect transversally, and $c^5$ intersects none of the other curves $V_3^2$ and $V_4^1$. Thus the component

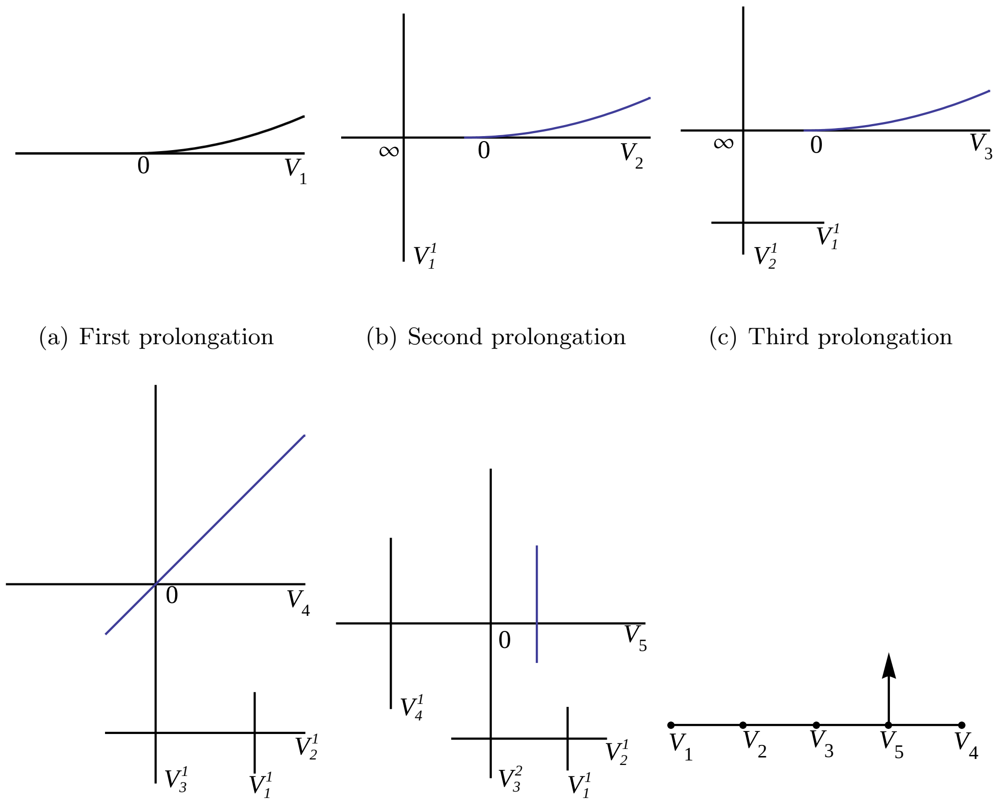
Table 4.2: The Prolongation of the curve $c : x(t) = t^2, y(t) = t^7$.

| Prol | KR coord | KR function |
|---|---|---|
| $c^1$ | $u_1 = \frac{dy}{dx}$ | $U_1(t) = c_1 t^5$ |
| $c^2$ | $u_2 = \frac{du_1}{dx}$ | $U_2(t) = c_2 t^3$ |
| $c^3$ | $u_3 = \frac{du_2}{dx}$ | $U_3(t) = c_3 t$ |
| $c^4$ | $u_4 = \frac{dx}{du_3}$ | $U_4(t) = c_4 t$ |
| $c^5$ | $u_5 = \frac{du_4}{du_3}$ | $U_5(t) = c_5$ |

curves form a normal system and the singularity is resolved. The configuration of the component curves of $P^5(c)$ is depicted with a resolution diagram in Figure 4.2(e). The dual graph contains 5 verticies, 4 edges and arrow stemming from the vertex representing $V_5$. The dual graph is depicted in Figure 4.2(f). We list the K-R coordinates and K-R coordinate functions corresponding to each of the prolongations in the Table. 4.2

### 4.1.3 Blow up of $(t^2, t^{2k+1})$

**Example 7.** We will work with the defining equation $x^{2k+1} = y^2$ and blow up the curve at the origin. Let the coordinates for $\mathbb{C}^2$ be given by $x, y$ and the affine coordinates for $\mathbb{C}P^1$ be given by $u_1, v_1$. We take $u_1 = \frac{u}{v}$ and $v_1 = \frac{v}{u}$ for homogeneous coordinates $[u; v]$ on $\mathbb{C}P^1$. We coordinatize the blow up of $\mathbb{C}^2$ at the origin by $X_1 = \{([u, v], x, y) \in \mathbb{C}P^1 \times \mathbb{C}^2 | \ xv = yu\}$. In the affine chart $[1, v_1]$, we have $u_1 = 1$ and $v_1 = \frac{v}{u}$, so that $y = xv_1$. The defining equation $x^{2k+1} = y^2$ becomes $x^2(x^{2k-1} - v_1^2) = 0$. In the other chart $[u_1, 1]$ the equation of the curve is $y^2(y^{2k-1}u_1^{2k+1} - 1) = 0$. These are the equations




(a) First prolongation (b) Second prolongation (c) Third prolongation

(d) Fourth prolongation (e) Fifth prolongation (f) Dual graph

Figure 4.2: The Prolongations of $(t^2, t^7)$

for the total transforms in their respective charts. After the first blow up, we have an exceptional curve with multiplicity 2, denoted by $E_1$. The coordinates for the blown up surface near the singular point are $(x, v_1)$. The exceptional curve is given by the equation $x = 0$, which is the $v_1$-axis. Since the singularity of the proper transform lies at the origin of the first chart, we blow up there.

Let $u_2, v_2$ be affine coordinates for the new $\mathbb{CP}^1$ corresponding to the second exceptional divisor. The blow up of $X_1$ at the singular point is given by $X_2 = \{([u, v], x, v_1) \mid xv = v_1 u\}$, locally over the singular point in $X_1$. In the affine chart $[1, v_2]$, we have $u_2 = 1$ so that $v_1 = xv_2$. The defining equation for the total transform becomes $x^4(x^{2k-3} - v_2^2) = 0$. In the chart $[u_2, 1]$, the defining equation is $v_2^2 v_1^4(v_2^{2k-1} v_1^{2k-3} - 1) = 0$. The singular point of the resulting curve lies in the chart $[1, v_2]$. Since the multiplicity of $x$ is 4, the exceptional fiber $E_2$, given by $x = 0$, has multiplicity equal to 4. In the other affine chart, the equation $u_2 = 0$ with multiplicity 2 gives us the blow up of $E_1$, which we now call $E_1^1$, with multiplicity equal to 2. The singular point of the proper transform lies in the first chart and we blow up again.

$X_3 = \{([u, v], x, v_2) \mid xv = uv_2\}$, locally over the singular point in $X_2$. In the affine chart $[1, v_3]$, the equation of the curve is $x^6(x^{2k-5} - v_3^2) = 0$, and in the chart $[u_3, 1]$, the equation is $v_2^6 u_3^4(v_2^{2k-5} u_3^{2k-3} - 1) = 0$. Our new exceptional divisor $E_3$ has multiplicity equal to 6, and $E_2^1$, the blow up of $E_2$ still with multiplicity equal to 4.

We continue in this fashion and note the pattern that the multiplicity of the exceptional curve increases by two and the degree of the proper transform decreases by two after every blow up. In the affine chart $[1, v_k]$, the equation of the curve is

$x^{2k}(x - v_k^2) = 0$. Our new exceptional divisor $E_k$ has multiplicity equal to $2k$. Now the curve is immersed but tangent to an exceptinal curve, so we blow up again.

$X_{k+1} = \{([u,v], x, v_k) \; |xv = uv_k\}$, locally over the singular point in $X_k$. In the affine chart $[1, v_{k+1}]$, the equation of the curve is $x^{2k+1}(1 - xv_{k+1}^2) = 0$, and in the chart $[u_{k+1}, 1]$, the equation is $v_k^{2k+1}u_{k+1}^{2k}(u_{k+1} - v_k) = 0$. Our new exceptional divisor $E_{k+1}$ has multiplicity equal to $2k + 1$, and $E_k^1$, the blow up of $E_k$ still with multiplicity equal to $2k$. The proper transform, and the exceptional curves all intersect in a point, this point is called a triple intersection point, and we must blow up one last time.

$X_{k+2} = \{([u,v], v_k, u_{k+1}) \; |v_k v = u_{k+1}u\}$, locally over the singular point in $X_{k+1}$. In the affine chart $[1, v_{k+2}]$, the equation of the curve is $v_k^{4k+2}v_{k+2}^{2k}(v_{k+2} - 1) = 0$, and in the chart $[u_{k+2}, 1]$, the equation is $u_{k+1}^{4k+2}u_{k+2}^{2k+1}(1 - u_{k+2}) = 0$. Our new exceptional divisor $E_{k+2}$ has multiplicity equal to $4k + 2$, and $E_{k+1}^1$, the blow up of $E_K$ with multiplicity equal to $2k + 1$. In either chart the proper transforms are smooth curves and intersect the corresponding exceptional curves transversally. In both charts the total transforms have only have normal crossing singularities, so we have reached a good resolution in $k + 2$ steps. We can describe the configuration of curves with the dual graph: we have $k + 2$ verticies, one for each exceptional curve, and $k + 1$ edges, one for each transverse intersection between the exceptional curves. We have an arrow stemming from the vertex which represents $E_{k+2}$, to indicate that the proper transform intersects the exceptional curve $E_{k+2}$. See Figure 4.3. We record the proper and total transforms in a table, see Table 4.3.

Table 4.3: The blow up of the curve $c : x(t) = t^2, y(t) = t^{2k+1}$.

| Blow up | total transform | proper transform |
|---|---|---|
| $B^1(c)$ | $x^2(x^{2k-1} - v_1^2)$ | $(x^{2k-1} - v_1^2)$ |
| $B^2(c)$ | $x^4(x^{2k-3} - v_2^2)$ | $(x^{2k-3} - v_2^2)$ |
| $B^3(c)$ | $x^6(x^{2k-5} - v_3^2)$ | $(x^{2k-5} - v_3^2)$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $B^k(c)$ | $x^{2k}(x - v_k^2)$ | $(x - v_k^2)$ |
| $B^{k+1}(c)$ | $v_k^{2k+1}u_{k+1}^{2k}(u_{k+1} - v_k)$ | $(u_{k+1} - v_k)$ |
| $B^{k+2}(c)$ | $v_k^{4k+2}v_{k+2}^{2k}(v_{k+2} - 1)$ | $(v_{k+2} - 1)$ |

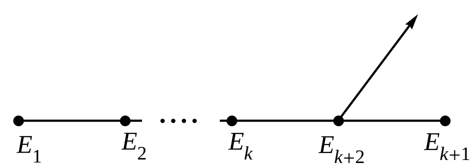



Figure 4.3: The dual graph for resolution by blow up of $(t^2, t^{2k+1})$

### 4.1.4 Prolongation of $(t^2, t^{2k+1})$

**Example 8.** The curve $x^{2k+1} = y^2$ is parameterized as $c(t) = (t^2, t^{2k+1})$. We introduce the fiber coordinate $u_1$ on $P^1(\mathbb{C}^2)$ by setting $[dx, dy] = [1, u_1]$ where $u_1 = dy/dx$ is the slope of the tangent curve. We compute with the coordinates $x, y, u_1 = dy/dx$ its first prolongation $c^1$ is $(t^2, t^{2k+1}, c_1 t^{2k-1})$. In these coordinates the vertical curve

$V_1$ is $(0,0,t)$. We see that $c^1$ is tangent to the vertical curve and still singular. See Figure 4.2(a). So we must prolong again.

We introduce the new coordinate $u_2 = du_1/dx$ which represents a fiber affine coordinate on $P^2(\mathbb{C}^2) \to P^1(\mathbb{C}^2)$. In the $x, y, u_1, u_2$ coordinates we find that $c^2 = (t^2, t^{2k+1}, c_1 t^{2k-1}, c_2 t^{2k-3})$ while $V_1^1 = (0,0,t,\infty)$, and $V_2$, the new vertical curve is given by $(0,0,0,t)$. The vertical curve $V_2$ is incident to the tangency curve $V_1^1$ by Definition 9, and we carry along the tangency curve $V_1^1$ through further prolongations by recording the intersection of $V_1^1$ with $V_2$ in the diagrams to come. The distribution $\Delta_2$ at level 2 is given in these coordinates by $dy - u_1 dx = 0$ and $du_1 - u_2 dx = 0$. The second prolongation $c^2$ is still singular. At $t = 0$, the prolonged curve $c^2$ is tangent to the vertical curve $V_2$, since $u_2$ is the lower order coordinate of $u_1, u_2$. So we prolong again. We introduce the fiber coordinate $u_3$ on $P^3(\mathbb{C}^2)$, near the point $c^3(0)$ by setting $[dx, du_2] = [1, u_3]$ which is to say $u_3 = du_2/dx$. In the coordinates $(x, y, u_1, u_2, u_3)$ the third prolongation $c^3$ is given by $(t^2, t^{2k+1}, c_1 t^{2k-1}, c_2 t^{2k-3}, c_3 t^{2k-5})$, where the $c_i$'s are nonzero constants. The tangency curve $V_2^1$ is given by $V_2^1 = (0,0,0,t,\infty)$, and $V_3$, the new vertical curve is given by $(0,0,0,0,t)$. The vertical curve $V_3$ and $V_2^1$ intersect normally at a point $q$ and the prolonged curve $c_3$ does not pass through $q$, so by Definition 9, $V_3$ and $V_2^1$ are incident. We carry along $V_2^1$ through further prolongations by recording its intersection with $V_3$. The curve $c^3$ is tangent to the vertical curve, so we prolong again.

We note the pattern that the order of the KR coordinate function drops by two and $V_k$ is incident to $V_{k-1}$ for the first $k$ prolongations. After the $k$th prolongation the curve $c^k$ is immersed but tangent to the vertical curve. For the $k+1$th prolongation we introduce the new coordinate $u_{k+1} = dx/du_k$. In the $x, y, u_1, u_2, u_3, \ldots, u_k, u_{k+1}$ coordinates we find that $c^{k+1} = (t^2, t^{2k+1}, c_1 t^{2k-1}, c_2 t^{2k-3}, c_3 t^{2k-5}, \ldots, c_k t, c_{k+1} t)$ while $V_k^1 = (0, 0, 0, 0, \ldots, 0, t, 0)$ and the new vertical curve $V_{k+1}$ is given by $(0, 0, 0, 0, 0, 0, \ldots, 0, t)$. The vertical curve $V_{k+1}$ is not incident to $V_k^1$, since $V_k^1$ and $c^k$ intersect. All three curves pass through the coordinate origin, and their tangents form three distinct lines, $du_k = 0, du_{k+1} = 0$ and $du_k = du_{k+1}$ within $\Delta_{k+1}(0, 0, 0, \ldots, 0)$. We have a triple intersection. One more prolongation is required to resolve the singularity according to the definition.

We find that $P^{k+2}(c) = c^{k+2} \cup V_k^2 \cup V_{k+1}^1 \cup V_{k+2}$. The prolongation $c^{k+2}$ and $V_{k+2}$ intersect transversally, and $c^{k+2}$ intersects none of the other curves $V_k^2$ and $V_{k+1}^1$. Thus the component curves form a normal system and the singularity is resolved. The dual graph contains $k+2$ verticies, $k+1$ edges, and an arrow stemming from the vertex representing $V_{k+2}$. The dual graph is depicted in Figure 4.4. We list the K-R coordinates and K-R coordinate functions corresponding to each of the prolongations in the Table 4.4

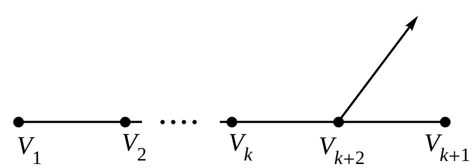



Figure 4.4: The dual graph for resolution by prolongation of $(t^2, t^{2k+1})$

Table 4.4: The Prolongation of the curve $c : x(t) = t^2, y(t) = t^{2k+1}$.

| Prol | KR coord | KR function |
|---|---|---|
| $c^1$ | $u_1 = \frac{dy}{dx}$ | $U_1(t) = c_1 t^{2k-1}$ |
| $c^2$ | $u_2 = \frac{du_1}{dx}$ | $U_2(t) = c_2 t^{2k-3}$ |
| $c^3$ | $u_3 = \frac{du_2}{dx}$ | $U_3(t) = c_3 t^{2k-5}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $c^k$ | $u_k = \frac{du_{k-1}}{du_k}$ | $U_k(t) = c_k t$ |
| $c^{k+1}$ | $u_{k+1} = \frac{dx}{du_k}$ | $U_{k+1}(t) = c_{k+1} t$ |
| $c^{k+2}$ | $u_{k+2} = \frac{du_{k+1}}{du_k}$ | $U_{k+2}(t) = c_{k+2}$ |

## 4.2 $E$ singularity $(t^3, t^7)$.

### 4.2.1 Blow up of $(t^3, t^7)$

**Example 9.** We blow up the curve $(t^3, t^7)$, with the defining equation $x^7 = y^3$, at the origin. Let the coordinates for $\mathbb{C}^2$ be given by $x, y$ and the affine coordinates for $\mathbb{C}P^1$ be given by $u_1, v_1$. We take $u_1 = \frac{u}{v}$ and $v_1 = \frac{v}{u}$ for homogeneous coordinates $[u; v]$ on $\mathbb{C}P^1$. We coordinatize the blow up of $\mathbb{C}^2$ at the origin by $X_1 = \{([u, v], x, y) \in \mathbb{C}P^1 \times \mathbb{C}^2 | \ xv = yu\}$. In the affine chart $[1, v_1]$, we have $u_1 = 1$ and $v_1 = \frac{v}{u}$, so that $y = xv_1$. The defining equation $x^7 = y^3$ becomes $x^3(x^4 - v_1^3) = 0$. In the other chart $[u_1, 1]$ the equation of the curve is $y^3(y^4 u_1^7 - 1) = 0$. In both charts, we have an exceptional curve with multiplicity 3, denoted by $E_1$. The coordinates for the blown up surface near the singular point are $(x, v_1)$. The exceptional curve is given by the equation $x = 0$, which is the $v_1$-axis. Since the singularity of the proper transform lies at the origin of the first chart, we blow up there. See 4.5(a).

Coordinates for the surface $X_1$ near the singular point of the blown-up curve are $x, v_1$. Let $u_2, v_2$ be affine coordinates for the new $\mathbb{CP}^1$ corresponding to the second exceptional divisor. The surface $X_2 = \{([u,v], x, v_1) \, |xv = v_1 u\}$, locally over the singular point in $X_1$. In the affine chart $[1, v_2]$, we have $u_2 = 1$ so that $v_1 = xv_2$. The defining equation for the total transform becomes $x^6(x - v_2^3) = 0$. In the chart $[u_2, 1]$, the defining equation is $u_2^3 v_1^6 (u_2^4 v_1 - 1) = 0$. The singular point of the resulting curve lies in the chart $[1, v_2]$. Since the multiplicity of $x$ is 6, the exceptional fiber $E_2$, given by $x = 0$, has multiplicity equal to 6. In the other affine chart, the equation $u_2 = 0$ with multiplicity 3 gives us the blow up of $E_1$, which we now call $E_1^1$, with multiplicity equal to 3. The singular point of the proper transform lies in the first chart and we blow up again. See Figure 4.5(b).

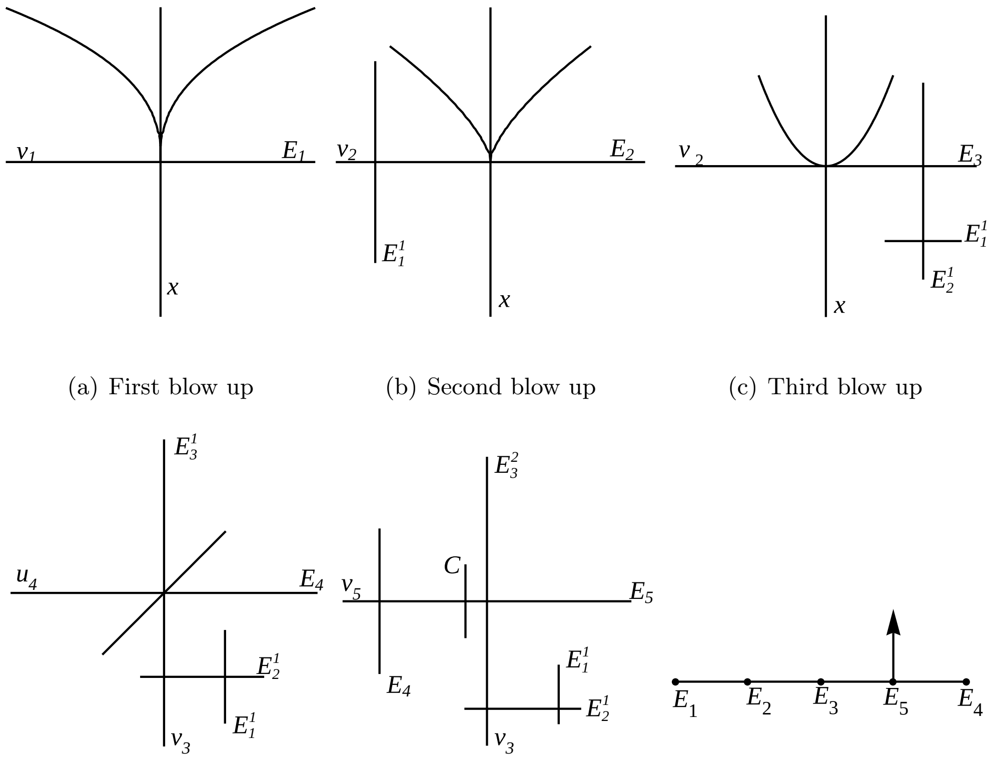
$X_3 = \{([u,v], x, v_2) \, |xv = uv_2\}$, locally over the singular point in $X_2$. In the affine chart $[1, v_3]$, the equation of the curve is $x^7(1 - x^2 v_3^3) = 0$, and in the chart $[u_3, 1]$, the equation is $v_2^7 u_3^6 (u_3 - v_2^2) = 0$. Our new exceptional divisor $E_3$ has multiplicity equal to 7, and $E_2^1$, the blow up of $E_2$ still with multiplicity equal to 6. Now, the curve is immersed, but we continue until our proper transform is transverse to all of the exceptional curves, and we have no triple intersection points. See Figure 4.5(c).

We blow up again. In the chart $[u_4, 1]$, we have $u_3^{14} u_4^7 (1 - u_3 u_4^2) = 0$. This chart contains both our new exceptional divisor $E_4$ with multiplicity equal to 14, as well as $E_3^1$ with multiplicity 7. In the chart $[1, v_4]$, we have $v_2^{14} v_4^6 (v_4 - v_2) = 0$. In the first chart, which contains the origin, all three curves intersect at the origin, and we call this a triple intersection point. We must blow up there one last time. See Figure 4.5(d).

In the affine chart $[1, v_5]$, the equation of the curve is $v_2^{21} v_5^6 (v_5 - 1) = 0$. In the chart $[u_5, 1]$ the equation is $v_4^{21} u_5^{14} (1 - u_5) = 0$. The affine chart $[1, v_5]$ contains the exceptional curve $E_5$ with multiplicity 21, as well as $E_2^3$ with multiplicity 6. The affine chart $[u_5, 1]$ contains the new exceptional curve $E_5$ and $E_4^1$ with multiplicity 14. In either chart the proper transforms are smooth curves and intersect the corresponding exceptional curves transversally. In both charts the total transforms have only have normal crossing singularities, so we have reached a good resolution in five steps. The resolution diagram is depicted in Figure 4.5(e). The dual graph has five verticies, one for each exceptional curve, and four edges, one for each transverse intersection between the exceptional curves. We have an arrow stemming from the vertex which represents $E_5$, to indicate that the proper transform intersects the exceptional curve $E_5$. See Figure 4.5(f). We record the proper and total transforms in a table, see Table 4.5.

Table 4.5: The blow up of the curve $c : x(t) = t^3, y(t) = t^7$.

| Blow up | total transform | proper transform |
|---|---|---|
| $B^1(c)$ | $x^3(x^4 - v_1^3)$ | $(x^4 - v_1^3)$ |
| $B^2(c)$ | $x^6(x - v_2^3)$ | $(x - v_2^3)$ |
| $B^3(c)$ | $v_2^7 u_3^6 (u_3 - v_2^2)$ | $(u_3 - v_2^2)$ |
| $B^4(c)$ | $v_2^{14} v_4^6 (v_4 - v_2)$ | $(v_4 - v_2)$ |
| $B^5(c)$ | $v_2^{21} v_5^6 (v_5 - 1)$ | $(v_5 - 1)$ |




(a) First blow up (b) Second blow up (c) Third blow up

(d) Fourth blow up (e) Fifth blow up (f) Dual graph

Figure 4.5: The Blow ups of $(t^3, t^7)$

### 4.2.2 Prolongation of $(t^3, t^7)$

**Example 10.** The curve $x^7 = y^3$ is parameterized as $c(t) = (t^3, t^7)$. We compute with the coordinates $x, y, u_1 = dy/dx$ its first prolongation $c^1$ is $(t^3, t^7, c_1 t^4)$. In these coordinates the vertical curve $V_1$ is $(0, 0, t)$. We see that $c^1$ is tangent to the vertical curve and still singular. See Figure 4.6(a). So we must prolong again.

We introduce the new coordinate $u_2 = du_1/dx$. In the $x, y, u_1, u_2$ coordinates we find that $c^2 = (t^3, t^7, c_1t^4, c_2t)$ while $V_1^1 = (0, 0, t, \infty)$, and $V_2$, the new vertical curve is given by $(0, 0, 0, t)$. The vertical curve $V_2$ is incident to the tangency curve $V_1^1$ and we carry along the tangency curve $V_1^1$ through further prolongations. See Figure 4.6(b). The distribution $\Delta_2$ at level 2 is given in these coordinates by $dy - u_1dx = 0$ and $du_1 - u_2dx = 0$. The second prolongation $c^2$ is an immersed curve. Since $u_2$ is the lower order coordinate of $u_1, u_2$, at $t = 0$ $c^2$ is tangent to the vertical curve $V_2$. We introduce the fiber coordinate $u_3$ on $P^3(\mathbb{C}^2)$, near the point $c^3(0)$ by setting $[dx, du_2] = [u_3, 1]$ so that $u_3 = dx/du_2$. In the coordinates $(x, y, u_1, u_2, u_3)$ the third prolongation $c^3$ is given by $(t^3, t^7, c_1t^4, c_2t, c_3t^2)$, where the $c_i$'s are nonzero constants. The tangency curve $V_2^1$ is given by $V_2^1 = (0, 0, 0, t, 0)$, and $V_3$, the new vertical curve is given by $(0, 0, 0, 0, t)$. See Figure 4.6(c). The curves $V_3$ and $V_2^1$ are not incident since the prolonged curve $c_3$ passes through their point of intersection. We see that $c^3$, though immersed, is tangent to the tangency curve. So we must prolong again.

We introduce the new coordinate $u_4 = du_3/du_2$. In the $x, y, u_1, u_2, u_3, u_4$ coordinates we find that $c^4 = (t^3, t^7, c_1t^4, c_2t, c_3t^2, c_4t)$ while $V_3^1 = (0, 0, 0, 0, t, \infty)$, the tangency curve $V_2^2 = (0, 0, 0, t, 0, 0)$ and $V_4$, the new vertical curve is given by $(0, 0, 0, 0, 0, t)$. See Figure 4.6(d). The vertical curve $V_4$ is incident to $V_3^1$. The vertical curve $V_4$, the tangency curve $V_2^2$, and the prolonged curve $c_4(t)$ all pass through the coordinate origin, and their tangents form three distinct lines, $du_3 = 0, du_4 = 0$ and $du_3 = du_4$ within $\Delta_4(0, 0, 0, 0, 0, 0)$. We have a triple intersection. One more prolongation is required to resolve the singularity according to the definition. We find that $P^5(c) = c^5 \cup V_2^2 \cup V_4^1 \cup V_5$. At level 5, we have that $c^5$ and $V_5$ intersect transversally, and $c^5$ intersects none of the

Table 4.6: The Prolongation of the curve $c : x(t) = t^3, y(t) = t^7$.

| Prol | KR coord | KR function |
|---|---|---|
| $c^1$ | $u_1 = \frac{dy}{dx}$ | $U_1(t) = c_1 t^4$ |
| $c^2$ | $u_2 = \frac{du_1}{dx}$ | $U_2(t) = c_2 t$ |
| $c^3$ | $u_3 = \frac{dx}{du_2}$ | $U_3(t) = c_3 t^2$ |
| $c^4$ | $u_4 = \frac{du_3}{du_2}$ | $U_4(t) = c_4 t$ |
| $c^5$ | $u_5 = \frac{du_4}{du_2}$ | $U_5(t) = c_5$ |

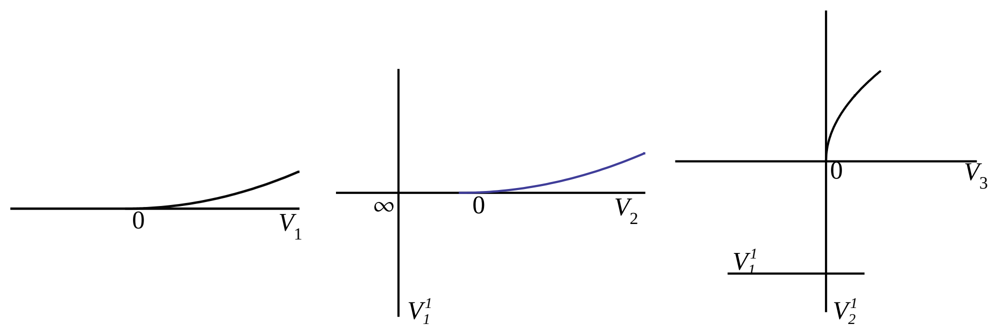
other curves $V_2^2$ and $V_4^1$. Thus the component curves form a normal system and the singularity is resolved. The configuration of the component curves of $P^5(c)$ is depicted with a resolution diagram in Figure 4.6(e). The dual graph is depicted in Figure 4.6(f). We list the K-R coordinates and K-R coordinate functions corresponding to each of the prolongations in the Table. 4.6

## 4.3 Quasi-Homogeneous Case $(t^m, t^n)$.

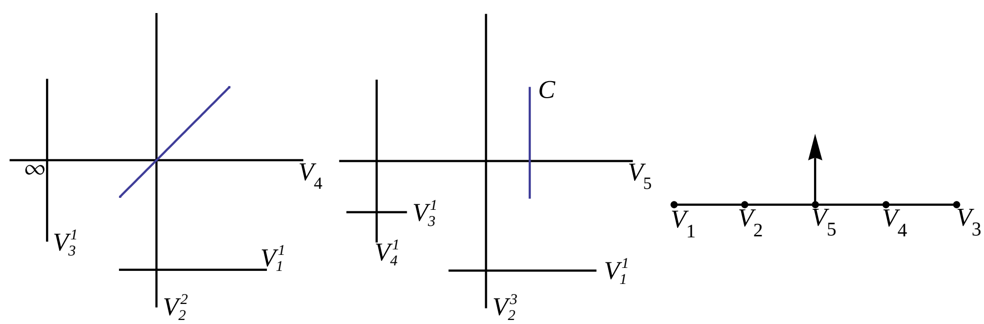
### 4.3.1 Prolongation of $(t^m, t^n)$

**Example 11.** A multigerm of a curve $c$ in $\mathbb{C}^3$ is called quasi-homogeneous if there exist weights $\lambda_1, \lambda_2, \lambda_3 > 0$ such that $c$ is $RL$-equivalent to a multigerm whose components have the form

$$x = at^{r\lambda_1},\ y = bt^{r\lambda_2},\ z = ct^{r\lambda_3},\quad a, b, c, r \in \mathbb{C}$$




(a) First prolongation (b) Second prolongation (c) Third prolongation




(d) Fourth prolongation (e) Fifth prolongation (f) Dual graph

Figure 4.6: The Prolongations of $(t^3, t^7)$

Let $c(t) = (t^m, t^n)$ with $m$ and $n$ relatively prime and $m < n$. The curve $c$ prolongs to a quasi-homogeneous curve. We compute with the coordinates $x, y, u_1 = dy/dx$ its first prolongation $c^1$ is $(t^m, t^n, c_1 t^{n-m})$. Applying the Euclidean Algorithm to the pair $(m, n)$, we have positive integers $q_1, \ldots, q_{k+1}, \ldots, r_1, \ldots, r_{k+1}$, such that

$$\begin{aligned} n &= q_1 m + r_1 \\ m &= q_2 r_1 + r_2 \\ &\vdots \\ r_{k-2} &= q_k r_{k-1} + r_k \\ r_{k-1} &= q_{k+1} r_k + r_{k+1} \end{aligned}$$

where $m > r_1 > r_2 > \ldots > r_k > r_{k+1} = 0$, and $r_k = (m, n) = 1$. Let $s_k = \sum_{i=1}^{k} q_i$. It takes $s_{k+1}$ steps to resolve this singularity. Since $n > q_1 m$, for the first $i$ prolongations, $i < q_1$ we introduce the new coordinate $u_i = \frac{du_{i-1}}{dx}$. In the $x, y, u_1, u_2, \ldots, u_i$ coordinates we find that $c^i = (t^m, t^n, c_1 t^{n-m}, c_2 t^{n-2m}, \ldots, c_i t^{n-im})$. When $i = q_1$, we find that $c^i = (t^m, t^n, c_1 t^{n-m}, c_2 t^{n-2m}, \ldots, c_i t^{r_1})$ and the $q_1$-coordinate is the lowest order coordinate. The distribution $\Delta_{q_1}$ at level $q_1$ is given in these coordinates by $du_{q_1-1} - u_{q_1} dx = 0$ and $du_{q_1-2} - u_{q_1-1} dx = 0$. The curve is still singular, so we prolong again. We introduce the new coordinate $u_{q_1+1} = \frac{dx}{du_{q_1}}$. In the $x, y, u_1, u_2, \ldots, u_{q_1+1}$ coordinates we find that $c^{q_1+1} = (t^m, t^n, c_1 t^{n-m}, \ldots, c_{q_1+1} t^{m-r_1})$.

Since $m > q_2 r_1$, for the next $i$ prolongations, $s_1 < i < s_2$ we introduce the new coordinate $u_i = \frac{du_{q_1+i}}{du_{q_1}}$. In the $x, y, u_1, u_2, \ldots, u_i$ coordinates we find that $c^i =$

$(t^m, t^n, c_1 t^{n-m}, c_2 t^{n-2m}, \ldots, c_i t^{m-ir_1})$. When $i = s_2$, we find that $U^i(t) = c_i t^{r_2}$ and the $s_2$-coordinate is the lowest order coordinate. We continue in this fashion and at level $s_k$, the curve is immersed. We introduce coordinates $u_{s_k} = \frac{du_{s_k-1}}{du_{s_{k-1}}}$, and we have $U_{s_k}(t) = c_{s_k} t^{r_k}$ with $r_k$ equal to 1. We continue one more level for resolution. We introduce the new coordinate $u_{s_{k+1}} = \frac{du_{s_{k+1}-1}}{du_{s_k}}$. In the $x, y, u_1, u_2, \ldots, s_{k+1}$ coordinates we find that $c^{s_{k+1}} = (t^m, t^n, c_1 t^{n-m}, \ldots,, c_{s_k} t, c_{s_{k+1}} t^{r_{k+1}})$ and $r_{k+1} = 0$. We list the sequence of prolongations in Table 4.7.

### 4.3.2 Blow up of $(t^m, t^n)$

**Example 12.** We blow up the curve $(t^m, t^n)$ with defining equation $x^n = y^m$ at the origin. This is an example of a curve with a Puiseux characteristic of length 2. The case of Puiseux characteristics of longer length is dealt with in [4]. We ingnore the exceptional divisors and focus our attention on the proper transform to count the steps to resolution. We coordinatize the blow up of $\mathbb{C}^2$ at the origin by $X_1 = \{([u, v], x, y) \in \mathbb{C}P^1 \times \mathbb{C}^2 | \; xv = yu\}$.

The affine chart $[1, v_1]$ contains the total transform $x^m(x^{n-m} - v_1^m) = 0$. We use the same integers $q_1, \ldots, q_{k+1}, r_1, \ldots, r_{k+1}$ from Example 11 given by the Euclidean algorithm. For $1 \leq i \leq s_1$, the proper transform of $B^i(c)$ is $(x^{n-im} - v_i^m)$. When $i = s_1$, we have the proper transform is $(x^{r_1} - v_i^m)$. Since $r_1 < m$, for $s_1 \leq i \leq s_2$, the proper transform takes the form $(u_{s_1+i}^{r_1} - v_{s_1}^{m-ir_1})$. We continue in this fashion, using the integers from the Euclidean algorithm applied to $m$ and $n$. After $s_k$ blow ups, the curve is immersed and has proper transform $(v_{s_k}^{r_k-1} - v_{s_{k-1}})$. It takes one more blow up

Table 4.7: The Prolongation of the curve $c : x(t) = t^m, y(t) = t^n$.

| Prolongation | KR coordinate | KR coordinate function |
|---|---|---|
| $c^1$ | $u_1 = \frac{dy}{dx}$ | $U_1(t) = c_1 t^{n-m}$ |
| $c^2$ | $u_2 = \frac{du_1}{dx}$ | $U_2(t) = c_2 t^{n-2m}$ |
| $c^3$ | $u_3 = \frac{du_2}{dx}$ | $U_3(t) = c_3 t^{n-3m}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $c^{s_1}$ | $u_{s_1} = \frac{du_{s_1-1}}{dx}$ | $U_{s_1}(t) = c_{s_1} t^{r_1}$ |
| $c^{s_1+1}$ | $u_{s_1+1} = \frac{dx}{du_{s_1}}$ | $U_{s_1+1}(t) = c_{s_1+1} t^{m-r_1}$ |
| $c^{s_1+2}$ | $u_{s_1+2} = \frac{du_{s_1+1}}{du_{s_1}}$ | $U_{s_1+2}(t) = c_{s_1+2} t^{m-2r_1}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $c^{s_2}$ | $u_{s_2} = \frac{du_{s_2-1}}{du_{q_1}}$ | $U_{s_2}(t) = c_{s_2} t^{r_2}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $c^{s_k}$ | $u_{s_k} = \frac{du_{s_k-1}}{du_{s_{k-1}}}$ | $U_{s_k}(t) = c_{s_k} t$ |
| $c^{s_{k+1}}$ | $u_{s_{k+1}} = \frac{du_{s_{k+1}-1}}{du_{s_k}}$ | $U_{s_{k+1}}(t) = c_{s_{k+1}}$ |

for a normal crossing singularity. The proper transform of $B^{s_{k+1}}(c)$ is $(v_{s_{k+1}} - 1)$. We record the proper transforms in a table, see Table 4.8.

### 4.3.3 Prolongation of $(t^3, t^{10} + t^{11})$

**Example 13.** $(t^3, t^{10}+t^{11})$ is an example of a curve that prolongs to a non-planar curve and the failure of intermediate step equivalences betwwen blow up and prolongation. A multigerm of a curve in $\mathbb{P}^1\mathbb{C}^2$ is planar if it's image belongs to a non-singular surface

Table 4.8: The blow up of the curve $c : x(t) = t^m, y(t) = t^n$.

| Blow up | proper transform |
| --- | --- |
| $B^1(c)$ | $(x^{n-m} - v_1^m)$ |
| $B^2(c)$ | $(x^{n-2m} - v_2^m)$ |
| $B^3(c)$ | $(x^{n-3m} - v_3^m)$ |
| $\vdots$ | $\vdots$ |
| $B^{s_1}(c)$ | $(x^{r_1} - v_{s_1}^m)$ |
| $\vdots$ | $\vdots$ |
| $B^{s_2}(c)$ | $(v_{s_2}^{r_1} - v_{s_1}^{r_2})$ |
| $\vdots$ | $\vdots$ |
| $B^{s_k}(c)$ | $(v_{s_k}^{r_{k-1}} - v_{s_{k-1}})$ |
| $B^{s_{k+1}}(c)$ | $(v_{s_{k+1}} - 1)$ |

of $\mathbb{P}^1\mathbb{C}^2$. The curve $y^3 - x^{10} - x^{11} - 3yx^7 = 0$ is parameterized as $c(t) = (t^3, t^{10} + t^{11})$. We compute with the coordinates $x, y, u_1 = dy/dx$ its first prolongation $c^1$ is $(t^3, t^{10} + t^{11}, \frac{10}{3}t^7 + \frac{11}{3}t^8)$. The curve $c(t) = (t^3, t^{10} + t^{11})$ prolongs to a non-planar curve $c^1(t) = (t^3, t^{10} + t^{11}, \frac{10}{3}t^7 + \frac{11}{3}t^8)$ in $\mathbb{P}^1\mathbb{C}^2$. Suppose that there exists $f = f(x, y, z)$ with $df(0, 0, 0) \neq 0$ and $f(x(t), y(t), z(t)) = 0$. Let $f = Ax + By + Cz + Dx^2 + \cdots$, equating coefficients in the polynomial equation $f(c^1(t)) = 0$, we have that $A, B, C = 0$. This contradicts the assumption that $df(0, 0, 0) = Adx + Bdy + Cdz \neq 0$. This example proves that a step-by-step diffeomorphism between the results of blow up and prolongation does not exist.

In the $x, y, u_1 = dy/dx$ coordinates the vertical curve $V_1$ is $(0, 0, t)$. We see that $c^1$ is tangent to the vertical curve and still singular. See Figure 4.7(a). So we must prolong again.

We introduce the new coordinate $u_2 = du_1/dx$. In the $x, y, u_1, u_2$ coordinates we find that $c^2 = (t^3, t^{10} + t^{11}, a_1t^7 + b_1t^8, a_2t^4 + b_2t^5)$ while $V_1^1 = (0, 0, t, \infty)$, and $V_2$, the new vertical curve is given by $(0, 0, 0, t)$. The vertical curve $V_2$ is incident to the tangency curve $V_1^1$, since both the curves intersect normally and since $c^2$ does not pass through their point of intersection. See Figure 4.7(b). We carry along the tangency curve $V_1^1$ through further prolongations by recording the intersection of $V_1^1$ with $V_2$ in the diagrams to come. The second prolongation $c^2$ is still singular and at $t = 0$ is tangent to the vertical curve $V_2$, since $u_2$ is the lower order coordinate of $u_1, u_2$. We introduce the fiber coordinate $u_3$ on $P^3(\mathbb{C}^2)$, near the point $c^3(0)$ by setting $[dx, du_2] = [1, u_3]$ which is to say $u_3 = du_2/dx$. In the coordinates $(x, y, u_1, u_2, u_3)$ the third prolongation $c^3$ is given by $(t^3, t^{10} + t^{11}, a_1t^7 + b_1t^8, a_2t^4 + b_2t^5, a_3t + b_3t^2)$, where the $a_i$'s and $b_i$'s are nonzero constants. The tangency curve $V_2^1$ is given by $V_2^1 = (0, 0, 0, t, \infty)$, and $V_3$, the new vertical curve is given by $(0, 0, 0, 0, t)$. See Figure 4.7(c). The curves $V_3$ and $V_2^1$ are incident since the prolonged curve $c^3$ does not pass through their point of intersection. We see that $c^3$, though immersed, is tangent to the vertical curve. So we must prolong again.

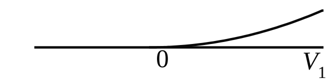
We introduce the new coordinate $u_4 = dx/du_3$. In the $x, y, u_1, u_2, u_3, u_4$ coordinates we find that $c^4 = (t^3, t^{10} + t^{11}, a_1t^7 + b_1t^8, a_2t^4 + b_2t^5, a_3t + b_3t^2, a_4t^2 + b_4t^3)$

while the tangency curve $V_3^1 = (0,0,0,0,t,0)$ and $V_4$, the new vertical curve is given by $(0,0,0,0,0,t)$. See Figure 4.7(d).

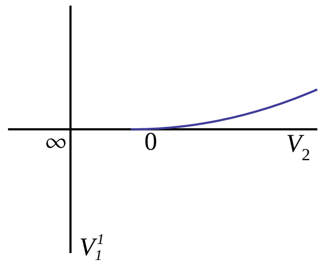
The vertical curve $V_4$, the tangency curve $V_3^1$, and the prolonged curve $c_4(t)$ all pass through the coordinate origin. Since $c_4(t)$ is tangent to the tangency curve, we prolong again.

We introduce the new coordinate $u_5 = du_4/du_3$. In the $x, y, u_1, u_2, u_3, u_4, u_5$ coordinates we find that $c^5 = (t^3, t^{10} + t^{11}, a_1t^7 + b_1t^8, a_2t^4 + b_2t^5, a_3t + b_3t^2, a_4t^2 + b_4t^3, a_5t + b_5t^2)$ while the tangency curve $V_3^2 = (0,0,0,0,t,0,0)$ and $V_5$, the new vertical curve is given by $(0,0,0,0,0,0,t)$. See Figure 4.7(e).

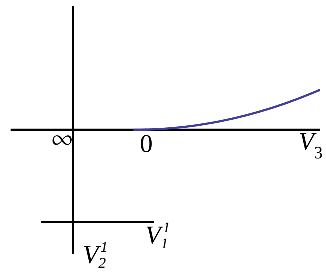
The vertical curve $V_5$, the tangency curve $V_3^2$, and the prolonged curve $c_4(t)$ all pass through the coordinate origin, and their tangents form three distinct lines. We have a triple intersection. One more prolongation is required to resolve the singularity according to the definition. We find that $P^6(c) = c^6 \cup V_3^3 \cup V_5^1 \cup V_6$. At level 6, we have that $c^6$ and $V_6$ intersect transversally, and $c^6$ intersects none of the other curves $V_3^3$ and $V_5^1$. Thus the component curves form a normal system and the singularity is resolved. The configuration of the component curves of $P^6(c)$ is depicted with a resolution diagram in Figure 4.7(f). The dual graph is depicted in Figure 4.7(g). We list the K-R coordinates and K-R coordinate functions corresponding to each of the prolongations in the Table. 4.9.



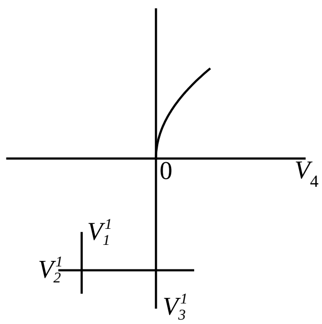

(a) First prolongation



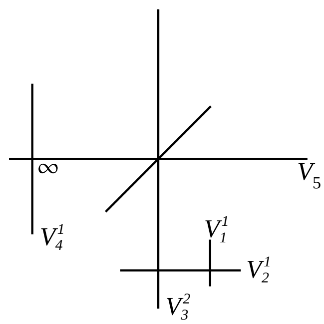

(b) Second prolongation



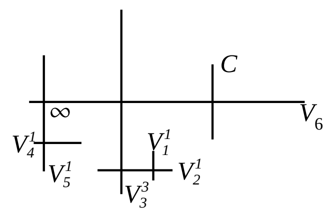

(c) Third prolongation



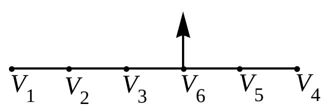

(d) Fourth prolongation




(e) Fifth prolongation




(f) Sixth prolongation




(g) Dual graph

Figure 4.7: The resolution by prolongation of $(t^3, t^{10} + t^{11})$

Table 4.9: The Prolongation of the curve $c : x(t) = t^3, y(t) = t^{10} + t^{11}$.

| Prol | KR coord | KR function |
|---|---|---|
| $c^1$ | $u_1 = \frac{dy}{dx}$ | $U_1(t) = a_1 t^7 + b_1 t^8$ |
| $c^2$ | $u_2 = \frac{du_1}{dx}$ | $U_2(t) = a_2 t^4 + b_2 t^5$ |
| $c^3$ | $u_3 = \frac{du_2}{dx}$ | $U_3(t) = a_3 t + b_3 t^2$ |
| $c^4$ | $u_4 = \frac{dx}{du_3}$ | $U_4(t) = a_4 t^2 + b_4 t^3$ |
| $c^5$ | $u_5 = \frac{du_4}{du_3}$ | $U_5(t) = a_5 t + b_5 t^2$ |
| $c^6$ | $u_6 = \frac{du_5}{du_3}$ | $U_6(t) = a_6 + b_6 t$ |

## 4.4 Blow up of $(t^3, t^{10} + t^{11})$

**Example 14.** The curve $(t^3, t^{10}+t^{11})$ has a singularity at the origin. We will work with the defining equation $y^3 - x^{10} - x^{11} - 3yx^7 = 0$. Let the coordinates for $\mathbb{C}^2$ be given by $x, y$ and the affine coordinates for $\mathbb{C}P^1$ be given by $u_1, v_1$. We take $u_1 = \frac{u}{v}$ and $v_1 = \frac{v}{u}$ for homogeneous coordinates $[u; v]$ on $\mathbb{C}P^1$. We coordinatize the blow up of $\mathbb{C}^2$ at the origin by $X_1 = \{([u, v], x, y) \in \mathbb{C}P^1 \times \mathbb{C}^2 |\ xv = yu\}$. In the affine chart $[1, v_1]$, we have $u_1 = 1$ and $v_1 = \frac{v}{u}$, so that $y = xv_1$. The defining equation $y^3 - x^{10} - x^{11} - 3yx^7 = 0$ becomes $x^3(v_1^3 - x^7 - x^8 - 3v_1x^5) = 0$. In the other chart $[u_1, 1]$ the equation of the curve is $y^3(1 - y^7u_1^{10} - y^8u_1^{11} - 3y^5u_1^8) = 0$. These are the equations for the total transforms in their respective charts. After the first blow up, we have an exceptional curve with multiplicity 3, denoted by $E_1$. The coordinates for the blown up surface near the singular point are $(x, v_1)$. The exceptional curve is given by the equation $x = 0$, which is the $v_1$-axis. See Figure 4.8(a). Since the singularity of the proper transform lies at the origin of the first chart, we blow up there.

Let $u_2, v_2$ be affine coordinates for the new $\mathbb{CP}^1$ corresponding to the second exceptional divisor. The blow up of $X_1$ at the singular point is given by $X_2 = \{([u,v], x, v_1)\ |xv = v_1 u\}$, locally over the singular point in $X_1$. In the affine chart $[1, v_2]$, we have $u_2 = 1$ so that $v_1 = xv_2$. The defining equation for the total transform becomes $x^6(v_2^3 - x^4 - x^5 - 3x^3v_2) = 0$. In the chart $[u_2, 1]$, the defining equation is $v_1^6u_2^3(1 - v_1^4u_2^7 - v_1^5u_2^8 - 3v_1^3u_2^5) = 0$. The singular point of the resulting curve lies in the chart $[1, v_2]$. Since the multiplicity of $x$ is 6, the exceptional fiber $E_2$, given by $x = 0$, has multiplicity equal to 6. In the other affine chart, the equation $u_2 = 0$ with multiplicity 2 gives us the blow up of $E_1$, which we now call $E_1^1$, with multiplicity equal to 3. The singular point of the proper transform lies in the first chart and we blow up again. See Figure 4.8(b).

$X_3 = \{([u,v], x, v_2)\ |xv = uv_2\}$, locally over the singular point in $X_2$. In the affine chart $[1, v_3]$, the equation of the curve is $x^9(v_3^3 - x - x^2 - 3xv_3) = 0$, and in the chart $[u_3, 1]$, the equation is $v_2^9u_3^6(1 - v_2u_3^4 - v_2^2u_3^5 - 3v_2u_3^3) = 0$. Our new exceptional divisor $E_3$ has multiplicity equal to 9, and $E_2^1$, the blow up of $E_2$ still with multiplicity equal to 6. The coordinates for the blown up surface near the singular point are $(x, v_1)$. Our singular point lies in the first chart, and we blow up there. See Figure 4.8(c).

$X_4 = \{([u,v], x, v_3)\ |xv = uv_3\}$, locally over the singular point in $X_3$. In the affine chart $[1, v_4]$, the equation of the curve is $x^{10}(x^2v_4^3 - 1 - x - 3v_4x) = 0$, and in the chart $[u_4, 1]$, the equation is $v_3^{10}u_4^9(v_3^2 - u_4 - v_3u_4^2 - 3v_3u_4) = 0$. Our new exceptional divisor $E_4$ has multiplicity equal to 10, and $E_3^1$, the blow up of $E_2$ still with multiplicity equal to 9. The coordinates for the blown up surface near the singular point are $(u_3, v_4)$. Our singular point lies in the second chart, and we blow up there. See Figure 4.8(d).

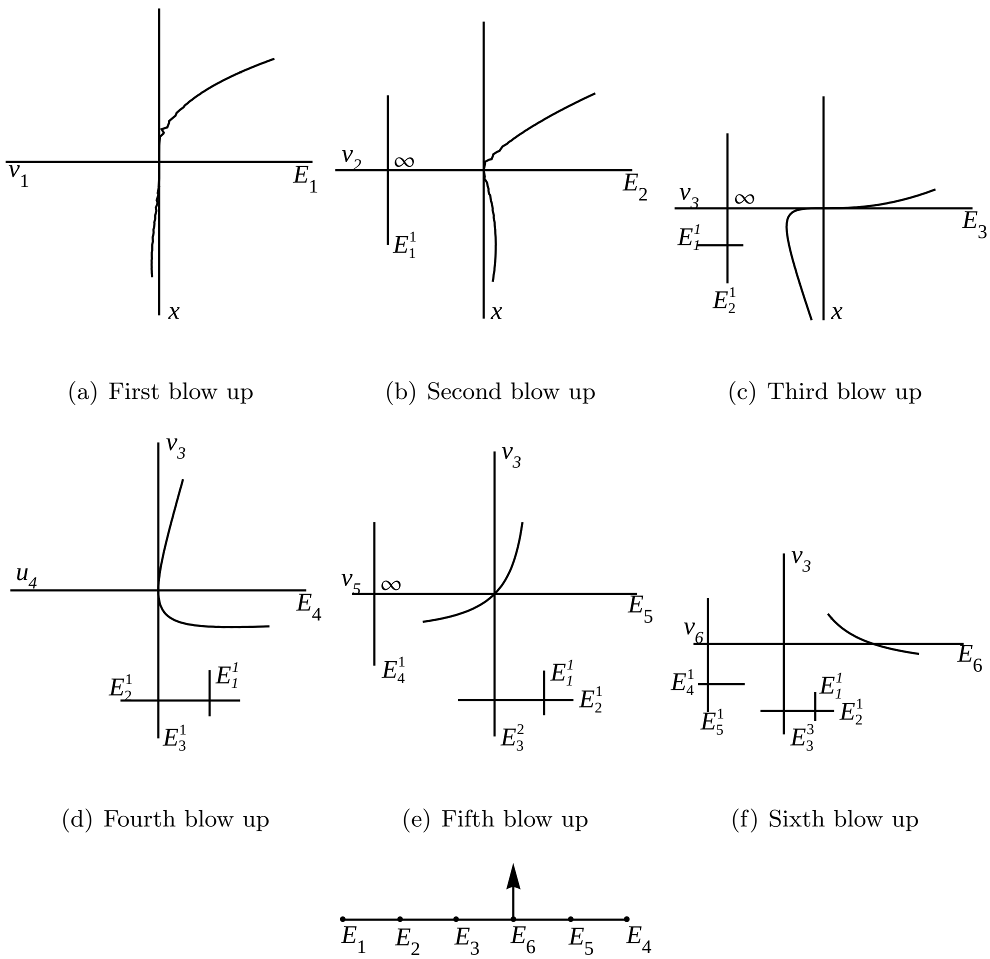
In the chart $[u_5, 1]$, we have $u_4^{20}u_5^{10}(u_4u_5^2 - 1 - u_462u_5 - 3u_4u_5) = 0$. This chart contains both our new exceptional divisor $E_5$ with multiplicity equal to 20, as well as $E_4^1$ with multiplicity 10. In the chart $[1, v_5]$, we have $v_3^{20}v_5^9(v_3 - v_5 - v_3^2v_5^2 - 3v_3v_5) = 0$. This chart contains both our new exceptional divisor $E_5$ with multiplicity equal to 20, as well as $E_3^2$ with multiplicity 9. In the second chart, all three curves intersect at the origin, and we call this a triple intersection point. We must blow up there one last time. See Figure 4.8(e).

In the affine chart $[1, v_6]$, the equation of the curve is $v_3^{30}v_6^9(1 - v_6 - v_3^3v_6^2 - 3v_3v_6) = 0$. In the chart $[u_6, 1]$ the equation is $v_5^{30}u_6^{20}(u_6 - 1 - v_5^3u_6^2 - 3v_5u_6) = 0$. The affine chart $[1, v_6]$ contains the exceptional curve $E_6$ with multiplicity 30, as well as $E_3^3$ with multiplicity 9. The affine chart $[u_6, 1]$ contains the new exceptional curve $E_6$ and $E_5^1$ with multiplicity 20. In either chart the proper transforms are smooth curves and intersect the corresponding exceptional curves transversally. In both charts the total transforms have only have normal crossing singularities, so we have reached a good resolution in six steps. The resolution diagram is depicted in Figure 4.8(f). We can also describe the configuration of curves with the dual graph: we have six verticies, one for each exceptional curve, and five edges, one for each transverse intersection between the exceptional curves. We have an arrow stemming from the vertex which represents $E_6$, to indicate that the proper transform intersects the exceptional curve $E_6$. See

Figure 4.8(g). We record the proper and total transforms in a table, see Table 4.10.

Table 4.10: The blow up of the curve $c : x(t) = t^3, y(t) = t^{10} + t^{11}$.

| Blow up | total transform | proper transform |
|---|---|---|
| $B^1(c)$ | $x^3(v_1^3 - x^7 - x^8 - 3v_1x^5)$ | $(v_1^3 - x^7 - x^8 - 3v_1x^5)$ |
| $B^2(c)$ | $x^6(v_2^3 - x^4 - x^5 - 3x^3v_2)$ | $(v_2^3 - x^4 - x^5 - 3x^3v_2)$ |
| $B^3(c)$ | $x^9(v_3^3 - x - x^2 - 3xv_3)$ | $(v_3^3 - x - x^2 - 3xv_3)$ |
| $B^4(c)$ | $v_3^{10}u_4^9(v_3^2 - u_4 - v_3u_4^2 - 3v_3u_4)$ | $(v_3^2 - u_4 - v_3u_4^2 - 3v_3u_4)$ |
| $B^5(c)$ | $v_3^{20}v_5^9(v_3 - v_5 - v_3^2v_5^2 - 3v_3v_5)$ | $(v_3 - v_5 - v_3^2v_5^2 - 3v_3v_5)$ |
| $B^6(c)$ | $v_3^{30}v_6^9(1 - v_6 - v_3^3v_6^2 - 3v_3v_6)$ | $(1 - v_6 - v_3^3v_6^2 - 3v_3v_6)$ |




(a) First blow up (b) Second blow up (c) Third blow up

(d) Fourth blow up (e) Fifth blow up (f) Sixth blow up

(g) Dual graph

Figure 4.8: The dual graph for resolution by blow up of $(t^3, t^{10} + t^{11})$

# Chapter 5

# Conclusion

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# Appendix A

# Some Ancillary Stuff

Ancillary material should be put in appendices, which appear after the bibliography.