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THE HAMILTONIAN DYNAMICS OF MAGNETIC CONFINEMENT AND INSTANCES OF QUANTUM TUNNELING

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Abstract

The Hamiltonian Dynamics of Magnetic Confinement and Instances of Quantum Tunneling

by

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We consider a class of magnetic fields defined over the interior of a manifold M which go to infinity at its boundary and whose direction near the boundary of M is controlled by a closed 1-form $\sigma_\infty \in \Omega^1(\partial M)$. We are able to show that charged particles in the interior of M under the influence of such fields can only escape the manifold through the zero locus of σ_∞. In particular in the case where the 1-form is nowhere vanishing we conclude that the particles become confined to its interior for all time. We also describe a class of magnetic fields defined on the unit disc which is strong enough to confine classical charged particles to the inside of the disc but fails to confine quantum particles, which provides evidence for the presence of quantum tunneling for such systems.

To Danyal,

To Sueli and Cezar

To Raphael, Renato and Andrea

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Part I

Introduction

Chapter 1

Magnetic dynamics

1.1 Preliminaries

We study the global asymptotics of the motion of a charged particle inside a manifold with boundary under the influence of a magnetic field $\mathbf{B}$ defined over its interior. We show that if the magnetic field goes to infinity fast enough at the boundary with a controlled nonzero direction which is tangent to the boundary, then every particle in its interior becomes confined for all time.

The understanding of magnetic confinement is incredibly valuable, most prominently because of its current usage in the construction of Tokamaks, which are torus-shaped devices used in fusion power generators. Our approximation excludes the possibility of collision of the interior particles with the Tokamak (represented here by the boundary of the manifold).

Our work is motivated by [6] where Colin de Verdière and Truc proved that a charged quantum particle becomes confined to the interior of a compact oriented manifold provided that the magnetic field goes to infinity fast enough at its boundary. In their work they pose the question of whether a classical analogue of their results would hold, which is the problem we address here.

Much work has been done on the analysis of the global behavior of solutions to Lorentz equations and related problems. In [11] Truc analyzes axially symmetric linear magnetic fields in $\mathbb{R}^3$ and finds an open set of initial conditions for which the solutions are bounded. In [3] Braun studies the Earth's magnetic field and establishes the existence of almost-periodic solutions. In [4] Castilho analyzes magnetic fields on Riemannian surfaces and finds conditions for trapping particles with large enough charge around level sets of the magnetic field. In [9] working in a quantum context, Montgomery shows that if a magnetic field in 2 dimensions vanishes nondegenerately along a closed curve, then its ground state concentrates along this curve as the ratio e/h of the charge over Planck's constant tends to infinity.

Most of the previous results concerning the classical system are based on a perturbative approach and are proven by applications of Moser's twist theorem for perturbations of integrable systems. Our strategy in this problem is different but not unrelated, by controlling the way the magnetic field tends to infinity at the boundary we're able to obtain a system of coordinates that shares enough of the properties of the action-angle coordinates in the integrable case and by taking advantage of conservation of energy we're able to establish confinement.

A magnetic field on a Riemannian manifold (M,g) is modeled by a closed 2-form $\mathbf{B} \in \Gamma\left(\bigwedge^2 T^*M\right)$. This form induces an antisymmetric endomorphism $Y : TM \to TM$ via the relation $\mathbf{B}(\cdot,\cdot) = g(\cdot, Y(\cdot))$. The corresponding equation of motion, called the Lorentz equation, for a particle of charge e and mass m moving in M under the influence of $\mathbf{B}$ is:

$$m\nabla_{\dot{q}}\dot{q} = eY_q(\dot{q}) \tag{1.1}$$

where we denote by $Y_q : T_qM \to T_qM$ the fiber-wise linear map to make the dependence of Y on the base point q explicit.

Definition 1. *We will call a solution $q(t)$ of equation 1.1 a* $\mathbf{B}$*-geodesic.*

Because Y is antisymmetric, the quantity $|\dot{q}|^2$ is an integral of motion of this system, since:

$$\frac{d|\dot{q}|^2}{dt} = 2g(\nabla_{\dot{q}}\dot{q}, \dot{q}) = (2e/m)g(Y\dot{q}, \dot{q}) = 0$$

Thus every solution has constant speed. However, unlike the geodesic flow, the dynamics on each level set $\{|\dot{q}|^2 = c\}$ are not simple reparametrizations of each other.

As an example, on $\mathbb{R}^3$ if $\vec{B}$ is a vector field we are able to encode it as the 2-form $\mathbf{B} = \langle \cdot, \cdot \times \vec{B} \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the Euclidian inner product. The closed condition $d\mathbf{B} = 0$ for the 2-form is equivalent to the divergence-free condition $\nabla \cdot \vec{B} = 0$ for the vector field. The endomorphism Y above is simply $Yv = v \times \vec{B}$ and equation (1.1) in this case takes the familiar form:

$$m\ddot{q} = e\dot{q} \times \vec{B}$$

For the 2 dimensional picture one may consider a magnetic field of the form $\vec{B} = (0, 0, B(x, y))$. This assumption forces particles in the xy-plane whose initial velocities are tangent to the plane to stay in this same plane for all time. The equation of a charged particle under the influence of this field is:

$$m\ddot{q} = -eB(q)J\dot{q} \tag{1.2}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the standard complex structure on $\mathbb{R}^2$.

If we consider a planar system with configuration space given by a bounded planar domain $\Omega \subset \mathbb{R}^2$ endowed with the standard Euclidian metric, and having smooth boundary $\partial\Omega$ and consider a magnetic field $\mathbf{B}(q) = B(q)dx \wedge dy$, then equation (1.1) for a charged particle moving in Ω under the influence of $\mathbf{B}$ reduces precisely to equation (1.2).

1.2 The Hamiltonian structure

In this section we describe how to lift the second order differential equation (1.1) on M to a Hamiltonian system on T^*M. The Hamiltonian vector field however will not preserve the standard symplectic form on T^*M, but a twisted version obtained by adding an appropriate multiple of the magnetic field.

We will first present this global formulation, then we will see that in the case where we can find a primitive 1-form (a magnetic potential) for $\mathbf{B}$, we may find a different Hamiltonian vector field which also lifts the Lorentz equation and that will indeed preserve the standard symplectic form on T^*M. This formulation will be useful for the momentum estimates used in our proofs.

Recall that on T^*M we may define the tautological 1-form $\alpha \in \Gamma(T^*(T^*M))$ by:

$$\alpha_p(\xi) = p(d\pi_p(\xi)), \quad \xi \in T_p(T^*M)$$

where $\pi : T^*M \to M$ is the base-point projection and $d\pi : T(T^*M) \to TM$ is its derivative. The canonical symplectic form on T^*M is then:

$$\omega_0 = d\alpha$$

On a trivialization (q,p) induced by coordinates on M the standard symplectic form has the simple expression $\omega_0 = dp_i \wedge dq^i$.

The twisted symplectic form is:

$$\omega_{\mathbf{B}} = \omega_0 + e\pi^*\mathbf{B}$$

We then define the Hamiltonian $H : T^*M \to \mathbb{R}$:

$$H(q,p) = \frac{|p|_g^2}{2m}$$

where $|\cdot|_g$ is the natural norm induced on T^*M by the metric g. Given the Hamiltonian we may use the symplectic form to define the Hamiltonian vector field $X_{\mathbf{B}}$ by:

$$\omega_{\mathbf{B}}(X_{\mathbf{B}},\cdot) = -dH$$

Definition 2. *We call the flow of $X_{\mathbf{B}}$ the* magnetic flow *of* $\mathbf{B}$.

A straightforward computation allows us to see that the integral curves of $X_{\mathbf{B}}$ are related to $\mathbf{B}$-geodesics in M.

Proposition 1. *Let $\gamma : [a,b] \to T^*M$ be an integral curve of $X_{\mathbf{B}}$. Then the curve $c : [a,b] \to M$ given by $c = \pi \circ \gamma$ is a $\mathbf{B}$-geodesic. Conversely, every $\mathbf{B}$-geodesic $c : [a,b] \to M$ is the projection to M of some integral curve $\gamma : [a,b] \to T^*M$ of $X_{\mathbf{B}}$.*

Now suppose that we are in the special case where there is a 1-form $\mathbf{A}$ satisfying $d\mathbf{A} = \mathbf{B}$, we call $\mathbf{A}$ a magnetic potential for $\mathbf{B}$ (notice that $\mathbf{A}$ always exists locally since $d\mathbf{B} = 0$). In this case we may describe an alternative Hamiltonian structure for this system in a simpler way by using the magnetic potential. We define the Hamiltonian $H_{\mathbf{A}} : T^*M \to \mathbb{R}$ by:

$$H_{\mathbf{A}}(q,p) = \frac{1}{2m}|p - e\mathbf{A}(q)|_g^2 \tag{1.3}$$

We then use the standard symplectic form to define the Hamiltonian vector field $X_{\mathbf{A}}$ on T^*M by:

$$\omega_0(X_{\mathbf{A}},\cdot) = -dH_{\mathbf{A}}$$

In natural coordinates since $\omega_0 = dp_i \wedge dq^i$ we may write the Hamiltonian vector field as:

$$X_{\mathbf{A}} = (\partial_{p_i} H_{\mathbf{A}})\partial_{q^i} - (\partial_{q^i} H_{\mathbf{A}})\partial_{p_i}$$

The equation for the magnetic flow takes the simple form of Hamilton's equations:

$$\begin{aligned} \dot{q}^i &= \partial_{p_i} H_{\mathbf{A}} \\ \dot{p}^i &= -\partial_{q_i} H_{\mathbf{A}} \end{aligned} \tag{1.4}$$

The same result of proposition 1 holds for integral curves of $X_{\mathbf{A}}$: they are all lifts of $\mathbf{B}$-geodesics in M.

The disadvantage of this alternative approach is that if the magnetic field is not exact it can only be applied locally, additionally this alternative Hamiltonian vector field will be dependent on the choice of magnetic potential, which means that if the magnetic field is not exact, we may not use it to define a global Hamiltonian vector field on T^*M.

Remark: The first of Hamilton's equations expresses the Lengendrian transform relating momentum and velocity. In this case we obtain:

$$m\dot{q} = (p - e\mathbf{A})^{\#}$$

Part II

Planar results

Chapter 2

Magnetic confinement on planar regions

2.1 Set up and main results

Let $\Omega \subset \mathbb{R}^2$ be an open bounded region with smooth boundary and let $\mathbf{B} \in \Gamma\left(\bigwedge^2 T^*\Omega\right)$ satisfying $d\mathbf{B} = 0$ be a magnetifc field. We will show that if $\mathbf{B}$ grows fast enough as it approaches the boundary, then a particle in Ω never reaches the boundary of the region in finite time. This implies in particular that the magnetic flow on $T^*\Omega$ is complete. In order to phrase our condition more precisely we introduce normal coordinates on a neighborhood of $\partial\Omega$.

Let $\mathcal{C}$ be a connected component of $\partial\Omega$, let L be its length and $\gamma : \mathbb{R}/L\mathbb{Z} \to \mathcal{C}$ be an arc length parametrization of this curve. Denote by $\nu(s)$ the inward-pointing normal vector of $\mathcal{C}$ and define the curvature $\kappa(s)$ by $\ddot{\gamma}(s) = \kappa(s)\nu(s)$. In this way we may define normal coordinates $x : (0,\varepsilon) \times \mathbb{R}/L\mathbb{Z} \to \Omega$ by

$$x(n,s) = \gamma(s) + n\nu(s) \tag{2.1}$$

For a small enough choice of $\varepsilon > 0$ the map x is a diffeomorphism, we denote its

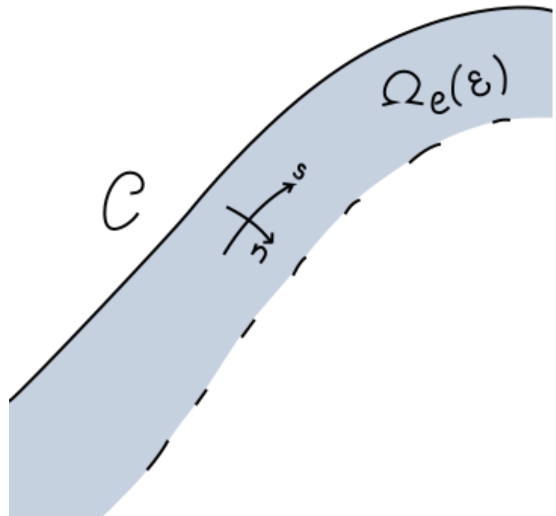


Figure 2.1: Normal coordinates on a neighborhood of one of the boundary components.

image by $\Omega_{\mathcal{C}}(\varepsilon) \subset \Omega$, which is a collar neighborhood for the boundary curve $\mathcal{C}$. We can now state our main theorem:

Theorem 1. *For every connected component $\mathcal{C}$ of $\partial\Omega$, write the magnetic field over $\Omega_{\mathcal{C}}(\varepsilon)$ as $\mathbf{B} = B(n,s)dn \wedge ds$ using normal coordinates. Suppose that $\mathbf{B}$ is smooth and that for every $\mathcal{C}$ we have:*

$$\lim_{n \to 0} \left| \int_n^{\varepsilon} B(m,s)dm \right| = \infty, \quad \forall s \in \mathbb{R}/L\mathbb{Z}. \tag{2.2}$$

Furthermore suppose that for all $\mathcal{C}$, there is a constant $D_{\mathcal{C}}$ such that:

$$\sup_s \int_0^{\varepsilon} \left| \frac{\partial B}{\partial s}(m,s) \right| dm < D_{\mathcal{C}} \tag{2.3}$$

Then, no $\mathbf{B}$-geodesic starting in Ω reaches the boundary $\partial\Omega$ in finite time.

Since the second cohomology of Ω vanishes (notice that Ω is homotopic to a bouquet of circles) the magnetic field $\mathbf{B}$ in this setting always possesses a globally defined potential $\mathbf{A}$.

This means we can freely use either of the descriptions given for the Hamiltonian structure of this system. Using the theorem above we obtain the following corollary:

Corollary 1. *Under the assumptions of Theorem 1 the Hamiltonian flow of $X_{\mathbf{B}}$ on $T^*\Omega$ is complete, moreover given any potential $\mathbf{A}$ of $\mathbf{B}$, the Hamiltonian flow of $X_{\mathbf{A}}$ is also complete.*

Proof. Let X denote either $X_{\mathbf{B}}$ or $X_{\mathbf{A}}$ and $H : T^*\Omega \to \mathbb{R}$ denote the corresponding Hamiltonian. Let $\gamma : I \to T^*\Omega$ be an integral curve of X with $I \subset \mathbb{R}$ its maximal domain of definition, let $\pi : T^*\Omega \to \Omega$ be the base point projection, let $H_0 = H(\gamma(0)) = H(\gamma(t))$ be the energy of γ and consider $c = \pi \circ \gamma$ the $\mathbf{B}$-geodesic obtained from projecting γ down to Ω.

If c is contained in a compact subset K of Ω then γ is contained in the compact subset $\pi^{-1}(K) \cap H^{-1}(H_0)$ of $T^*\Omega$, hence γ must be defined for all time, that is $I = \mathbb{R}$. Otherwise, c must approach the boundary, and by the previous theorem it must take infinite time to do so, in particular it must be defined for all time and therefore γ is defined for all time. $\square$

By restricting the form of the magnetic field we may give a more quantitative description of the boundary behavior of the charged particle. Denote the cotangent coordinates induced by the normal coordinates by (n, s, p_n, p_s). In the next result we give an explicit lower bound for the distance a particle in $\Omega_C(\varepsilon)$ must keep from the boundary in any finite amount of time.

Theorem 2. *Let C be a component of $\partial\Omega$, let $K = sup_s|\kappa(s)|$ be the maximum curvature of C, let $K' = sup_s|\kappa'(s)|$ and let $\varepsilon > 0$ be small so that we can define normal coordinates on $\Omega_C(\varepsilon)$ and such that $\varepsilon < \delta/K$, for some $0 < \delta < 1$. Suppose the magnetic field has the form:*

$$B(n,s) = \frac{M}{n^{\alpha}} + f(n,s), \quad \alpha \geq 1 \tag{2.4}$$

with $|f| \leq C_f$ a bounded smooth function with integrable s-partial

$$\sup_s \int_0^\varepsilon \left| \frac{\partial f}{\partial s}(m,s) \right| dm \leq D_C \tag{2.5}$$

Let $q(t)$ be a $\mathbf{B}$-geodesic contained in $\Omega_C(\varepsilon)$ for $0 \leq t \leq T$ with energy H_0. Then:

$$dist(q(t), \mathcal{C}) \geq \begin{cases} (\varepsilon^{-(\alpha-1)} + (\alpha-1)\ell(T))^{-\frac{1}{\alpha-1}} & \text{if } \alpha > 1 \\ \\ \varepsilon e^{-\ell(T)} & \text{if } \alpha = 1 \end{cases} \tag{2.6}$$

where:

$$\ell(T) = D_0 + D_1 T \tag{2.7}$$

with constants given by:

$$\begin{aligned} D_0 &= \frac{C_f \varepsilon}{|M|} + \frac{|p_s(0)| + \sqrt{2mH_0}(1+\delta)}{e|M|} \\ D_1 &= \sqrt{\frac{2H_0}{m}} \cdot \frac{D_C}{|M|(1-\delta)} + \frac{2H_0 K' \varepsilon}{e|M|(1-\delta)} \end{aligned} \tag{2.8}$$

2.2 Comparison to previous results

We now compare the conditions of Theorem 1 to the ones in [6]. To this purpose we will take Ω to be the unit disc for simplicity, we'll parametrize the boundary by $\gamma(s) = (\cos(s), \sin(s))$, let $r = \sqrt{x^2 + y^2}$, $n = 1 - r$ and set $e = m = 1$. We use both cartesian and normal coordinates to express the magnetic field:

$$\mathbf{B} = B(x,y) dx \wedge dy = \tilde{B}(n,s) dn \wedge ds \tag{2.9}$$

with this notation we have $\tilde{B}(n,s) = (n-1)B(x,y)$, since $dx \wedge dy = (n-1) dn \wedge ds$. The conditions we impose on the magnetic field in order to guarantee completeness are not stronger or weaker than the ones in [6]. One major difference is that we suppose the boundary of our domain to be smooth, while [6] works with more general domains. Aside from that, using the above notation, their condition amounts to:

$$|B(x,y)| \geq \frac{1}{n^2} \tag{2.10}$$

To begin the comparison with the hypothesis of Theorem 1, we consider magnetic fields of the form:

$$\tilde{B}(n,s) = \frac{M}{n^\alpha} \tag{2.11}$$

in cartesian coordinates we have:

$$B(x,y) = \frac{M}{(n-1)n^\alpha} \sim \frac{M}{n^\alpha} \tag{2.12}$$

asymptotically as $n \to 0$. For this type of magnetic field, while the conditions from [6] require $|M| \geq 1$ and $\alpha \geq 2$, our result is more flexible and only requires $M \neq 0$ and $\alpha \geq 1$.

On the other hand, the results from [6] do not require any control on the dependence of **B** on the variable s, while Theorem 1 assumes that the s-partial derivative of **B** is not too wild (i.e. condition (2.3) in Theorem 1). For example the magnetic field

$$B(x,y) = \frac{2+\sin(s)}{n^2} = \frac{2n+y}{n^3} \tag{2.13}$$

satisfies their hypothesis but not ours, since $\partial B/\partial s$ is not integrable along normal rays. A very interesting problem is whether or not one might be able to remove hypothesis (2.3) from Theorem 1.

2.3 Examples

Here are some examples of regions Ω and magnetic fields B that satisfy our hypothesis. Again we denote $r = \sqrt{x^2+y^2}$.

1. We take $\Omega = \{q \in \mathbb{R}^2, |q| < R\}$, the disc of radius R with a magnetic field

$$B(x,y) = \frac{M}{(R-r)^{\alpha}} + f(x,y) \tag{2.14}$$

with f a smooth bounded function on the closed disc, $\alpha \geq 1$ and $M \neq 0$.

2. Take the domain to be an annulus $\Omega = \{q \in \mathbb{R}^2, R_1 < |q| < R_2\}$ with magnetic field:

$$B(x,y) = \frac{M_2}{(R_2-r)^{\alpha_2}} + \frac{M_1}{(r-R_1)^{\alpha_1}} + f(x,y) \tag{2.15}$$

with similar assumptions as before on f and the constants M_i, R_i and α_i.

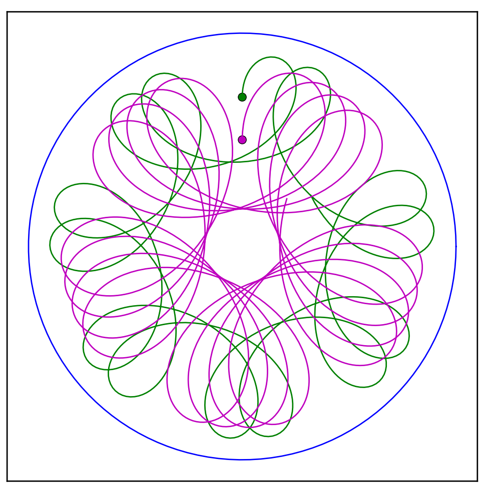
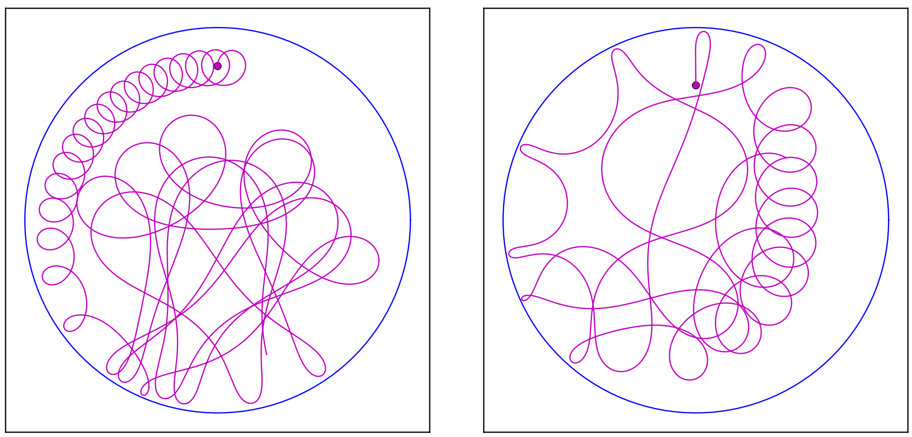


Figure 2.2: $B(x,y) = \frac{1}{1-r}$

We experimented numerically with many examples in the unit disc. We can see in Figure 2.2 a couple of trajectories of charged particles in a magnetic field that depends only on $r = |q|$. The dynamics in this case is totally integrable. In Figure 2.3 we can see a couple of additional examples of magnetic fields satisfying our hypothesis, notice how the particles always bounce back from the boundary.



(a) $B = \frac{1}{1-r} + 7y + 5x^2$ (b) $B = \frac{1}{1-r} + 10x - 2x^2 - 10y^3$

Figure 2.3: More examples in the unit disc

Finally Figure 2.4 shows three different trajectories of particles under a magnetic field which is integrable as $|q| \to 1$, this is an example that doesn't fit our hypothesis. One of the trajectories exhibits a similar behavior to the trajectories in Figure 2.2 and it is confined inside the disc, while the two other trajectories look like arcs with cuspidal endpoints at the boundary, it seems like the magnetic field is not strong enough to push these two trajectories away from the boundary curve.

When experimenting numerically with non integrable magnetic fields, we could not

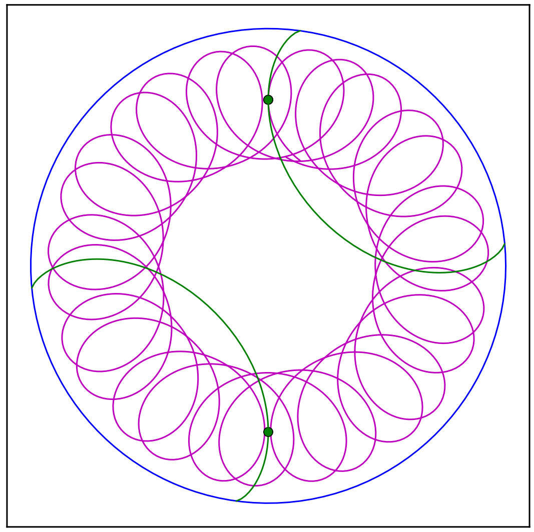


Figure 2.4: $B(x,y) = \frac{1}{\sqrt{1-r}}$

find trajectories that approach the boundary as the “cusp-like” trajectories in Figure 2.4, which is a good confirmation of our result, but a relevant point is that our experiments suggest these “cusp-like” orbits might still take infinite time to reach the boundary.

The equations were solved numerically by using the odeint integrator in the SciPy library for Python, which itself is an implementation of the LSODA integrator from the FORTRAN library ODEPACK. The figures were created with the matplotlib library for Python.

2.4 Proofs of main results

As a first step we establish an estimate for the growth of the magnetic potential along a solution $q(t)$. We then apply this result in the proof of both Theorems 1 and 2.

We start by using normal coordinates to find a convenient potential for the magnetic field $\mathbf{B} = B(n,s)dn \wedge ds$, let us define $\mathbf{A} = A(n,s)ds$ where

$$A(n,s) = -\int_n^{\varepsilon} B(m,s)dm. \tag{2.16}$$

Since $\partial A/\partial n = B$ we have the desired identity $d\mathbf{A} = \mathbf{B}$. Notice that the Euclidian metric can be written in normal coordinates as $dn^2 + (1-\kappa(s)n)^2 ds^2$, where again $\kappa(s)$ denotes the curvature of $\mathcal{C}$, so that the Hamiltonian in these coordinates reads:

$$H_{\mathbf{A}}(q,p) = \frac{p_n^2}{2m} + \frac{(p_s - eA(n,s))^2}{2m(1-\kappa(s)n)^2} \tag{2.17}$$

The strategy of our proof is to show that the potential $A(n,s)$ must remain bounded along a magnetic trajectory that stays close to the boundary. This in turn implies that the particle cannot approach the boundary since the non integrability condition (2.2) in the theorem can be expressed in terms of the potential as:

$$\lim_{n\to 0} |A(n,s)| = \infty \tag{2.18}$$

Our results are based on the following:

Proposition 2. *Assume that the magnetic field* $\mathbf{B}$ *satisfies the hypothesis of Theorem 1. Let* $K = \sup_s |\kappa(s)|$, $K' = \sup_s |\kappa'(s)|$ *and fix* δ *satisfying* $0 < \delta < 1$. *Furthermore choose* ε *small enough so that we may define normal coordinates on* $\Omega_{\mathcal{C}}(\varepsilon)$ *and such that* $\varepsilon < \delta/K$. *If* $q(t) \subset \Omega_{\mathcal{C}}(\varepsilon)$ *is a trajectory with energy* H_0, *then:*

$$|A(q(t))| \leq C_0 + C_1|t| \tag{2.19}$$

where $C_0, C_1 > 0$ *are the following explicit constants:*

$$C_0 = \frac{|p_s(0)| + \sqrt{2mH_0}(1+\delta)}{e}, \quad C_1 = \sqrt{\frac{2H_0}{m}} \cdot \frac{D_{\mathcal{C}}}{1-\delta} + \frac{2H_0 K' \varepsilon}{e(1-\delta)} \tag{2.20}$$

Proof. Notice that by equation (2.17), a particle with energy H_0 must satisfy:

$$\frac{(p_s - eA(n,s))^2}{2m(1-\kappa(s)n)^2} \le H_0 \tag{2.21}$$

By our assumptions we have the following bound:

$$|p_s - eA(n,s)| \le \sqrt{2mH_0}(1-\kappa(s)n) \le \sqrt{2mH_0}(1+\delta) \tag{2.22}$$

which shows that we can bound $A(n,s)$ by finding a bound for p_s instead. In order to do that, we use Hamilton's equation:

$$\dot{p}_s = -\frac{\partial H}{\partial s} = \frac{(p_s - eA(n,s))}{m(1-\kappa(s)n)^2} \cdot e\frac{\partial A}{\partial s} - \frac{(p_s - eA(n,s))^2}{m(1-\kappa(s)n)^3} \cdot \kappa'(s)n \tag{2.23}$$

Now, notice that $|\partial A/\partial s| < D_C$ since:

$$\begin{aligned}
\left|\frac{\partial A}{\partial s}(n,s)\right| &= \left|\int_n^{\varepsilon} \frac{\partial B}{\partial s}(m,s)dm\right| \\
&\le \int_n^{\varepsilon} \left|\frac{\partial B}{\partial s}(m,s)\right| dm \\
&\le \int_0^{\varepsilon} \left|\frac{\partial B}{\partial s}(m,s)\right| dm \\
&\le \sup_s \int_0^{\varepsilon} \left|\frac{\partial B}{\partial s}(m,s)\right| dm \\
&= D_C
\end{aligned}$$

so we obtain the following bound along $q(t)$:

$$|\dot{p}_s| \le \sqrt{\frac{2H_0}{m}} \cdot \frac{eD_C}{1-\delta} + \frac{2H_0K'\varepsilon}{1-\delta} = eC_1 \tag{2.24}$$

Integrating inequality (2.24) from time 0 to time t we obtain the following:

$$|p_s(t)| \le |p_s(0)| + eC_1|t| \tag{2.25}$$

We deduce now from (2.22):

$$
\begin{aligned}
|A(q(t))| &\leq \frac{|p_s(t) - eA(q(t)|}{e} + \frac{|p_s(t)|}{e} \\
&\leq \frac{\sqrt{2mH_0}(1+\delta)}{e} + \frac{|p_s(0)|}{e} + C_1|t| \\
&= C_0 + C_1|t|
\end{aligned} \tag{2.26}
$$

which is the estimate we wanted. $\square$

We move on to the proof of the main theorem:

Proof. (*of Theorem 1*): Suppose that a **B**-geodesic $q:[0,T]\to\Omega$ reaches the boundary at time T at a connected component $\mathcal{C}$. By the first condition in our theorem, i.e. equation (2.2), we must have:

$$\lim_{t\to T} A(q(t)) = \infty \tag{2.27}$$

we may suppose further,without loss of generality, that $q(t)$ lies in $\Omega_{\mathcal{C}}(\varepsilon)$ for all $t\in[0,T]$ for some ε that satisfies the hypothesis of propostion 2. We have then:

$$|A(q(t))| \leq C_0 + C_1 T \tag{2.28}$$

Hence we conclude that A is bounded along the trajectory, which contradicts (2.27) so that no such **B**-geodesic may exist. $\square$

Finally, in order to prove Theorem 2 we express the distance from a particle $q(t)$ to the boundary component $\mathcal{C}$ using normal coordinates.

Proof. (*of Theorem 2*): Notice that if $q(t)$ is in $\Omega_{\mathcal{C}}(\varepsilon)$ and is given in normal coordinates by $(n(t), s(t))$, we have:

$$\operatorname{dist}(q(t), \mathcal{C}) = n(t) \tag{2.29}$$

since $n(t)$ is the length of the line segment perpendicular to $\mathcal{C}$ connecting $q(t)$ to $\mathcal{C}$.

In order to obtain a lower bound for the distance between $q(t)$ and the boundary we use proposition 2 to obtain such a bound for $n(t)$ instead.

Given the form of the magnetic field $B(n,s)$ we obtain the following estimate for $A(q(t))$:

$$\begin{aligned} |A(q(t))| &= \left| -\int_{n(t)}^{\varepsilon} \frac{M}{m^{\alpha}} + f(m,s)dm \right| \\ &\geq \left| \int_{n(t)}^{\varepsilon} \frac{M}{m^{\alpha}} dm \right| - NC_f \end{aligned} \tag{2.30}$$

Moreover, since the magnetic field B satisfies the conditions of Proposition 2, we also have:

$$|A(q(t))| \leq C_0 + C_1|t| \leq C_0 + C_1 T \tag{2.31}$$

Putting these two estimates together we obtain:

$$\left| \int_{n(t)}^{\varepsilon} \frac{1}{m^{\alpha}} dm \right| \leq \ell(T) \tag{2.32}$$

Evaluating the integral on the left hand side of (2.32) explicitly we arrive at the result of Theorem 2. $\square$

Chapter 3

Evidence of quantum tunneling

3.1 Relations to Quantum Systems

We discuss in this section an example that illustrates an interesting dichotomy between the behavior of a classical system and its quantization. To be more precise, we will describe a family of magnetic fields $\mathbf{B}_\alpha$ defined over the unit disc for which the Hamiltonian dynamics of a classical particle under the influence of $\mathbf{B}_\alpha$ is complete, but the dynamics of a quantum particle under the influence of the same field is not.

From now on denote by Ω the unit disc in $\mathbb{R}^2$

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = r^2 < 1\} \tag{3.1}$$

Given a constant $\alpha > 0$ consider the magnetic field:

$$\mathbf{B}_\alpha(x,y) = \alpha \frac{r-2}{(r-1)^2} dx \wedge dy \tag{3.2}$$

If we choose a potential 1-form $\mathbf{A} = A_x dx + A_y dy$ for $\mathbf{B}_\alpha$, we may define the magnetic

Schrödinger operator $\hat{H}_{\mathbf{A}}$ as the quantization of the classical Hamiltonian defined in equation (1.3), i.e.:

$$\hat{H}_{\mathbf{A}} = -(\partial_x - ieA_x)^2 - (\partial_y - ieA_y)^2 \tag{3.3}$$

Where we set $\hbar = 1$ for convenience, which can be accomplished by choosing appropriate units.

A quantum particle inside Ω is modeled by a function $\psi_0(x)$ in $L^2(\Omega)$, with $|\psi_0|_{L^2(\Omega)} = 1$ and the corresponding system is governed by Schrödinger's equation:

$$i\frac{\partial \psi}{\partial t} = \frac{1}{2m}\hat{H}_{\mathbf{A}}\psi \tag{3.4}$$

Definition 3. *We say that the quantum system is complete or that the quantum particle is confined by the magnetic field* $\mathbf{B}$ *if the operator* $\hat{H}_{\mathbf{A}}$ *with domain* $C_0^\infty(\Omega)$ *of smooth functions compactly supported in* Ω *is essentially self-adjoint (this property does not depend on the choice of potential* $\mathbf{A}$*, see [6]).*

Since the unitary maps between the deficiency spaces of the operator $\hat{H}_{\mathbf{A}}$ can be interpreted as boundary conditions for Schrödinger's equation 3.4, we observe that an operator that is essentially self-adjoint can be interpreted as a Hamiltonian for which Schrödinger's equation can be solved without need for boundary conditions. If $\hat{H}_{\mathbf{A}}$ is essentially self-adjoint, there is a unique self-adjoint extension of $\hat{H}_{\mathbf{A}}$ and by Stone's theorem a unique strongly continuous unitary one-parameter subgroup $(U_t)_{t\in\mathbb{R}}$ so that $\psi(t,x) = U_t\psi_0(x)$ is a solution to equation (3.4).

Notice that the solutions to this equation are truly confined. Since the operators U_t are unitary we have

$$|\psi_t|_{L^2(\Omega)} = |U_t\psi_0|_{L^2(\Omega)} = |\psi_0|_{L^2(\Omega)} = 1 \tag{3.5}$$

which means that for any time t, the particle is observed inside of Ω with probability 1, since $|\psi_t|^2$ is the probability density for the position operators.

In [6] Colin de Verdière and Truc notice that while the operator $\hat{H}_{\mathbf{A}}$ associated to $\mathbf{B}_\alpha$ is essentially self-adjoint for $\alpha > \sqrt{3}/2$, for values of α in the range $0 < \alpha < \sqrt{3}/2$ the operator is not essentially self-adjoint in $C_0^\infty(\Omega)$. So that in the latter case, in order to solve Schrödinger's equation we must impose boundary conditions on the solutons of equation 3.4 (or equivalently choose a self-adjoint extension of the symmetric operator $\hat{H}_{\mathbf{A}}$). This means that as time evolves certain wave functions will interact with the boundary, so not all quantum particles are confined to Ω. In this sense we'll say the quantum dynamics is not complete

On the other hand by parametrizing the boundary by arclength through $\gamma(s) = (\cos(s), \sin(s))$, using normal coordinates (n, s) we can express the magnetic field by (notice $n = 1 - r$):

$$\mathbf{B}_\alpha = \alpha \left(\frac{1}{n^2} - 1 \right) dn \wedge ds \tag{3.6}$$

Which fits the hypothesis of Theorem 1 as long as $\alpha \neq 0$ and hence for a classical particle, the dynamics is complete.

One interesting remaining question is to prove that for magnetic fields $\mathbf{B}_\alpha$ for which the corresponding Hamiltonian operator is not essentially self-adjoint in $C_0^\infty(\Omega)$, there are self-adjoint extensions of the Hamiltonian that yield solutions ψ_t in $L^2(\mathbb{R}^2)$ for which the norm restricted to Ω decreases with time, so that some particles would in fact "leak" through the boundary walls, which would confirm the phenomenon of quantum tunneling for these systems.

Part III

Higher dimensional results

Chapter 4

Magnetic confinement on manifolds

4.1 Magnetic fields with σ_∞-blow up

In this section we describe the class of magnetic fields whose magnetic flow we shall analyze. What we require from these fields is that they diverge to infinity at the boundary at a fast enough rate and that on a neighborhood of ∂M, they are controlled by a closed 1-form σ_∞ defined on the boundary. To precisely phrase this condition we must first fix normal coordinates at a neighborhood of ∂M as in the planar case.

Let $D = M \setminus \partial M$. Notice that for $\varepsilon > 0$ small enough the neighborhood

$$\Omega_\varepsilon = \{q \in D \mid \mathrm{dist}(q, \partial M) < \varepsilon\}$$

is a collar neighborhood of the boundary and possesses normal coordinates

$$\begin{array}{rccc} \phi: & (0,\varepsilon) \times \partial M & \to & \Omega_\varepsilon \\ & (n,x) & \mapsto & \exp_x(n\nu(x)) \end{array}$$

where $\nu(x)$ denotes the unit inward normal vector to the boundary.

We denote by $\pi_{\partial M} : \Omega_\varepsilon \to \partial M$ and $\pi_\nu : \Omega_\varepsilon \to (0,\varepsilon)$ the natural projections $\pi_{\partial M} = \mathrm{pr}_2 \circ \phi^{-1}$ and $\pi_\nu = \mathrm{pr}_1 \circ \phi^{-1}$. Using these coordinates we also obtain that over Ω_ε the metric looks like a warped product:

$$g = dn^2 + g_{\partial M}(n)$$

where $g_{\partial M}(n)$ is a Riemannian metric on the fibers $\pi_\nu^{-1}(n)$. Additionally, the distance function from the boundary is smooth over this neighborhood, since it is given by:

$$\mathrm{dist}(p, \partial M) = n(p) = \pi_\nu(p) \tag{4.1}$$

The normal coordinates ϕ also induce a splitting over Ω_ε of the tangent and cotangent bundles. For ease of notation denote $T_\varepsilon \partial M := \pi_{\partial M}^* T\partial M$ the subbundle of vectors tangent to the fibers $\pi_\nu^{-1}(n)$, similarly denote $T_\varepsilon^* \partial M := \pi_{\partial M}^* T^* \partial M$. We then obtain orthogonal splittings:

$$T\Omega_\varepsilon \cong \mathbb{R}\partial_n \oplus T_\varepsilon \partial M, \quad T^*\Omega_\varepsilon \cong \mathbb{R}dn \oplus T_\varepsilon^* \partial M,$$

Consider now the Hodge star operator $* : \bigwedge^k T^*M \to \bigwedge^{n-k} T^*M$ and the natural map induced by the Riemannian metric $G : TM \to T^*M$ given by $Gv = g(v, \cdot)$. We define the magnetic $(n-2)$-vector field by:

$$\mathbb{B} = \left(\bigwedge^{n-2} G^{-1}\right) * \mathbf{B} \in \Gamma\left(\bigwedge^{n-2} TM\right)$$

We now give some definitions in order to build a simple model for a confining magnetic field in D, later we shall define a class of admissible perturbations that will not affect the confining property of such fields.

Definition 4. *We will say that a magnetic field* $\mathbf{B}$ *defined over* Ω_ε *is* ∂M-tangent *if the corresponding magnetic* $(n-2)$*-vector field* $\mathbb{B}$ *is tangent to the fibers* $\pi_\nu^{-1}(n)$ *that is:*

$$\mathbb{B} \in \Gamma(\bigwedge^{n-2} T_\varepsilon \partial M)$$

In dimension $n = 3$, $\mathbb{B} = \vec{B}$ is an actual vector field and it corresponds to the classical notion of magnetic field in $\mathbb{R}^3$. In this case the magnetic field $\mathbf{B}$ is ∂M-tangent if and only if the vector field $\vec{B}$ is tangent to the fibers $\pi_\nu^{-1}(n)$.

Since $T_\varepsilon \partial M$ is a vector bundle of rank $n-1$, we can see that over points $q \in \Omega_\varepsilon$ where $\mathbf{B}_q \neq 0$ we have that $\mathbb{B}$ is in fact an $(n-2)$-blade of this bundle (that is, a homogeneous tensor in the space $\bigwedge^{n-2} T_\varepsilon \partial M$) and it represents a field of hyperplanes inside the fibers of $T_\varepsilon \partial M$, in fact these hyperplanes are exactly $\ker \mathbf{B}_q$.

Notice that if $\mathbf{B}$ is ∂M-tangent, there is a unique 1-form $\sigma \in \Gamma(T_\varepsilon^* \partial M)$ such that $\mathbf{B} = dn \wedge \sigma$. Since $\mathbf{B}$ must be closed, the 1-form σ should satisfy:

$$d\sigma \wedge dn = -d\mathbf{B} = 0$$

Furthermore we have whenever $\mathbf{B}_q \neq 0$

$$\ker \mathbf{B} = \ker\left(\sigma|_{T_\varepsilon \partial M}\right)$$

Definition 5. *Given a* ∂M*-tangent magnetic field* $\mathbf{B} = dn \wedge \sigma$*, with* $\sigma \in \Gamma(T_\varepsilon^* \partial M)$*. Let* $S = G^{-1}\sigma$ *and if* $\mathbf{B}_q \neq 0$ *define* $\overline{S}_q = S_q/|S_q|$*. We call* $\overline{S}_q$ *the* direction *of* $\mathbf{B}$.

Next we provide a description of the magnetic force of a ∂M-tangent magnetic field.

Proposition 3. *Let* $\mathbf{B} = dn \wedge \sigma$ *be a* ∂M*-tangent magnetic field. Given a point* $q \in \Omega_\varepsilon$*, if* $\mathbf{B}_q = 0$

the magnetic force $Y_q\dot{q}$ is zero for all $\dot{q} \in T_qM$. If $\sigma_q \neq 0$, then:

$$Y_q\partial_n = -|\sigma_q|\overline{S}_q, \quad Y_q\overline{S}_q = |\sigma_q|\partial_n, \quad Y_qW = 0 \textit{ for } W \in \ker\mathbf{B}$$

That is, the force is zero on particles with velocity $\dot{q}$ tangent to the $(n-2)$-planes $\ker\mathbf{B}$*, and it acts by a 90 degrees rotation composed with a scaling by a factor of $|\sigma|$ for velocities in span$(\partial_n, \overline{S})$.*

Proof. We simply notice that since $\mathbf{B} = dn \otimes \sigma - \sigma \otimes dn$ we have:

$$g(\partial_n, Y_q\overline{S}) = \mathbf{B}_q(\partial_n, \overline{S}) = |\sigma|$$

and that for $U \in T_qM, W \in \ker\mathbf{B}_q$ we obtain:

$$g(U, Y_qW) = \mathbf{B}_q(U, W) = 0$$

$\square$

This description makes it plausible that if $|\sigma|$ approaches infinity at the boundary then particles that try to exit the region (which inevitably would have some nontrivial ∂_n-component in their velocity) will be pushed sideways in the direction $\overline{S}$ by a very strong magnetic force and would not be able to leave the region.

Definition 6. *We say that a magnetic field $\mathbf{B}_\infty$ is ∂M-*regular *if there is a 1-form $\sigma_\infty \in \Omega^1(\partial M)$ on the boundary of M and a function $f : [0,\varepsilon] \to \mathbb{R}$ such that writing $\sigma_\varepsilon := \pi^*_{\partial M}\sigma_\infty$ over Ω_ε we have:*

$$\mathbf{B} = f(n)dn \wedge \sigma_\varepsilon$$

We call σ_∞ the asymptotic 1-form of $\mathbf{B}_\infty$.

Remark 1: Notice that this condition is equivalent to requiring that the direction $\overline{S}$ is constant along the normal rays $\exp_x(n\nu(x))$ (whenever $\mathbf{B}_q \neq 0$) and that if $\mathbf{B}_q = 0$ for some $q \in \Omega_\varepsilon$ then $\mathbf{B} = 0$ along the normal ray going through q .

Remark 2: Denote by $S_\infty = G^{-1}\sigma_\infty$. Notice that if f is nowhere zero on $(0,\varepsilon]$ in order for $\mathbf{B}$ to be closed we must have $d\sigma_\infty = 0$ which in terms of the vector field S_∞ reads $\nabla_g \times S_\infty = 0$. Here we compute the curl using the metric g restricted to ∂M and using the general formula for a Riemannian manifold M and vector field $V \in \Gamma(TM)$:

$$\nabla_g \times V = \left(\bigwedge^{n-2} G^{-1}\right) * dGV \in \Gamma(\bigwedge^{n-2} TM)$$

We will study magnetic fields that are perturbations of ∂M-regular fields. In the next definition we state which are the admissible perturbations. Given a k-form ω and a vector V we denote the contraction with V by $\iota_V\omega$, that is:

$$\iota_V\omega(W_1,\ldots,W_{k-1}) = \omega(V,W_1,\ldots,W_{k-1})$$

We also denote by $\mathcal{L}_V\omega$ the Lie derivative along V.

Definition 7. *Given a 1-form $\sigma_\infty \in \Omega^1(\partial M)$, let $\sigma_\varepsilon = \pi^*_{\partial M}\sigma_\infty$ and $S_\varepsilon = G^{-1}\sigma_\varepsilon$. We say that a magnetic field $\mathbf{B}_{per}$ is a σ_∞-*perturbation *if $\iota_{S_\varepsilon}\mathbf{B}_{per}$ and $\mathcal{L}_{S_\varepsilon}\mathbf{B}_{per}$ are bounded on Ω_ε and there is a function $h : (0,\varepsilon] \to \mathbb{R}$ satisfying $\int_0^\varepsilon nh(n)dn < \infty$ such that $|\mathbf{B}_{per}|_q, |\nabla\mathbf{B}_{per}|_q \leq h(n(q))$.*

Remark 1: The function h in the above definition should have a growth rate of the form $h(n) \sim 1/n^{2-\delta}$ for some $\delta > 0$.

Remark 2: If a magnetic field $\mathbf{B}$ is C^1-bounded, that is if $|\mathbf{B}|$ and $|\nabla\mathbf{B}|$ are both bounded on D, then it is automatically a σ_∞-perturbation for any choice of $\sigma_\infty \in \Omega^1(\partial M)$.

We now provide the definition for the class of magnetic fields we shall study. These are fields defined on the interior $D = M \setminus \partial M$ which go to infinity with asymptotic direction S_∞ in a controlled way.

Definition 8. *Given a closed 1-form $\sigma_\infty \in \Omega^1(\partial M)$, a magnetic field* $\mathbf{B}$ *defined over D has* σ_∞-blow up *if there is a ∂M-regular magnetic field $\mathbf{B}_\infty = f(n)dn \wedge \sigma_\varepsilon$ with $\sigma_\varepsilon = \pi^*_{\partial M}\sigma_\infty$ and $f(n) : (0, \varepsilon] \to \mathbb{R}$ satisfying $\int_0^\varepsilon f(n)dn = \pm\infty$, and a σ_∞-perturbation $\mathbf{B}_{per}$ such that over Ω_ε we have $\mathbf{B} = \mathbf{B}_\infty + \mathbf{B}_{per}$.*

In preparation for the main theorem we will need the following small lemma:

Lemma 1. *Let $c : I \to D$ be a* $\mathbf{B}$*-geodesic with $I \subset \mathbb{R}$ its maximal domain of definition. Suppose $T^\infty = \sup(I) \neq \infty$, then the limit $x^\infty = \lim_{t \nearrow T^\infty} c(t)$ exists and it belongs to ∂M. Similarly, if $T^{-\infty} = \inf(I) \neq -\infty$, the limit $x^{-\infty} = \lim_{t \searrow T^{-\infty}} c(t)$ exists and it belongs to ∂M.*

Proof. First notice that by the same argument used in corollary 1 if either of the limits $x^{\pm\infty}$ exist then they must belong to ∂M, since for example if $T^\infty \neq \infty$ and $x^\infty \notin \partial M$ then the future of the $\mathbf{B}$-geodesic would be contained in a compact subset of D and therefore it would have to be defined for all future time, contradicting $T^\infty \neq \infty$.

Now, since $|\dot{c}(t)|$ is constant, the curve c is Lipschitz which means that the sequence $(c(T^\infty - 1/n))_{n \in \mathbb{N}}$ is Cauchy and hence must converge to some x^∞ which clearly must be the desired limit. □

Given $\sigma_\infty \in \Omega^1(\partial M)$, we denote its zero locus by:

$$Z(\sigma_\infty) = \{x \in \partial M \mid (\sigma_\infty)_x = 0\}$$

We may now state our theorem:

Theorem 3. *Let* $\mathbf{B}$ *be a magnetic field defined on D with* σ_∞*-blow up. If a* $\mathbf{B}$*-geodesic* $c(t)$ *in D reaches the boundary in finite time in the future, then* $x^\infty \in Z(\sigma_\infty)$*, similarly if it approaches* ∂M *in finite time in the past then* $x^{-\infty} \in Z(\sigma_\infty)$.

Finally, as a consequence of theorem 3 we obtain the following:

Theorem 4. *Let* σ_∞ *be a non-vanishing 1-form on* ∂M*. Let* $\mathbf{B}$ *be a magnetic field on* $D = M \setminus \partial M$ *with* σ_∞*-blow up, then every* $\mathbf{B}$*-geodesic in D is defined for all time. In particular any* $\mathbf{B}$*-geodesic must take infinite time to approach the boundary.*

As before we obtain completeness of the magnetic flow as a corollary of theorem 4

Corollary 2. *Let* $\mathbf{B}$ *be a magnetic field satisfying the hypothesis of Theorem 4. Then, the Hamiltonian flow of* $X_{\mathbf{B}}$ *on* T^*D *is complete.*

Proposition 4. *For any closed 1-form* $\sigma_\infty \in \Omega^1(\partial M)$*, the space of magnetic fields with* σ_∞*-blow up is a nontrivial open set in the space of closed 2-forms on D with the uniform* C^1 *topology.*

Proof. It is clear from the second remark after definition 7 that this set is open in the uniform C^1 topology it is only necessary to show that it is nontrivial.

For that, given a closed 1-form $\sigma_\infty \in \Omega^1(\partial M)$, again let $\sigma_\varepsilon = \pi^*_{\partial M}\sigma_\infty$ and choose any smooth function $f(n) : (0,\varepsilon] \to \mathbb{R}$ with the property that $\int_0^\varepsilon f(n)dn = \pm\infty$ and such that there is some $0 < \delta < \varepsilon$ so that for $n > \delta$, $f(n) = 0$. Define the 2-form

$$\mathbf{B} = f(n)dn \wedge \sigma_\varepsilon$$

This form is closed, since:

$$d\mathbf{B} = d(f(n)dn) \wedge \sigma_\varepsilon - f(n)dn \wedge d(\sigma_\varepsilon) = 0$$

and it extends naturally to a closed 2-form defined on the whole M which has σ_∞-blow up. $\square$

4.2 Toroidal Domains

In order to confine every particle to the interior of our manifold using a magnetic force we see from theorem 3 that it is sufficient to find a nowhere vanishing closed 1-form over the boundary ∂M. This of course imposes a topological constraint on the manifold M.

Definition 9. *We call M a* toroidal domain *if its boundary ∂M carries a nowhere vanishing closed 1-form.*

Our goal now will be to describe the toroidal condition in a more geometric way. We will say that ∂M is fibered over the circle if there is a submersion $s : \partial M \to S^1$. Notice that in this case ∂M carries a non-vanishing closed 1-form, simply by considering the pullback $s^* d\theta$ of any non-vanishing 1-form $d\theta$ on the circle. Conversely, we have the following:

Proposition 5. *A manifold M is a toroidal domain, if and only if its boundary is fibered over the circle.*

Proof. Let σ_∞ be a nowhere vanishing 1-form on ∂M, we want to construct a submersion from ∂M to S^1. Let $H_1(\partial M, \mathbb{Z})$ be the first singular homology group of ∂M, which is finitely generated since ∂M is compact.

Let $\mathrm{Tor}(H_1(\partial M,\mathbb{Z}))$ be its subgroup of torsion elements and denote

$$H_1^f(\partial M,\mathbb{Z}) = H_1(\partial M,\mathbb{Z})/\mathrm{Tor}(H_1(\partial M,\mathbb{Z})).$$

This is a free $\mathbb{Z}$-module and so we may choose a collection $c_1,\dots,c_k$ of closed 1-cycles that form a $\mathbb{Z}$-basis for it. Choose then a collection of closed 1-forms $\omega^1,\dots,\omega^k$ dual to the c_i's so that:

$$\int_{c_i}\omega^j = \delta_i^j$$

where δ_i^j denotes the Kronecker delta, $\delta_i^i = 1$ and $\delta_i^j = 0$ if $i \neq j$.

Now given numbers $e_1,\dots,e_k$ consider the 1-form:

$$\sigma_e = \sigma + e_i\omega^i$$

Its periods over the basis of $H_1^f(\partial M,\mathbb{Z})$ are:

$$\int_{c_i}\sigma_e = \left(\int_{c_i}\sigma\right) + e_i$$

We may then choose the e_i's small enough so that σ_e is still non-vanishing and such that all these periods are rational. By multiplying σ_e by a large enough integer $N >> 0$ we obtain a closed 1-form $\sigma^* = N\sigma_e$ that is still non-vanishing and such that all of its periods over the c_i's are in fact integers.

Denote by

$$H = \left\{ \int_c \sigma^* \,\middle|\, c \text{ is any closed 1-cycle in } \partial M \right\}$$

The set H forms a discrete subgroup of $\mathbb{R}$ (it is contained in $\mathbb{Z}$) so $\mathbb{R}/H \cong S^1$. Now, fix a base point $p_0 \in \partial M$ and consider the map

$$\begin{aligned} s: \quad \partial M \quad &\to \quad \mathbb{R}/H \\ p \quad &\mapsto \quad \int_{p_0}^{p} \sigma^* \quad \mod H \end{aligned}$$

The integral on the right is defined along any path connecting p_0 and p, its values can only differ by an element of H so s is a well defined map.

Finally a straightforward computation in local charts using straight lines for paths allows one to prove it is also a submersion. $\square$

Remark: Notice in particular that by the Poincaré-Hopf theorem if M is toroidal then the Euler characteristic of the boundary must vanish.

4.3 Examples

Before getting into the proofs of the main theorems, let's discuss some examples where this result can be applied. Let us start by analyzing some examples of toroidal domains in different dimensions.

4.3.1 Surfaces

In dimension 2, any surface with non-empty boundary is a toroidal domain (since their boundary is simply a union of disjoint circles). Consider for example the unit disc in $\mathbb{R}^2$; according to the theorem if we choose a magnetic field $\mathbf{B} = B(x,y)dx \wedge dy$ that has the form:

$$B(x,y) = f(r) + b(x,y)$$

where $b(x,y)$ is any smooth function with bounded derivative defined over the unit disc and $f(r)$ is a function of the radius alone satisfying $\int_{1-\varepsilon}^{1} rf(r)dr = \pm\infty$, then every **B**-geodesic is confined to the interior of the unit disc for all time.

4.3.2 3-dimensional Solid Tori

In dimension 3 a toroidal domain must have its boundary consisting of a disjoint union of tori. In the simpler case of a solid torus (possibly the most important case since this is the shape of Tokamaks), we may write the 2-form in terms of vector fields and the result implies that every $\vec{B}$-geodesic is confined as long as the magnetic field has the form

$$\vec{B} = f(n)\vec{X} + \vec{B}_b \tag{4.2}$$

where on a neighborhood of the boundary $\vec{X}$ agrees with the extension of a vector field defined on the boundary torus diffeomorphic to a constant vector field, $\vec{B}_b$ is a smooth magnetic field defined on the open solid torus with bounded derivatives and $f(n)$ is a function satisfying $\int_0^{\varepsilon} f(n)dn = \pm\infty$ for some $\varepsilon > 0$ small enough.

4.3.3 Tubular Neighborhoods

For higher dimensional examples one may consider any manifold of the form $M = X \times S^1$ where X is a compact oriented manifold with boundary. Since such manifolds are clearly toroidal we deduce that there are confining magnetic fields defined on $D = M \setminus \partial M$.

In particular if one considers a closed simple curve C inside some given orientable manifold of dimension n and take M to be a closed tubular neighborhood of this curve, then M is diffeomorphic to an orientable $(n-1)$-disc bundle over S^1, since over the circle the orientability of a bundle implies its triviality this disc bundle must be trivial, so we may deduce that $M \cong D^{n-1} \times S^1$ and M is therefore toroidal.

4.3.4 Flat Circle Bundles

In the same spirit as the previous example we may also consider a base manifold X with boundary and take M to be any circle bundle over X with a flat connection α. The connection 1-form is closed (since it is flat) and never-vanishes, so M is a toroidal domain. In this case the 2-form B_∞ near the boundary can be written as

$$\mathbf{B}_\infty = f(n)dn \wedge \alpha$$

where α denotes the connection 1-form.

Our theorem then implies that any flat circle bundle over a manifold with boundary carries confining magnetic fields.

Remark: Recall that since the holonomy of a loop is homotopy invariant on a flat bundle, there is a correspondence between S^1-bundles with a flat connection (M,α) over a compact base X and representations $\pi_1(X) \to U(1)$.

4.3.5 Log-Symplectic Magnetic Fields

In this section we describe a class of examples of magnetic fields with σ_∞-blow up which are symplectic in the interior D. These magnetic fields arise naturally from a special class of Poisson manifolds called log-symplectic manifolds. We first recall some basic definitions.

Given $\alpha \in \bigoplus^k TM$, write $\alpha = (a_1, \ldots, a_k)$ and denote by $\overline{\alpha} = a_1 \wedge \cdots \wedge a_k$ the corresponding homogeneous k-vector field. Denote the i-th deletion of α by

$$D_i(\alpha) = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k).$$

Definition 10. *The Schouten-Nijenhuis bracket* $[\cdot,\cdot] : \bigwedge^{\bullet} TM \times \wedge^{\bullet} TM \to \wedge^{\bullet} TM$ *of multivector fields is uniquely defined by its action on homogeneous elements. Given* $\alpha \in \bigoplus^k TM$ *and* $\beta \in \bigoplus^l TM$ *we define:*

$$[\overline{\alpha}, \overline{\beta}] = \sum_{i,j} (-1)^{i+j} [a_i, b_j] \wedge \overline{D_i(\alpha)} \wedge \overline{D_j(\beta)}$$

where $[a_i, b_j]$ *denotes the Lie bracket of vector fields.*

Definition 11. *A Poisson structure on a manifold* M *is a bi-vector field* $\pi \in \Gamma(\bigwedge^2 TM)$ *satisfying the Jacobi identity* $[\pi, \pi] = 0$.

Given a Poisson manifold (M, π) we have a map $\Pi : T^*M \to TM$ given by $\Pi(\lambda) = \pi(\lambda, \cdot)$. If the manifold M has dimension $2n$ we may consider the $2n$-vector field $\pi^n = \wedge^n \pi \in \Gamma(\bigwedge^{2n} TM)$. We then define the *singular locus* of π by:

$$Z = \{p \in M \mid \pi_p^n = 0\}$$

We also call the complement $D = M \setminus Z$ the *symplectic locus* of π. Over the symplectic locus the map Π is invertible and we may use its inverse to define the symplectic form $\omega(v, w) = \left(\Pi^{-1}(v)\right)(w)$. The fact that this form is closed is a consequence of the Jacobi identity $[\pi, \pi] = 0$. We sometimes denote $\omega = \pi^{-1}$.

Definition 12. *A log-symplectic manifold is an even dimensional Poisson manifold* (M, π) *such* *that π^n only has nondegenerate zeroes. That is the section π^n of the line bundle $\bigwedge^{2n}TM$ is transversal to the zero section.*

The nondegeneracy of π^n implies that the singular locus Z is a hypersurface of M and the symplectic locus D is a dense open subset. For more information on log-symplectic manifolds and many examples see [5], for some of the structure theory see [7].

We now consider the magnetic field $\mathbf{B} = \pi^{-1}$ defined on D. The situation here is analogous to the case of a manifold M with boundary, except now the singular locus Z plays the role of the boundary ∂M. We will describe how Theorem 4 can be applied in this case to show that a $\mathbf{B}$-geodesic in D may never reach the singular locus in finite time. Notice that the vanishing of π^n along Z, translates to the blowing up of $\mathbf{B}$ along the singular locus.

Let (M, g, π) be a compact orientable Riemannian log-symplectic manifold and let $NZ = TZ^{\perp}$ be the normal bundle of Z. One may prove that since M is orientable the normal bundle NZ must be trivial. Fix a unit normal vector $\nu \in \Gamma(NZ)$ and denote by n the induced fiber coordinate. We have the following local form on a neighborhood of the singular locus (we refer the reader to [7] for a proof of this theorem):

Theorem 5. *Let (M, g, π) be a compact orientable Riemannian log-symplectic manifold. There is a nowhere vanishing closed 1-form $\sigma \in \Gamma(T^*Z)$ and a closed 2-form $\eta \in \Gamma(\bigwedge^2 T^*Z)$ such that $\sigma \wedge \beta^{n-1} \neq 0$. Furthermore there is a neighborhood of Z in M which is symplectomorphic to a neighborhood of the zero section of NZ with symplectic form $d(\log|n|) \wedge \sigma + \beta$.*

This means that close to Z the magnetic field $\mathbf{B}$ has the form:

$$\mathbf{B} = \frac{1}{|n|} dn \wedge \sigma + \beta$$

Which is a magnetic field of σ-blow up, since $1/|n|$ is non-integrable as $n \to 0$ and β is a C^∞-bounded 2-form, so that it is a σ-perturbation. We then obtain the following:

Corollary 3. *Let (M, g, π) be a compact orientable Riemannian log-symplectic manifold with magnetic field $\mathbf{B} = \pi^{-1}$. Any $\mathbf{B}$-geodesic in D is defined for all time and never reaches the singular locus Z.*

4.4 Proof of the Main Theorem

We prove Theorem 3 by contradiction. Let $\mathbf{B}$ be a magnetic field with σ_∞-blow up for some 1-form $\sigma_\infty \in \Omega^1(\partial M)$ and assume there is some $\mathbf{B}$-geodesic $c : I \to D$ with $I \subset \mathbb{R}$ its maximal domain of definition. Suppose $T^\infty = \sup(I) \neq \infty$, by lemma 1 we know that there is a limit $x^\infty = \lim_{t \nearrow T^\infty} c(t) \in \partial M$ which we assume by contradiction does not lie in $Z(\sigma_\infty)$.

Before proceeding, we must choose a chart of M at x^∞, $\mathbf{q} : \overline{U} \to \mathbb{R}^d$ which is adapted to σ_∞, we also denote $U = \overline{U} \cap D$.

Lemma 2. *Let $x^\infty \in \partial M \setminus Z(\sigma_\infty)$. There is a chart of M at x^∞, $\mathbf{q} : \overline{U} \to \mathbb{R}^d$ satisfying:*

1. *$U \subset \Omega_\varepsilon$ and $(\sigma_\varepsilon)_u = (\pi^*_{\partial M}\sigma_\infty)_u \neq 0$ for all $u \in U$.*
2. *If we denote $\mathbf{q}(u) = (q^1(u), \ldots, q^d(u))$, then $q^1(u) = n(u) = dist(u, \partial M)$ and denoting $\theta(u) = q^2(u)$ we have: $d\theta = \sigma_\varepsilon$ and $S_\varepsilon = G^{-1}\sigma_\varepsilon = |\sigma_\varepsilon|^2 \partial_\theta$*
3. *There is some $r > 0$ such that $q(U) = \{q \in \mathbb{R}^d \mid |q| < r, \quad q^1 > 0\}$*

Proof. We start by choosing $\overline{U}$ small enough so that $(\sigma_\varepsilon)_u \neq 0$ for all $u \in \overline{U}$, the function $n(u) = \text{dist}(u, \partial M)$ is smooth and such that we can find a primitive $\theta : \overline{U} \to \mathbb{R}$ for σ_ε, that is

$d\theta = \sigma_\varepsilon$.

Next we choose the remaining coordinates by noticing that the distribution

$$D = \ker dn \cap \ker \sigma_\varepsilon \leq T\overline{U}$$

is integrable since both 1-forms are closed. In order to see this notice that if α is a 1-form and X, Y are local vector fields in $\ker \alpha$, the formula:

$$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$$

reduces to $\alpha([X,Y]) = 0$ if $d\alpha = 0$, which means that $\ker \alpha$ is integrable.

We then choose the remaining coordinates $q^3, \ldots, q^d$ so that the leaves of D are defined by setting n and θ to be constant.

This in turn implies that S_ε is perpendicular to $\partial_n, \partial_{q^3}, \ldots \partial_{q^d}$ and a simple calculation shows that it must have the form required $S_\varepsilon = |\sigma_\varepsilon|^2 \partial_\theta$.

Finally by making $\overline{U}$ possibly a bit smaller we can obtain condition 3. which finishes our proof. $\square$

By translating the time parameter we may focus on the tail end of the curve $c(t)$ and assume that it is defined for $0 \leq t < \tau^\infty$ with $x^\infty = \lim_{t \to \tau^\infty} c(t)$ and that $c(t)$ is completely contained in the open set U.

In these coordinates a magnetic field with σ_∞-blow up looks like:

$$\mathbf{B} = f(n) dn \wedge d\theta + \mathbf{B}_{\text{per}}$$

We may also choose a convenient magnetic potential $\mathbf{A}$ for $\mathbf{B}$. In order to do that define $F(n)$ by

$$F(n) = -\int_n^{\varepsilon} f(m)dm \tag{4.3}$$

Since $F(n)$ is an antiderivative of $f(n)$ we may choose:

$$\mathbf{A} = F(n)d\theta + \mathbf{A}_{\text{per}} \tag{4.4}$$

where $\mathbf{A}_{\text{per}}$ is a smooth 1-form defined over the domain of the chart with $d\mathbf{A}_{\text{per}} = \mathbf{B}_{\text{per}}$. For our estimates we will need to choose a primitive $\mathbf{A}_{\text{per}}$ in a way that is again adapted to σ_∞. We show in the following lemma that such a choice is possible.

Lemma 3. *There is a primitive $\mathbf{A}_{per} = a_i dq^i$ of $\mathbf{B}_{per}$ defined over the chart $q : U \to \mathbb{R}^d$ such that its coefficient $a_\theta = a_2$ is bounded and the θ-derivatives of all coefficients $\partial_\theta a_i$ are bounded.*

Proof. Following the idea in the standard proof of Poincaré's lemma, using our chart from lemma 2 we consider the negative radial vector field defined over U by:

$$V_q = -q^i \frac{\partial}{\partial q^i}$$

Its flow is simply $\phi_t(q) = e^{-t}q$. Notice that we have $\phi_t(U) \subset U$ for $t \geq 0$ by condition 3. in lemma 2, so the forward flow of V remains inside U.

Now we define an averaging operator $h : \Omega^k(U) \to \Omega^k(U)$ by:

$$h\omega = -\int_0^{\infty} \phi_t^* \omega dt$$

A straightforward computation using Cartan's formula allows us to show that given any k-form ω, the $(k-1)$-form $h\iota_V\omega$ is always one of its primitives. Here the notation $\iota_V\omega$ stands for the contraction with V.

We may then choose

$$\mathbf{A}_{\text{per}} = h\iota_V \mathbf{B}_{\text{per}}.$$

We will show that this choice of A_{per} satisfies the conditions required by the lemma. Let's denote the coefficients of $\mathbf{B}_{\text{per}}$ by:

$$\mathbf{B}_{\text{per}} = \left[b_{ij} dq^i \wedge dq^j\right]_{i<j} = \left[b_{ij} dq^i \otimes dq^j - b_{ij} dq^j \otimes dq^i\right]_{i<j} = b_{ij} dq^i \otimes dq^j$$

where we make $b_{ij} = -b_{ji}$ when $i > j$ and $b_{ii} = 0$. We then compute:

$$\begin{aligned}
\mathbf{A}_{\text{per}} &= -\int_0^\infty \phi_t^* \iota_V \mathbf{B}_{\text{per}} dt \\
&= -\int_0^\infty \phi_t^* (-q^i b_{ij} dq^j) dt
\end{aligned}$$

Since $\phi_t^* dq^i = e^{-t} dq^i$ we obtain:

$$\mathbf{A}_{\text{per}} = -\int_0^\infty (-e^{-t} q^i b_{ij}(e^{-t} q) e^{-t} dq^j) dt$$

so that the coefficients of $\mathbf{A}_{\text{per}}$ obey the formula:

$$a_i(q) = q^j \int_0^\infty e^{-2t} b_{ij}(e^{-t} q) dt$$

and changing variables $s = e^{-t}$ we obtain:

$$a_i(q) = q^j \int_0^1 s b_{ij}(sq) ds$$

Since $|\mathbf{B}_{\text{per}}|$ and $|\nabla \mathbf{B}_{\text{per}}|$ are bounded by a function h satisfying $\int_0^\varepsilon n h(n) dn < \infty$ we see that the integral defining the coefficients a_i converges and

$$\partial_k a_i(q) = q^j \int_0^1 s \partial_k b_{ij}(sq) ds$$

Furthermore because $\iota_{S_\varepsilon}\mathbf{B}_{\text{per}}$ and $\mathcal{L}_{S_\varepsilon}\mathbf{B}$ are bounded and in this chart $S_\varepsilon = |\sigma_\varepsilon|^2\partial_\theta$ we deduce that $a_\theta = q^j \int_0^1 s b_{\theta j}(sq)ds$ is bounded since

$$\iota_{S_\varepsilon}\mathbf{B}_{\text{per}} = |\sigma_\varepsilon|^2\iota_{\partial_\theta}\mathbf{B}_{\text{per}} = |\sigma_\varepsilon|^2 b_{\theta i}dq^i$$

Similarly the θ-derivatives $\partial_\theta a_i$ are all bounded in U as well. □

From now on we choose $\mathbf{A}_{\text{per}}$ to be the primitive provided by the lemma.

Let's introduce some notation in order to carry out a few local computations. Using the trivialization of T^*M induced by the chart $\mathbf{q}$ on M, write:

$$p = p_i dq^i \quad \mathbf{A} = A_i dq^i \quad \mathbf{A}_{\text{per}} = a_i dq^i$$

As before we will also denote A_2 by the more suggestive notation A_θ. Notice that by the condition required from $f(n)$ we have that $|\mathbf{A}_\theta(c(t))| \to \infty$ as $t \to t_{\max}$, since

$$\begin{aligned} A_\theta(c(t)) =& F(n(c(t))) + a_\theta(c(t)) \\ =& -\int_{n(c(t))}^{\varepsilon} f(m)dm + a_\theta(c(t)) \end{aligned}$$

and a_θ is bounded. We are able to derive a contradiction by proving that A_θ may not go to infinity in finite time. This is due to the following:

Proposition 6. *Let $c(t)$ be a* $\mathbf{B}$*-geodesic, using the chart above one has:*

$$|A_\theta(c(t))| \leq C_0 + C_1|t| \tag{4.5}$$

for some positive constants $C_0, C_1 > 0$.

Proof: Notice that by Hamilton's equations (1.4) we have:

$$|\dot{p}_\theta(t)| = \left|\frac{\partial H}{\partial \theta}\right|$$

By using expression (1.3) for the Hamiltonian we obtain in charts:

$$H = \frac{1}{2m} g^{ij}(p_i - eA_i)(p_j - eA_j)$$

We then derive the following formula for $|\dot{p}_\theta|$:

$$\left| \frac{1}{2m}(\partial_\theta g^{ij})(p_i - eA_i)(p_j - eA_j)) - \frac{e}{m} g^{ij}(p_i - eA_i)\left(\frac{\partial A_j}{\partial \theta}\right) \right|$$

Since H is constant along the trajectory, the terms above of the form $(p_i - eA_i)$ are bounded along $c(t)$ and since g is smooth and defined over the closed domain M, the terms g^{ij} and $\partial_\theta g^{ij}$ are also bounded.

Lastly, notice that $\partial_\theta A_i = \partial_\theta a_i$ since for $i \neq 2$, in fact $A_i = a_i$, and for $i = 2$, we have $A_\theta = F(n) + a_\theta$. We conclude that those terms are also bounded since A_{per} was chosen so that its coefficients had bounded θ-derivatives. This means that $|\dot{p}_\theta|$ is bounded along $c(t)$.

Integrating this inequality we obtain that $|p_\theta|$ is bounded by a linear function

$$|p_\theta(c(t))| \leq C_0 + C_1|t|$$

and finally since $p_\theta - eA_\theta$ is bounded we obtain

$$|A_\theta(c(t))| \leq C_0 + C_1|t|$$

with possibly different constants C_0, C_1. This finishes the proof of the proposition and the proof of the theorem.

Chapter 5

Conclusion

We have seen that the topology of a manifold seems to play a role on whether it may carry a confining magnetic field or not. One very interesting question that remains after this work is to make this dependence on the topology more explicit and hopefully more refined.

In order to proceed in that direction one must perform a more detailed study of the dynamics of **B**-geodesics close to a zero of σ_∞. Notice for example, that if there is a geodesic $c(t)$ in D (by that we mean a geodesic for the Riemannian metric) exiting the interior in finite time and such that $\dot{c}(t) \in \ker \mathbf{B}$, then $c(t)$ is also a **B**-geodesic and the magnetic field **B** would fail to confine particles to the interior D. One interesting property one could ask from the magnetic field that would prevent such phenomena is to require that $\ker \mathbf{B} = 0$ everywhere, that is, to ask that **B** is symplectic.

One could try to address for example some of the following questions: Can we find confining magnetic fields on domains which are not toroidal? If a magnetic field is not confining, could one still prove that almost all **B**-geodesics would still be trapped in the interior of the manifold assuming that $Z(\sigma_\infty)$ has measure zero?

Another interesting direction for future work is to study whether there are possible improvements to the class of magnetic fields considered on Theorem 4. Could we relax the requirements on the σ_∞-perturbations? Is there a non-perturbative way of describing these magnetic fields?

Working through some examples it also seems reasonable to expect that, in the case where the magnetic field does not go to infinity at the boundary, one might still be able to trap particles with low-energy to the interior D. This would not be possible for any given magnetic field but it is interesting to look for a class of magnetic fields for which one may always find a threshold value $h > 0$ for which particles with energy smaller than h would be confined to the interior of the manifold for all time.

One also wonders whether a similar confinement result would hold for a more general Yang-Mills field in place of a magnetic field. The structure of the problem is similar to the magnetic case, but we are still unsure whether the non-abelian nature of the Gauge group might affect the strategy we have used to establish confinement.

In the quantum context there are also many interesting questions left. Can one find an analytic proof of quantum tunneling for the not essentially self-adjoint magnetic Schrödinger operators coming from the family of magnetic fields $\mathbf{B}_\alpha$ presented in Section 3.1? That is, in that case could one prove that there is a self-adjoint extension of the Hamiltonian for which there are particles whose wave function are initially supported inside of the domain $\Omega \subset \mathbb{R}^2$ but who will at some future time leak through the boundary?

A more daunting but very interesting undertaking would be to also analyze the semiclassical properties of this system more precisely. Given a magnetic field, when does the quantum completeness of the Hamiltonian operator $\hat{H}_{\mathbf{A}}$ imply the completeness of the Hamiltonian vector field $X_{\mathbf{B}}$? Conversely when does the completeness of the Hamiltonian vector field imply the completeness of the Hamiltonian operator? When the implication is not possible, can one prove the presence of quantum tunneling for some of the self-adjoint extensions of the Hamiltonian operator?

Additionally, as in the classical context, could one generalize the result on quantum confinement proved in [6] to Yang-Mills Schödinger operators?

There are many interesting questions that still remain to be answered around the subject of magnetic confinement, this thesis is but a small step towards a deeper understanding of the geometry of this problem. Hopefully many more steps will be taken in the future towards building not only a clearer picture of the geometry and dynamics, but also of the quantum mechanics of these systems.

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