Value Sets of Points in the Monster Tower

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Chapter 1

Introduction

In this work we study the classification of points in the The Monster Tower, which is a tower of $\mathbb{P}^1$ bundles over a manifold M. The Monster Tower construction over the plane $\mathbb{C}^2$, or $\mathbb{R}^2$ is of particular interest, as it contains all rank 2 Goursat distributions (see Chapter 1 of [19]), and can be seen as a space whose points contain compactified curvilinear data of jets of singular analytic plane curve germs.

It had been previously established that the points in the Monster Tower could be coarsely stratified by a set of code words, where the words consisted of three letters. This was first observed by Mormul in [20], and eventually by Montgomery and Zhitomirskii in [19] and Colley, Kennedy, and Shanbrom in [9]. One generally studies this Monster Tower, and its points, by studying irreducible singular analytic plane curve germs on the base, or what we call Legendrian curves on the first level. In the case of singular plane curves, certain words that stratify the points in the Monster Tower also have the potential to topologically classify plane curve germs on the base.

O. Zariski in [25] and later in [26] gives us the remarkable fact that analytic plane curve germs of the form $f(X,Y) = 0$, with $f \in \mathbb{C}[[X,Y]]$ are of the same topological type if and only if they share the same set of discrete invariants, known as the semigroup of the curve. This semigroup consists of valuations of regular functions $\mathcal{O} := \mathbb{C}[[X,Y]]/f$ on the curve germs with the same topological type. There are other discrete topological invariants of the curve, most of which can be found in C.T.C. Wall's book [24]. One invariant of particular importance is known as the Puiseux Characteristic, and consists of essential exponents, that are closely related to the Puiseux Expansion of the curve (see Wall Section 3.1 in [24], or I.2 in [26]).

There is a finer discrete analytic invariant, which we call the value set of the plane curve germ, and it is given by the valuations of differential one-forms $\mathcal{O}d\mathcal{O}$ on the curve. These were first studied by Zariski in [25] and [26], and they have been studied in more detail by Delorme in [10] and [11], and more recently by Hefez and Hernandes in [13] and [15], Perraire in [21] and Almirón in [3].

Since points in the Monster Tower can be stratified via code words, and topological classes of singular plane curve germs also have a code word, we can assign a semigroup to a point in the Monster Tower using this code word to semigroup correspondence. We can similarly ask whether one can further stratify the points in the Monster Tower using the value sets of plane curve germs.

A large part of this work is dedicated to this question, and answers that if the point is what we will later define to be regular, then it indeed has a set of value sets associated to it. We find these value sets are limited by those value sets associated to a single topological class. We further show that at some levels in the Monster Tower, certain points will have only one value set associated to them.

Another large portion of this paper is dedicated to studying the value sets of plane curve germs in detail, namely the generic value set of a topological class of plane curve germs. We will also dedicate some effort to working with Legendrian curve germs, as they are a natural object of study when it comes to the Monster Tower.

The main results of this work include associating value sets of plane curve germs to points in the Monster Tower, giving a recursive formula for the generic value set of a topological class of plane curve germs with coprime semigroup, and finding new contact invariants of Legendrian curves at level 1 of the Monster Tower. We now outline the course of the paper.

1.1 Outline of the Paper

In Chapter 2 we state all of the preliminaries and the work done before this paper that helps us achieve our results. The first section of the preliminaries is dedicated to the Monster Tower and its basic properties. The second section involves developing the basics of plane curve germs and their semigroups and value sets. Finally in the last section we define what a Legendrian curve is, and discuss its associated semigroup.

In Chapter 3 we establish the connection between jets of plane curve germs and points in the Monster Tower. We do this by associating a certain set of plane curve germs to each point in the Monster Tower. We find that certain points have a well-defined set of value sets associated to them. We give explicit results in terms of which points have only one value set associated to them in the Monster Tower. We also determine which points have the generic value set associated to them, given that their code word is of a certain type. The main result of our paper is stated at the end of this chapter, which fully describes the points that have generic associated value sets of a topological class with coprime semigroup. We will need results from the following chapter to complete the picture

We then move on to Chapter 4, where we study in more detail value sets of plane curve germs. In this chapter we present what we call the Coordinated Mancala game for a coprime semigroup. We prove that every game we play results in a value set of some curve with the given semigroup, and in fact every curve with a given coprime semigroup has a value set that corresponds to a Coordinated Mancala game. We find that the minimal game gives us the generic value set of a topological class of plane curves germs with coprime semigroup. We then complete what is essentially the main result of the paper by providing a recursive formula for the minimal generators of the generic value set of a topological class of plane curve germs with coprime semigroup.

The last part of this chapter is dedicated to partially classifying the set of code words mentioned above that have only one value set associated to them. Here we give rather robust results and determine a large set of words which we call Λ-simple, and have the property above of only one value set.

Finally in Chapter 5 we determine some contact invariants of Legendrian curve germs, and establish that there is a notion of a Zariski invariant of the Legendrian curve (see Chapter III in [26] or Theorem 2 in [13]). In this chapter we also give some results that relate to the Legendrian semigroup of the Legendrian curve. This chapter concludes the results of this work, and all is left is the conclusion and further speculations in Chapter 6.

Chapter 2

Preliminaries

2.1 The Monster Tower

Here we introduce one of the main objects of study known as the Monster Tower, or in some cases the Semple Tower. The construction we are going to introduce can be done with any smooth manifold M as its base. Here we will stick to the complex plane $\mathbb{C}^2$ for the sake of simplicity, but really any field of characteristic 0 will do. Other constructions of this tower can be found in 1.1 in [19] over the real plane, or over an arbitrary base in section 5 in [9].

2.1.1 The Basic Construction of the Monster Tower

The construction of the Monster Tower starts at level 0 where we define $M(0) := \mathbb{C}^2$. It is a sequence of $\mathbb{C}P^1$ bundles

$$\dots M(n+1) \to M(n) \to \dots \to M(1) \to M(0),$$

where at each level n we inductively define a rank 2 distribution Δ_n over the Monster $M(n)$, which we will sometimes refer to as the *focal distribution,* or *focal bundle* as they do in [9]. We begin with $\Delta_0 := TM(0) = T\mathbb{C}^2$. We now define $M(1) := \mathbb{P}(T\mathbb{C}^2)$, the projectivization of the tangent bundle of $M(0)$. It is straightforward that $M(1) = \mathbb{C}P^1 \times \mathbb{C}^2$.

Next we construct Δ_1. One way to define Δ_1 locally is to use coordinates on $M(1)$ given by (x, y, y'), where y' records the slope of a line in $\mathbb{C}^2$ passing through (x, y). It can also be seen as the affine coordinate $[1 : y'] \in \mathbb{C}P^1$. Then $\Delta_1 = \ker(dy - y'dx)$ locally. Similarly we can reverse the roles of x and y and locally define $\Delta_1 = \ker(dx - x'dy)$ where in this case we have the affine coordinate x' for $[x' : 1] \in \mathbb{C}P^1$. Anywhere these charts overlap we have $y' = 1/x'$. Alternatively we consider the natural projection map

$$\pi \colon T\mathbb{C}^2 \to \mathbb{C}^2$$

This map induces a map

$$\bar{\pi}\colon M(1) \to M(0) := \mathbb{C}^2.$$

There is then a differential map $d\bar{\pi}\colon TM(1) \to TM(0)$. Any point $\tilde{p} \in M(1)$ is of the form $\tilde{p} = (p, l)$ where $p \in \mathbb{C}^2$ and $l \in \mathbb{C}P^1$ is a line through p in $T_pM(0)$. We define

$$\Delta_1|_{\tilde{p}} := (d\bar{\pi}_{\tilde{p}})^{-1}(l).$$

That is, it is the set of vectors in $T_{\tilde{p}}M(1)$ that map to the line l in $T_pM(0)$. We then define

$$M(2) := \mathbb{P}(\Delta_1),$$

the projectivization of Δ_1. The construction continues inductively: we define $M(n) := \mathbb{P}(\Delta_{n-1})$ and Δ_n is obtained by considering preimages of lines $l \subset \Delta_{n-1}(p)$ in $T_pM(n-1)$ where $p \in M(n-1)$ under the differential of the induced projection map $\bar{\pi}\colon M(n) \to M(n-1)$. We then define

$$\Delta_n|_{\tilde{p}} := (d\bar{\pi}_{\tilde{p}})^{-1}(l),$$

where $\tilde{p} = (p, l)$ as before.

A good exercise is to verify that in the case of $M(1)$ our general construction of Δ_1 agrees with the local construction of Δ_1 defined by $\ker(dy - y'dx)$.

In the following section we will show how for analytic curves $\gamma\colon \mathbb{C} \to M(n)$, with γ *tangent* to the distribution, i.e. such that $\gamma'(t) \in \Delta_n$, we can define a procedure called lifting or prolonging γ that results in an analytic curve germ in $M(n+1)$, tangent to Δ_{n+1}. We will later on work with what we call Legendrian curve germs, which can be though of as lifts of singular analytic plane curve germs up into the 1st level of the Monster Tower. We can carry out a similar procedure for a local symmetry $\Phi\colon U \subseteq M(n) \tilde{\to} \tilde{U} \subseteq M(n)$, which we also call lifting or prolonging. Let us see how this is done.

2.1.2 Lifting and Projecting Analytic Curves and Symmetries in the Monster Tower

We saw from our previous section that the Monster Tower has the potential to capture information about derivatives of plane curves. We now give a procedure that allows us to lift certain types of curves into the Monster in a way that records higher order curvilinear data. These curves are a special type of analytic curves called integral curves, for which we give a definition below.

Definition 1. We say that an analytic curve $\gamma\colon \mathbb{C} \to M(n)$ that is *tangent* to Δ_n, i.e. $\gamma'(t) \in \Delta_n$ for all $t \in \mathbb{C}$, is an *integral* curve.

These integral curves are the curves that we can successfully lift up to higher levels of the Monster, and the procedure will continue to give us integral curves at each level. Note that an analytic plane curve is automatically an integral curve at level 0. Let us now formally define the notion of lifting these curves into the Monster Tower.

Definition 2. We define the *lift* or *first prolongation* of an integral curve $\gamma\colon \mathbb{C} \to M(n)$, denoted γ^1, to be the curve in $M(n+1)$ given by the following:

1. If $\gamma'(t) \neq 0$ then $\gamma^1(t) = (\gamma(t), \overline{\gamma'(t)}) \subset M(n+1)$, where $\overline{\gamma'(t)}$ is the line in $\Delta_n|_{\gamma(t)}$. containing $y'(t)$.
2. If for some t_0 we have $\gamma'(t_0) = 0$, then since γ is analytic in $M(n)$, the point $\gamma(t)$ is an isolated singular point. Hence there exists some punctured neighborhood of t_0 such that $\gamma'(t) \neq 0$. We define $\gamma^1(t_0) := \lim_{t \to t_0} \gamma^1(t)$.

Theorem 2.5 in [19] guarantees that the limit above is well defined, and that γ^1 is analytic and is an integral curve in $M(n+1)$. We can continue the process iteratively to obtain higher prolongations. We denote the *kth* prolongation of a curve in $M(n)$ by $\gamma^k\colon \mathbb{C} \to M(n+k)$. Let us give an example of this procedure starting with a plane curve.

Example 1. Let γ be the plane curve germ given by the following parameterization:

$$x(t) = t^4, \quad y(t) = t^9 + t^{10}.$$

We see that the coordinates on $M(1)$ are given by (x, y, y') and so our new lifted curve is given by

$$x(t) = t^4, \quad y(t) = t^9 + t^{10}, \quad y'(t) = \frac{9}{4}t^5 + \frac{5}{2}t^6.$$

where $y' = dy/dx$ as before.

We also have the ability to project integral curves in $M(n)$ down to the lower level $M(n-1)$. If γ is an integral curve in $M(n)$ we will define the projection down to $M(n-1)$ by $\gamma_1(t) := \bar{\pi}(\gamma(t))$, where $\bar{\pi}\colon M(n) \to M(n-1)$ is the induced projection map. It is again an integral curve, and so we may continue the process. We will denote the kth projection map as $\bar{\pi}_k\colon M(n) \to M(n-k)$. We with also denote the kth projection of γ as γ_k.

Definition 3. Let the diffeomorphism $\Phi\colon M(n) \to M(n)$ be called a *symmetry* of $M(n)$ if we have that $\Phi_*(\Delta_n) = \Delta_n$. We call $\Phi\colon U \to \tilde{U}$ a local symmetry if it sends the open set $U \tilde{\to} \tilde{U}$ in $M(n)$ with $\Phi_*(\Delta_n|_U) = \Delta_n|_{\tilde{U}}$

Definition 4. The *prolongation* of a local symmetry Φ of $M(n)$ is the local symmetry of $M(n+1)$ given by $\Phi^1(p, l) := (\Phi(p), d\Phi(l))$.

Theorem 1. (1.1 in [19]) *For $i > 1$, every local symmetry at level i is the prolongation of a symmetry at level $i-1$.*

This theorem asserts that we need only study local symmetries of $M(1)$, since any other symmetry at a higher level is obtained by prolongations of lower level symmetries until we get to level 1. We do not go down to level 0, as it is not sufficient to capture all symmetries of the Monster. Indeed the set of lifted or prolonged local symmetries of the plane $\mathbb{C}^2$ up to level 1 is strictly contained in the local contact transformations of $M(1)$ ([19], and see example 2 below).

From this theorem we get also obtain the notion projecting local symmetries of the Monster. Indeed Theorem 1 shows that for any $n > 1$ any local symmetry Φ of $M(n)$ has a unique local symmetry of $M(n-1)$, that we will denote Φ_1, such that $(\Phi_1)^1 = \Phi$. We will call Φ_1 the *projection* of Φ to $M(n-1)$. We can continue to do this down as far as level 1. We will denote the kth projection of Φ, for any $1 \leq k \leq n-1$ by Φ_k, which will be a local symmetry of $M(n-k)$.

We now define what it means for two points in $M(n)$ to be equivalent.

Definition 5. We say that two points $p, q \in M(n)$ are *equivalent* if there is a local symmetry Φ of $M(n)$ such that $\Phi(p) = q$. We similarly say two integral curves are *locally equivalent* if there is a local symmetry that takes one to the other.

By Theorem 1 we must have that two points p and q are equivalent in $M(n)$ if and only if there is a symmetry at level 1 that prolongs to level n and takes p to q. Sometimes we want to restrict our attention to just those local symmetries that come from lifting local analytic isomorphisms of the plane. Let us give a formal name to the equivalency of points under these lifted analytic isomorphisms, as it will be used quite often in the following chapters.

Definition 6. Let $p, q \in M(n)$ be called *analytically equivalent*, or *a-equivalent* , if Φ in Definition 5 is the nth prolongation of a local analytic isomorphism of the plane.

Note that a-equivalence implies equivalence. However as mentioned above, there are examples of symmetries in the first level of the Monster $M(1)$ that are not lifted analytic plane isomorphisms. Let us give an example of one such symmetry of $M(1)$.

Example 2. Consider the local transformation given by $\Phi\colon (M(1), 0) \to (M(1), 0)$ defined by

$$(x, y, y') \mapsto (y', xy' - y, x) =: (\tilde{x}, \tilde{y}, \tilde{y}')$$

One can quickly check that this is a diffeomorphism of $M(1)$, since it is in fact an involution. We compute the pullback on the contact form:

$$\begin{aligned}
\Phi^*(d\tilde{y} - \tilde{y}'d\tilde{x}) &= d(xy' - y) - xdy' \\
&= xdy' + y'dx - dy - xdy' \\
&= -(dy - y'dx)
\end{aligned}$$

This shows that the one-form defining the distribution Δ_1 is preserved up to sign, and therefore the distribution is preserved, and Φ is a local symmetry of $M(1)$, also called a *contactomorphism*. Clearly Φ is not a lifted analytic isomorphism of the plane.

This gives us the example we were looking for, and later we show that this transformation is important to consider when it comes to moving curves in a fiber of $M(1) \to M(0)$ to curves that are not entirely in that fiber.

Now that we have defined prolongation and equivalence of curves, we wish to state a couple of facts about local symmetries and integral curves given to us by [19]:

Proposition 1. (2.6 and 2.9(i) in [19]) *Projection and prolongation of local symmetries and integral curve germs are inverses.*

Proposition 2. (2.9(ii) in [19]) *Projections and prolongations of integral curves commute with local symmetries: $(\Phi \circ \gamma)^k = \Phi^k \circ \gamma^k$ for any k, and $(\Phi \circ \gamma)_k = \Phi_k \circ \gamma_k$ if $k \leq n-1$.*

These two propositions along with Theorem 1 give us the following theorem from Montgomery and Zhitomirskii.

Theorem 2. (2.2 in [19]) *Let $\gamma, \tilde{\gamma} \colon (\mathbb{C}, 0) \to (M(n), p)$ be two integral curve germs such that their $(n-1)$st projections are non constant curve germs. Then γ and $\tilde{\gamma}$ are equivalent if and only if γ_{n-1} is equivalent to $\tilde{\gamma}_{n-1}$.*

2.1.3 RVT Code Words for Points and Curves in The Monster Tower

We can stratify the points in the Monster Space $M(n)$ by using a code word that has the same length n. We only need three distinct symbols to form our alphabet. These symbols are R for regular, V for vertical, and T for tangent. These three letters correspond to three types of directions in the distribution. We now explicate what we mean by types of directions. These symbols have been adopted from [8] and [19], but can be originally credited to Mormul in [20] where instead he uses the letters C and S in place of R and V respectively.

We will do most of our work on a-equivalence, and use definitions that fit this criteria. When it is appropriate, we will note the difference between regular equivalence and a-equivalence.

Definition 7. A point $p \in M(n)$ is said to be a *nonsingular* point if there exists a nonsingular plane curve $\gamma(t)$ such that $\gamma^n(0) = p$ and $\gamma'(0) \neq 0$. Otherwise we say that p is *singular*.

The author notes that the terminology may seem off in the sense of algebraic geometry. However we have that the group of germs of plane diffeomorphisms fixing the origin acts on the fiber of the Monster Tower over the origin. This action generates a-equivalence. In other words, two points of the Monster Tower over the origin are a-equivalent if and only if they lie on the same orbit of this group action. When a group acts on a space, one has a single generic or principal orbit type whose points we call "regular" and a subvariety of smaller orbits which we call "singular". For example, if we act on square matrices by conjugation by invertible ones, then the matrices with distinct eigenvalues form the regular points, and all those with double eigenvalues form the singular points. It is with this group theoretic perspective in mind, rather than the algebraic geometric perspecitve, that we use this 'nonsingular / singular' terminology. In our case, the nonsingular points form a single orbit.

We see from the preceding definition that p can be realized by evaluating the prolongation of a nonsingular curve in the plane. In [19] they instead ask for a nonsingular integral curve germ $\gamma : (\mathbb{C}, 0) \to M(1)$ such that $\gamma^{n-1}(0) = p$. In the latter definition, this allows for nonsingular curve germs completely contained in the fiber of $M(1) \to M(0)$, as well as a nonsingular curve tangent to such a fiber. We will call this notion of nonsingular points *contact nonsingular points* When considering nonsingular curves in $M(1)$ there is no reason to distinguish immersed curves along a fiber from any other immersed curves (see Remark 2.18 in [19]). This is due to the fact that we can use a symmetry of $M(1)$ similar to the one presented in Example 2 to transform a nonsingular curve germ in the fiber to one not even tangent to the fiber.

Theorem 2.15 in [19] shows us that all contact nonsingular points in $M(n),\ n \geq 1$, are equivalent. We now present a similar theorem that shows that all nonsingular points are as well.

Theorem 3. *All nonsingular points in $M(n)$ are a-equivalent.*

Proof. All nonsingular plane curves are locally equivalent to a line by the inverse function theorem. Hence all nonsingular plane curves are locally equivalent to each other, and therefore for any two nonsingular curves in the plane we have a local symmetry of $M(0)$ taking one to the other. We can prolong this symmetry and these curves up to level n. Using the propositions from the previous chapter we have the results we desire. $\square$

Here is a point where we can make a distinction between what we are calling a-equivalence of points and equivalence in the broader sense. We have for instance that if we restrict ourselves to a-equivalence, then not all points in $M(2)$ are equivalent, as we have some critical points. On the other hand, if we allow symmetries of $M(1)$ to define equivalence of points, then all points on $M(2)$ are equivalent (2.14 in [19]).

Since all nonsingular points are essentially the same, we now wish to turn to singular points and coarsely stratify these points by RVT code words. How shall we assign a letter at each level to our point? Let us begin by defining what a critical directions and critical curves are.

Definition 8. An integral curve $\gamma \subset M(n)$ is called *critical* if its projection γ_n is a constant curve.

Definition 9. An integral curve $\gamma \subset M(n)$ is called *vertical* if γ_1 is a constant curve (i.e. $n = 1$ in the prior definition).

We can now define some special directions in the distribution at level n. To do so we define critical lines and points in the Monster Tower.

Definition 10. (Critical and Regular Lines and Points) Let $p \in M(n)$ We call $l \subset \Delta_n(p)$ a *critical* line if there exists an immersed critical integral curve germ $\gamma\colon (\mathbb{C}, 0) \to (M(n), p)$ such that $\gamma'(0) \in l$. The point $(p, l) \in M(n+1)$ is then called *critical* as well. All other lines in $\Delta_n(p)$ are called *regular*, and similarly for points in $M(n+1)$.

new def here for vertical line or put in pvs def?

Definition 11. An immersed integral curve germ $\gamma\colon (\mathbb{C}, 0) \to M(n)$ is called *regular* if $\gamma'(0)$ is in a regular direction.

Montgomery and Zhitomirskii prove Proposition 2.31 in [19], which states that every prolongation of a regular integral curve germ $\gamma\colon (\mathbb{C}, 0) \to M(n)$ is regular. It follows by induction that the prolongation $\gamma^k\colon (\mathbb{C}, 0) \to M(n+k)$ and the point $\gamma^k(0)$ for all $k \geq 0$ must be regular as well.

Note the distinction between regular and nonsingular points. Regular points need only that there exists a regular curve at that level. Nonsingular points require a curve that is regular at the base (in the plane, or, if we use contact nonsingular points, in $M(1)$).

As it turns out there are two types of critical directions. Roughly there is the vertical direction, which comes from vertical curves, and there is the tangency direction, which comes from all other critical curves. It is important to note as well that if $p \in M(n)$ is regular, then $\Delta_n(p)$ has only one critical direction, namely the vertical direction. On the other hand if p is critical, then $\Delta_n(p)$ has two critical directions, vertical and tangent. We sum this up with a proposition:

Proposition 3. (2.41 in [19]) *Let $p \in M(n)$ and $n \geq 2$. If p is a regular point then $\Delta_n(p)$ contains no tangency lines, hence only the one critical line in the vertical direction (fiber direction). If p is critical, then $\Delta_n(p)$ contains exactly two critical lines: the vertical line and the tangency line.*

We can relate these three directions−regular, vertical and tangent−to what we call divisors at infinity (see section 8 in [9]). One can consider the immersed vertical curves that lie completely in the fiber of $M(n-1) \to M(n-2)$ at each point in $M(n-1)$. From there we can form the line bundle from these vertical curves using vectors tangent to them. The projectivization of this bundle sits inside of $M(n)$, and is called the *nth divisor at infinity*, $I_n \to M(n)$, where we borrow the notation from section 9 in [9].

We may take the kth prolongation of these immersed vertical curves and again take their tangent vectors at each point. These vectors will be in what we call the tangency direction in the distribution Δ_{n+k-1}. We denote their projectivization as $I_n[k] \subset M(n+k)$. As noted by Colley, Kennedy and Shanbrom in section 9 of [9] we get a tower of identical spaces with growing codimension:

$$\dots I_n[k] \to I_n[k-1] \to \dots \to I_n \to M(n-1),$$

where we call $I_n[k]$ the kth prolongation of I_n. With these relations between regular, vertical and tangency directions, and what we call divisors at infinity, we are ready to assign a unique RVT code word to a point $p \in M(n)$.

Assigning RVT Code Words to Points: Given a point $p \in M(n)$, we would like to assign to it a unique code word of length n consisting of the letters R, V and T. We will call this the RVT code word of the point p. Let us define the process by which one assigns this code word to a point in the Monster Tower. We define a critical point $p = (\bar{\pi}(p), l) \in M(n)$ *vertical* if $l \subset \Delta_{n-1}(\bar{\pi}(p))$ is in the vertical direction, and *tangent* if l is in the tangency direction. Alternatively we can call p vertical if $p \in I_n$, and tangent if there exists some $j + k = n$, such that $p \in I_j[k]$.

We assign the RVT code word to $p \in M(n)$ recursively as follows. Assume we have formed the word for $\bar{\pi}_1(p)$. Then the RVT code word for p is given by adding an R, V, or T to the end of the RVT code word of $\bar{\pi}_1(p)$, depending on whether p is a regular, vertical or tangent point in $M(n)$, respectively. Beginning with the empty word at the base and assigning an R for any point in $M(1)$, a word of length k is assigned to each point in $M(k)$.

By Proposition 3 it follows that we cannot record a T for the last symbol of the word for p unless we have the symbol V or T for $\bar{\pi}_1(p)$, that is if $\bar{\pi}_1(p)$ is already critical. This restricts the words that are possible for our points.

One way to assign an RVT code word to a point is if we have an integral curve germ γ in $M(n)$ such that $\gamma(0) = \tilde{p}$. If we do have such a curve, then we can take the tangent vectors of each projection $\bar{\pi}_k(\gamma)$, and find their directions.

Assigning RVT Code Words to Analytic Plane Curve Germs: We are also able to assign an RVT code word to an analytic plane curve germ γ in the following way. We can lift γ into the Monster tower and at each level n determine if the prolongation $\gamma^n(0)$ is a regular, vertical, or tangency point. We can then record an R, V or T for the nth letter of the RVT word depending respectively on the type of point. This is the same as looking at the tangent direction of the $(n-1)$st prolongation of γ, and determining its type.

If the plane curve germ is analytic and irreducible, then there is a level at which we will only record R's from there on out. We will call this the *regularization level* of a singular plane curve germ. This is proved in Theorem 2.36 in [19], which states that every well-parameterized, analytic integral curve germ has a finite level of regularization. Let us also say a plane curve germ is *regularized* in the Monster if we have prolonged it to its regularization level.

Let us define the RVT code word of singular curve germ at the base to be the *critical* RVT code word, i.e. the one ending in V or T, that we assign to its prolongation at its regularization level in the Monster Tower. Any regular integral curve germ through a point $p \in M(n)$ will therefore have the same RVT code word. For completeness, we assign the empty word to any nonsingular plane curve germ. Let us see an example of how to assign a code word to an analytic plane curve germ.

Example 3. Let $\gamma(t)$ be the curve germ given by

$$\gamma: \; x(t) = t^4, \quad y(t) = t^9.$$

The tangent vector is horizontal at $t = 0$ so we record an R. Now we take the derivative

$$y' = dy/dx = \frac{9}{4}t^5$$

so that $\gamma^1 = (t^4, t^9, \frac{9}{4}t^5)$. A quick check shows that $(\gamma^1)'(0)$ is in a regular direction. We then record another R. Now we retain x again, and find that in this chart

$$\gamma^2 = \left(t^4, t^9, \tfrac{9}{4}t^5, \tfrac{45}{16}t\right).$$

The tangent vector of γ^2 at $t = 0$ is in the vertical (fiber) direction. We record a V for our word at level 3, and anticipate a need to change our retained coordinate. We now see that if we make the regular choice of coordinates we have $\gamma^3 = (t^4, t^9, \frac{9}{4}t^5, \frac{45}{16}t, \frac{45}{64}\frac{1}{t^3})$. It follows that $\gamma^3(0)$ cannot be expressed in this chart, and so we choose to retain y'' instead and our new coordinate is $x' = dx/dy''$. Thus

$$\gamma^3 = \left(t^4, t^9, \tfrac{9}{4}t^5, \tfrac{45}{16}t, \tfrac{64}{45}t^3\right).$$

in the chart with coordinates x, y, y', y'', x'. Since the tangent vector $(0, 0, 0, \frac{45}{16}, 0)$ is still in the direction of a fiber, but at a lower level, we record a T. We continue onward now retaining y'' at each step, and one can verify that we will record one more T and then R's from there on out. This implies our code word is $RRVTT$ for our curve γ.

The example gives a set of coordinates in which we may express our point $\gamma^n(0)$, and $\gamma^n(t)$ locally. These are special coordinates that can be found by observing the type of point $\pi_k(p)$ is for $p \in M(n), k < n$, or in the case of a curve, the type of point $\gamma^k(0)$ is at each level.

There is an algorithm that gives a one-to-one correspondence between RVT code word and Puiseux Characteristic, a topological invariant of the curve (see section 2.2). Colley, Kennedy and Shanbrom devote section 12 in [9] to this algorithm where it is illustrated with multiple examples. This gives the following proposition.

Proposition 4. *Two curve germs $\gamma, \tilde{\gamma}\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ are topologically equivalent if and only if they have the same RVT code word.*

Every point $p \in M(n)$ has a uniquely assigned RVT code word. If we find a plane curve germ γ with $\gamma^n(0) = p$ and γ^n regular, then γ have the same critical RVT code word as p, i.e. the word assigned to p truncated by removing any R's at the end. Later we show that for each $p \in M(n)$ there exists such a curve, γ. These facts along with Proposition 4 give us a coarse stratification of the Monster Tower through equi-singularity classes, equivalently critical RVT code words, of plane curves in the following way:

Proposition 5. *If p, q are equivalent, and their RVT code words both start with two R's, then they will have the same RVT code word. If $p, q \in M(n)$ are a-equivalent points in the Monster Tower, then they have the same RVT code word.*

Proof. The first part is Montgomery and Zhitomirskii's Proposition 3.4 in [19]. The second claim is a matter of restricting to a-equivalence, which allows for a V at level 2. In this case if we have two points $p, q \in M(n)$ that are a-equivalent under an analytic isomorphism of the plane Φ, then, as we will show later on, it is not difficult to construct two plane curve germs γ and $\tilde{\gamma}$ such that $\gamma^n(0) = p$ and $\tilde{\gamma}^n(0) = q$, and $\Phi(\gamma) = \tilde{\gamma}$. Since Φ is an analytic isomorphism, it is a homeomorphism, and so we have that γ and $\tilde{\gamma}$ have the same RVT code word by Proposition 4. $\square$

The converse is immediately false, as there is known to be moduli of plane curves and Legendrian curve germs at level 1 (see [6]) when it comes to equivalence classes of those curves under analytic isomorphism or contactomorphism. It follows by prolongation of these curves into the Monster that there must be points that are not equivalent, but still have the same RVT code word. This will become more clear as we formalize the connection between jets of plane curve germs (or Legendrian curve germs) and points in the Monster Tower.

Our next section is devoted to the special coordinates related to the RVT code word of a point in the Monster. These are the ones we found in Example 3.1.1, sometimes referred to as Kumpera-Ruiz coordinates (see [16]).

2.1.4 Coordinates on the Monster and Finding a Plane Curve That Prolongs Through a Given Point

We would now like to introduce special coordinates on the Monster Tower. In [19] and [9] they relate these coordinates to Kumpera-Ruiz coordinates (KR-coordinates), which appear in [16]. A good demonstration of these coordinates and how they work can be found in Section 8 in [9], and 8.1 in [19]. We will give an explanation here as well.

To express a point $p \in M(n)$ in a set of coordinates, we need to make a choice of a specific chart. If $U \subset \mathbb{C}^2$ is an neighborhood in $\mathbb{C}$, then on $U(n) \subset M(n)$ there are 2^n charts that look like $U \times \mathbb{C}^n$. How we make this choice will depend on the RVT code word of p, however the choice itself is between only two possibilities, what are called the *ordinary* or *inverted* choice in [9]. It should be noted that for a general we point, we can see it in every chart. However there are some points that can only appear in certain charts and not others. It is for these points that we need to define special coordinates.

Let us assume we are given a point $p \in M(n)$. We will give a recursive process to determine which chart to express p in. We define at each level $1 \leq k \leq n$ a new coordinate n_k, a retained coordinate r_k and a deactivated coordinate d_k as follows. Suppose we are given n_{k-1}, r_{k-1}, then we define

1. The *ordinary* choice to be $n_k := dn_{k-1}/dr_{k-1}$, $r_k := r_{k-1}$ and $d_k := n_{k-1}$.

2. The *inverted* choice to be $n_k := dr_{k-1}/dn_{k-1}$, $r_k := n_{k-1}$ and $d_k := r_{k-1}$.

For any $p \in M(n)$ in the case where $k = 0$ we may arbitrarily choose n_0 and r_0 so long as dn_0/dr_0 is finite at p. There is no d_0. We then make the ordinary choice. Now suppose $0 < k < n$ and let us determine how we will make the choice for $k+1$. We assume that we are given n_k and r_k for our point p. If we find that $dr_k \neq 0$ on $\Delta_k(\bar{\pi}_{n-k}(p))$, we make the ordinary choice for $k+1$, otherwise we make the inverted choice. We can record our coordinate chart with a series of o's and i's. We call the n_k and r_k the *active coordinates* at level k. The coordinates on each chart at level $M(k)$ are the coordinates $r_0, n_0, n_1, \ldots, n_k$.

For an alternate set of names, set $x = r_0$, $y = n_0$ and $y' = n_1$, so that $y' = dy/dx$. As noted in section 8 of [9] from here on out we have that the active coordinates are $x^{(i)}$ and $y^{(j)}$ for some nonnegative integers i and j. If $x^{(i)}$ is the retained coordinate we define $y^{(j+1)} = dy^{(j)}/dx^{(i)}$ as the new coordinate. Similarly if $y^{(j)}$ is the retained coordinate, $x^{(i+1)} = dx^{(i)}/dy^{(j)}$ becomes the new coordinate. At each step we get a pair of active coordinates that can be thought of as a parameterization of a plane curve germ itself, at the point $(x^{(i)}(0), y^{(j)}(0))$. Let us give a name to this process of obtaining this pair of active coordinates.

Definition 12. We call the process outlined above, where we obtain a new pair of active coordinates $x^{(i)}, y^{(j)}$ in $M(n)$ to be the *directional blowup* at level n of the plane curve germ γ.

Let us now show how one can use the RVT code word of a point or a curve germ to decide whether or not to make the ordinary or inverted choice at each level.

Using the RVT code word to choose a chart:

Suppose that we know the RVT code word of a given point $p \in M(n)$. From this we know what type of point $\bar{\pi}_{n-k}(p)$ is for all $0 \leq k \leq n$ in terms of regular, vertical or tangent. If we encounter a V in position K, then we must make the inverted choice, since $dr_{k-1} = 0$ on the specified line $\Delta_{k-1}(\bar{\pi}_{n-k+1}(p))$. This is reflecting the fact that ∂_{r_k} is in the vertical direction, i.e., that p lies on the divisor at infinity I_k. If we encounter a T in position k, then we must make the ordinary choice, since $dr_{k-1} = 0$ on the specified line $\Delta_{k-1}(\bar{\pi}_{n-k+1}(p))$. In other words, ∂_{n_k} is in the tangency direction and thus p sits on the prolongation of a divisor at infinity from a lower level. If we encounter an r we may make the ordinary choice, since $dr_{r_k} \neq 0$ on the specified line $\Delta_{k-1}(\bar{\pi}_{n-k+1}(p))$. The inverted choice is also allowed if $d_{n_k} \neq 0$, but we conventionally make the ordinary choice if we encounter an R. Let us illustrate this with an example:

Example 4. Consider the word $RRVTTR$, and some point $p \in M(6)$ that has this code word. We first choose x and y so that $y' = dy/dx$ is not vertical. We can again make an ordinary choice for our chart on $M(2)$ so that x will be our retained coordinate and y' our deactivated one. This implies $y'' = dy'/dx$ is our new coordinate. This coordinate for p is expressible as a scalar in this chart because the fiber is in the regular R direction.

We now encounter our first V and need to make the inverted choice. Thus we choose our new coordinate to be $x' = dx/dy''$ and retain y''. This leaves x as our now deactivated coordinate. The rest are T's and R's, so our retained coordinate will always be y'', and our new coordinates will be $x^{(j)} = dx^{(j-1)}/dy''$ for $j = 2, 3, 4$. At our point p we have $x^{(2)} = x^{(3)} = 0$, but the value of $x^{(4)}$ is nonzero.

Going from a point to a plane curve:

Given a point $p \in M(n)$ and a chart chosen based off of the RVT code word of p as above (or the more simple o's and i's), we can construct a plane curve germ γ such that $\gamma^n(0) = p$, and γ^n is regularized. This is a finite inductive process that starts with a regular parameterization for the active coordinates of the chart we have chosen for p. In general we have either $y^{(j+1)} = dy^{(j)}/dx^{(i)}$ or $x^{(i+1)} = dx^{(i)}/dy^{(j)}$ as described above. Therefore, in terms of differential one-forms, we must have that

$$y^{(j+1)}dx^{(i)} = dy^{(j)} \text{ or } x^{(i+1)}dy^{(j)} = dx^{(i)},$$

and we can integrate to get $y^{(j)}$ or $x^{(i)}$. This is not difficult if we have a parameterization for the active coordinates, as we can pull everything back in terms of t. Let us illustrate this with an example.

Example 5. We will use the chart from our previous example in this section

$$(x, y, y', y'', x', x'', x''', x^{(4)})$$

and assume that $p = (0, 0, 0, 0, 0, 0, 0, 1)$ in our chart. We see from the example that our active coordinates must be $x^{(4)} = dx'''/dy''$ and y''. To find a plane curve that lifts through p we must have that $x^{(4)}(0) = 1$, and $y''(0) = 0$. To make it so the prolongations of the lifted curve is regular from level 6 on, we will choose a nonsingular parameterization for $x^{(4)}(t)$ and $y''(t)$.

We will keep it as simple as possible, and choose

$$x^{(4)}(t) = 1 + t, \quad y''(t) = t.$$

This way we have $dy'' = dt$ and our other conditions are satisfied as well. Since $x^{(4)} = dx'''/dy''$, then $x^{(4)}dy'' = dx'''$ so that

$$x'''(t) = \int_0^t x^{(4)}dy'' = \int_0^t (1 + s)ds = t + \frac{1}{2}t^2.$$

It is clear that $x'''(0) = 0$, which is necessary for the point p. We can continue on in this manner to find

$$\begin{aligned}
x''(t) &= \int_0^t x''' dy'' = \int_0^t (s + \frac{1}{2}s^2) ds = \frac{1}{2}t^2 + \frac{1}{3!}t^3, \\
x'(t) &= \int_0^t x'' dy'' = \frac{1}{3!}t^3 + \frac{1}{4!}t^4, \\
x(t) &= \int_0^t x' dy'' = \frac{1}{4!}t^4 + \frac{1}{5!}t^5, \\
y'(t) &= \int_0^t y'' dx = \frac{1}{30}t^5 + \frac{1}{144}t^6 \\
y(t) &= \int_0^t y' dx = \frac{1}{1,620}t^9 + \frac{11}{43,200}t^{10} + \frac{1}{38,016}t^{11}
\end{aligned}$$

Our plane curve germ becomes

$$(x(t), y(t)) = \left(\frac{1}{4!}t^4 + \frac{1}{5!}t^5, \frac{1}{1,620}t^9 + \frac{11}{43,200}t^{10} + \frac{1}{38,016}t^{11}\right).$$

The multiplicity of x and y in this case is not a surprise. It is hopefully clear that our plane curve will have the desired regularization level of level 6, and that it indeed prolongs through the point p. This concludes our example.

This example suggests to us that in general we can construct a plane curve which has a regularized prolongation at level n and prolongs through the any point $p \in M(n)$. This is indeed the case, as the process in the above example generalizes to any RVT code word, or choice of coordinate charts using our conventions quite nicely.

So far, we have shown that there exists a coarse stratification of the Monster Tower using RVT code words. We have also shown the connection of these RVT code words to singular plane curve germs that prolong through points with our given word, and hence have the same critical word as our point. We have further demonstrated, using our special coordinates, a way to construct a plane curve germ that prolongs through $p \in M(n)$ and has a regularized prolongation at that level. It is because of these facts that we will focus more and more on the invariants of plane curve germs and their Legendrian counterparts at level 1.

Therefore the main objective of this writing is to answer whether or not we can further stratify the Monster Tower using finer discrete invariants of plane or Legendrian curve germs. There are several candidates for these finer invariants. One of them is the Legendrian Semigroup of the lifted Legendrian curve, which we will explain shortly. Another is the valuations of one forms on the plane curve germs that lift through our given point. These all have potential to help us understand points in the Monster Tower. We will see as well in the next chapter that points in the monster and sets of certain sized jets of analytic curve germs have a natural identification, and so studying points in the Monster is the same as studying jets of planar or Legendrian curve germs.

We end this section on the Monster Tower with an exploration of $M(1)$ and the integral curves germs at this level.

2.1.5 Level 1 of the Monster Tower and Legendrian Curve Germs

The first level of the Monster Tower $M(1) = \mathbb{C}^2 \times \mathbb{C}P^1$ is a complex three dimensional manifold. We can use the contact form $dy - y'dx$ to define our distribution Δ_1 locally. We call a local symmetry of $M(1)$ a *(local) contactomorphism.*

Definition 13. An integral curve germ $\gamma\colon (\mathbb{C}, 0) \to M(1)$ is called a *Legendrian* curve germ.

A contactomorphism, or germ of a contactomorphism, takes a Legendrian curve germ to another Legendrian curve germ. It has been established by M&Z in [19] that studying equivalence of points in the Monster Tower is the same as studying contact equivalence of certain jets of Legendrian curve germs. To show this we can use Theorem 1 in part, and the rest is obtained by studying Legendrian curve germs that prolong through our given points. We will show later in a similar fashion that studying a-equivalence of points is the same as studying analytic equivalence of jets of plane curves up to a certain specific jet.

Theorem 4.12 in [19] asserts that any critical Legendrian curve germ in $M(1)$ is contact equivalent to the one-step prolongation of a singular analytic plane curve germ $\gamma(t) = (t^p, t^m + h.o.t.)$ such that $m > 2p$, where $h.o.t.$ stands for *higher order terms* in regards to powers of t. This allows us to assign an RVT code word to our Legendrian curve germ, namely the RVT code word of the plane curve germ γ. The RVT code word of a Legendrian curve will always have two R's at the beginning.

We give an example of two plane curve germs that are not analytically equivalent, but whose Legendrian lifts are contactomorphic in $M(1)$.

Example 6. Let $\gamma(t) = (t^m, t^{m+1})$, with $m > 1$. This curve is singular, and so is not analytically equivalent to the curve germ $(t, 0)$. If we lift the curve we get that

$$\gamma^1(t) = \left(t^m, t^{m+1}, \tfrac{m+1}{m} t\right).$$

This curve is locally diffeomorphic to the curve $(t, 0, 0)$ which is also a contact curve. Therefore by Zhitomirskii's Lemma in [27], we have that these two curves must be contactomorphic as well. Explicitly we can first take the contactomorphism

$$\begin{aligned} x &\mapsto x - \left(\tfrac{m}{m+1} y'\right)^m, \\ y &\mapsto \tfrac{m}{m+1} y - \left(\tfrac{m}{m+1}\right)^{m+2} (y')^{m+1}, \\ y' &\mapsto \tfrac{m}{m+1} y'. \end{aligned}$$

This will give us the curve $(0,0,t)$ which is clearly a regular curve, and is contactomorphic to $(t,0,0)$ under the transformation

$$x \mapsto y', \quad y \mapsto xy' - y, \quad y' \mapsto x.$$

Since m was arbitrary it follows that though (t^m, t^{m+1}) and $(t^k, t^k + 1)$ are not analytically equivalent if $k \neq m$, their lifts to $M(1)$ are both contactomorphic to $(t,0,0)$ and hence to each other.

The transformation above, which was used in Example 2 is the one we used to take an immersed curve that is in the fiber of $M(1) \to M(0)$ to an immersed curve that projects down to an immersed curve on the base. This base is now the plane given by coordinates $(y', xy' - y)$ but yet still gives the same construction of the Monster from level 1 onwards.

Now that we have started to establish a connection between equivalence of points in the Monster Tower and equivalence of plane curve germs or Legendrian curve germs, it is time that we start to discuss some finer analytic (resp. contact) invariants of plane (resp. Legendrian) curve germs. Our next section is devoted to these invariants with some information on the analytic classification of irreducible plane curve germs.

2.2 Discrete Invariants and Moduli of Plane and Legendrian Curve Germs

From our previous section we begin to see why studying the invariants of irreducible plane branches and Legendrian curve germs is such an important topic when studying points in the Monster Tower. Soon we will formally develop the intimate relation between points in the Monster, Analytic plane curve germs, and their Legendrian lifts. We devote this section to some of the basics of invariants of plane curve germs under local Analytic transformations of the plane, and invariants of Legendrian curves germs in $M(1)$. Let us now set the stage for our work in this section.

2.2.1 Theorem of Puiseux and Topological Invariants of Plane Branches

Let us define what we mean by an irreducible plane branch of an analytic curve germ. We follow along with the preliminaries in [26]. Denote $\mathbb{C}[[X,Y]]$ as the ring of all formal power series in two variables over $\mathbb{C}$. Further denote $\mathbb{C}\{X,Y\}$ to be the ring of all convergent power series in X and Y near $0 \in \mathbb{C}^2$.

Definition 14. We say an element $f \in \mathbb{C}[[X,Y]]$ defines an *(irreducible) analytic plane branch* γ given by $f(X,Y) = 0$ if f is irreducible in $\mathbb{C}[[X,Y]]$, and $f \in \mathbb{C}\{X,Y\}$.

From here on out we assume, unless otherwise stated, that if we write plane curve germ, we are writing about an irreducible analytic plane branch like f in the definition above. We now define the main equivalence classes of plane branches we wish to work with.

Definition 15. Let $\gamma, \gamma' \colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ be two irreducible plane curve germs. We say γ and γ' are *topologically equivalent* if there exists a germ of a homeomorphism $T \colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $T(\gamma) = \gamma'$ as germs. If T is also an analytic isomorphism, we say that γ and γ' are *analytically equivalent* as curve germs.

We recall that germs are equivalence classes of pairs $[C, U]$ where C is a curve (or in the case of T a homeomorphism) defined on U, and U an open set in $\mathbb{C}^2$ containing 0. The equivalence relation requires curves to agree on the intersection of their two paired open sets. One can check quickly that both topological and analytic equivalence are equivalence relations on the set of analytic curve germs in the plane.

Let us denote the equivalence class of γ under topological equivalence by $L(\gamma)$ as in [26]. This is known as the *equisingularity class* of of γ. Let us further denote the analytic class of γ as $A(\gamma)$. It is always the case that $A \subseteq L$ for any γ. Therefore if we write $\mathcal{A}$ as the set of all analytic equivalence classes, and $\mathcal{L}$ as the set of all topological classes, then we can consider the quotient space $\mathcal{L}/\mathcal{A}$, which is a moduli space of plane branches.

In [26], Zariski notes that $\mathcal{L}/\mathcal{A}$ can be endowed with a (generally nonseparable) topology, but does not have the "structure of an algebraic set or even a scheme." He then poses the problem of the moduli space: construct a map from $L(\gamma)$ to a finite dimensional affine space so that any two analytically equivalent germs will have the same image in this space (see Section 2.2 in [26]).

One can wonder how to study such a space and form a map with much algebraic structure. As it turns out there is a theorem by Puiseux which allows us to find in some coordinates, a very nice form for γ. Once we are given this form, called the Puiseux expansion, we can extract a discrete topological invariant from the curve germ, known as the Puiseux characteristic.

We need some important notation and facts about the algebraic structure of plane branches in $\mathbb{C}[[X, Y]]$. First we note that $\mathbb{C}[[X, Y]]$ is a local ring with unique maximal ideal $\mathfrak{m} := (X, Y)$. Any $f \in \mathfrak{m}$ defines a plane curve germ γ at the origin via the vanishing locus $f = 0$. If $f \in \mathfrak{m}^2$, then it defines a *singular* curve germ. We denote the local ring of regular functions on γ as $\mathcal{O} := \mathbb{C}[[X, Y]]/f$. The curve γ is analytic and so has a unique tangent line in the plane. A linear isomorphism of the plane allows us to rotate it and scale it if necessary, so that our tangent line becomes $Y = 0$, and thus the leading term of f becomes $f_p(X, Y) = Y^p$, for some p.

We next use the Weierstrauss Preparation Theorem, which allows us to write f up to a unit in the form

$$f(X, Y) = Y^p + F_1(X)Y^{p-1} + \cdots + F_0(X)$$

where each $F_i \in \mathbb{C}[[X]]$, and has order of vanishing strictly larger than i. We call p the multiplicity of the branch, and it is our first topological invariant. Note that f defines a *singular* plane branch γ whenever $p > 1$. Zariski reminds us that the above polynomial, called the Weierstrauss polynomial is also irreducible in $\mathbb{C}((X))$. Puiseux's Theorem uses these algebraic facts, and several others to give us a set of coordinates x and y that allow us to write f in a nice parameterized form. We state the essential part of Puiseux's Theorem that concerns us in this work.

Theorem 4. (Puiseux Expansion) *Let γ be an irreducible plane curve germ with multiplicity $p > 1$. Then there is a positive integer $m > p$ such that $p \nmid m$ and a set of coordinates (x, y) on $\mathbb{C}^2$ such that γ is represented by*

$$\gamma : x(t) = t^p, \quad y(t) = t^m + \sum_{i>m} a_i t^i$$

A perhaps more algebraic way to state the theorem is that there is an injection of $\mathcal{O} \hookrightarrow \mathbb{C}[[t]] = \overline{\mathcal{O}}$, such that $x \mapsto t^p$, and $y \mapsto t^m + \sum_{i>m} a_i t^i$. Here $\overline{\mathcal{O}}$ is the integral closure of $\mathcal{O}$. Wall in [24] gives us a way to go back and forth between the equation $f(X, Y) = 0$, and $\gamma(t) = (x(t), y(t))$. The forward direction of obtaining a parameterization is a well-known procedure, known as the Newton-Puiseux Method, which requires the Newton Polygon. The reverse direction involves a method using matrices of coefficients in a *good* parameterization, which is given by lemma 2.3.1 in [24].

The integer m is a topological invariant as is the *Puiseux Characteristic* (PC), which is the set of *essential exponents* in the Puiseux expansion of the curve germ γ. It is outlined in Chapter II, section 3 of [26] how one is to obtain this characteristic from any irreducible plane curve germ. We denote this set of exponents as

$$PC(\gamma) = (\beta_0; \beta_1, \dots, \beta_g),$$

where in general we have $p = \beta_0$ and $m = \beta_1$. It is a well known topological invariant of the curve and is one of many that can be found in C.T.C Wall's book [24] in 4.3.8 (p. 85), which include the proximity diagram, multiplicity sequence, and the proximity matrix, to name a few.

If $\gcd(p, m) = 1$, then the Puiseux Characteristic of γ is $(p; m)$. In general for longer Puiseux Characteristics we have that $\gcd(\beta_0; \beta_1, \dots, \beta_g) = 1$, but not necessarily pairwise. We define $e_1 := \gcd(p, m)$ and then recursively define $e_i := \gcd(e_{i-1}, \beta_i)$. The set of e_i always decreases so that $e_1 > e_2 > \cdots > e_g = 1$ for an irreducible plane curve germ.

An equally important topological invariant for plane branches is the set denoted as $\Gamma := v(\mathcal{O})$, where v is the function given by taking the valuations (orders of vanishing) of elements of $\mathcal{O}$ using the injection $\mathcal{O} \hookrightarrow \mathbb{C}[[t]]$. In particular we have $v(x) = p$ and $v(y) = m$. Γ has the algebraic structure of a numerical semigroup, which is a monoid under addition, and has a finite complement in the nonnegative integers. We will denote the minimal generators of Γ as ν_i and write

$$\Gamma = \langle \nu_0, \nu_1, \dots, \nu_g \rangle,$$

where in general the minimal generators of Γ are the same in number as the essential exponents. We also have that in general $\nu_0 = p$ and $\nu_1 = m$.

The set Γ can be obtained from the PC, and vice-versa due to II.3.9 in [26]. The theorem also provides that $\gcd(\nu_0, \nu_1, \ldots, \nu_g) = 1$, and $\gcd(e_{i-1}, \nu_i) = e_i$ as well. These facts are precisely what makes Γ a numerical semigroup. Therefore $\Gamma \subset \mathbb{Z}_{\geq 0}$ has a finite complement, and we call this set the *gaps* of Γ. We distinguish the greatest gap $\max(\mathbb{Z}_{\geq 0} \backslash \Gamma)$ and call this the *Frobenius number*. It is one less than the *conductor* $c(\Gamma)$, the minimum value for which all integers greater than or equal to it are contained in Γ, and simply denoted as c if Γ is clear. Alternatively one can write that $c(\Gamma)$ is the minimum value in Γ such that $\Gamma \cap (\mathbb{N}_0 + c) = \mathbb{N}_0 + c$

The conductor is also the *Milnor number* of the curve Γ, which is given by

$$\mu := \dim_{\mathbb{C}} \big(\mathbb{C}[[X, Y]]/(f_x, f_y)\big),$$

where γ is the curve given by $f = 0, with f \in \mathfrak{m}$ as before, and f_x, f_y are the partials with respect to x and y. Thus we often denote the conductor as $\mu = c(\Gamma)$. In terms of deformation theory, it is also the dimension of the versal deformation space (see [4]). For more on deformation theory, see [12], or the appendix by Tessier in [26].

Zariski notes in II.2 in [26] that the number of gaps of Γ is given by $\mu/2$, precisely half of the conductor of the semigroup Γ. These semigroups are often referred to as symmetric semigroups. Upon studying the gap structure this can become rather clear as to why this name fits well.

Zariski also gives a formula for μ in terms of the e_i defined above. Equation II.2.1 in [26] tells us that

$$\mu = \sum_{i=0}^{g} e_i(e_i - 1).$$

and alternatively on page 13 he writes

$$\mu = e_{g-1}\nu_g - \beta_g - (p - 1).$$

In the particular case that $\Gamma = \langle p, m\rangle$ we have that $\mu = (p - 1)(m - 1)$.

Theorem II.3.9 in [26] gives us a one-to-one correspondence between Puiseux Characteristic and Semigroup of a plane branches. The remarkable fact proved by Zariski is stated in the following theorem in regards to these two invariants.

Theorem 5. (Zariski, 3.3 in [26]) *Two branches are in the same equisingularity class if and only if they have the same Γ, if and only if they have the same Puiseux Characteristic.*

To connect this to the Monster tower, we note that there is also an algorithm (see [9]) to go from RVT code word to PC and vice-versa as well. Thus there is a one-to-one correspondence between PC, Γ and the critical RVT code word (ending in a V or T) of a branch, making the RVT code word a topological invariant of the branch as well. In some ways this suggests that there is some type of topological equivalence of points in the Monster tower, even though there is no way to lift homeomorphisms into the tower.

Let us now illustrate this process of Puiseux Characteristic to semigroup Γ to RVT code word with a relatively simple but nontrivial example.

Example 7. Consider any curve γ with $PC = (4; 6, 7)$. To determine the semigroup Γ we note that $\text{lcm}(4, 6) = 12$. Our semigroup is then $\Gamma = \langle 4, 6, 13\rangle$. This is reflecting that 7 is one more than 6, and so when our valuations of x^3 and y^2 meet up at 12 they will cancel to the valuation 13 with the right linear combination.

Indeed every curve in $L(\gamma)$ is topologically equivalent to the curve $x(t) = t^4$, $y(t) = t^6 + t^7$. We see in $\mathcal{O}$ that the only way to get a valuation outside of the semigroup $\langle 4, 6\rangle$ is to consider the minimal valuations in $v(\mathcal{O})$ that are shared by monomials in x and y. The first is clearly $v(x^3) = v(y^2) = 12$. One can check that $v(y^2 - x^3) = 13$.

It is left to the reader to follow the prolongation procedure outlined in the previous section on the Monster Tower for the curve written above. When one goes through this procedure, one finds that the RVT code word for this curve is $RVRVR\ldots$.

This concludes our example and our section on the topological invariants of curve germs. We now look to a finer analytic invariant of the curve that is also discrete, and is a semimodule of the semigroup Γ.

2.2.2 Value Sets of Plane Curve Germs

A finer discrete analytic invariant of our plane curve germ γ is the set of valuations of (non-torsion) differential one-forms $\mathcal{O}d\mathcal{O}$ on the curve, denoted $\Lambda(\gamma) := v(\mathcal{O}d\mathcal{O})$, or just Λ if there is no confusion about the curve. Similarly to before we will use the pullback of these forms into $\mathbb{C}[[t]]dt$, and take their order of vanishing.

We will generally use the convention that is used by Hefez and Hernandes in [13] where we insist that $v(dg) = v(g)$ for all $g \in \mathfrak{m}$. This is equivalent to demanding that $v(dx) = p$ and $v(dy) = m$, or that $v(dt) = 1$. With this convention we have that $\Gamma^* \subseteq \Lambda$ for any γ, where $\Gamma^* = \Gamma \setminus \{0\}$. Furthermore Λ is a Γ-semimodule, i.e. for any $l \in \Lambda$, $a \in \Gamma$ we have $l + a \in \Lambda$. As with our semigroup Γ we will denote the conductor of a semimodule Λ as $c(\Lambda)$. As before $c(\Lambda)$ is the minimum value for which $\mathbb{N}_0 + c(\Lambda) \subset \Lambda$. We also call the $\mathbb{N} \setminus \Lambda$ the *gaps* of Λ.

Alternatively to Λ we can consider the valuations of $\mathcal{O} + y'\mathcal{O}$ where $y' = dy/dx$. We will call $\mathcal{O} + y'\mathcal{O}$ the *Delorme module*, after Charles Delorme (see [10], [11]), and the set $\Delta = v(\mathcal{O} + y'\mathcal{O})$ the *Delorme semimodule.* Λ and Δ are isomorphic as Γ-semimodules, since $\Lambda = \Delta + p$. We can see this via the fact that $\mathcal{O}d\mathcal{O} = \mathcal{O}dx + \mathcal{O}dy$.

Both of these semimodules are, in practice, rather difficult to compute given a singular plane curve germ. There are several algorithms that can help us compute the value set (see [15]). In practice, these algorithms often rely on some sort of reduced form, such as the Zariski short form in III.1 in [26]. We present a particularly useful form for plane curve germs that we intend to use in several places throughout the paper.

Normal Form of Hefez and Hernandes:

Zariski in Proposition III.1.2 of [26] gives for any curve germ γ an analytically equivalent curve germ called a short parameterization of γ. This short parameterization eliminates all powers of t in Γ from the parameterization of $y(t)$, leaving only powers of t that are gaps in Γ with nonzero coefficients. This is provides us with a polynomial for the parameterization of $y(t)$ with no exponent greater than μ. We can further eliminate powers of t to pick a convenient representative of the analytic class of our curve germ.

In [13] Hefez and Hernandes prove Theorem 2.1, their main result, called the 'Normal Forms Theorem.' It gives a particularly useful form for our plane curve γ. The theorem states that γ as a plane curve germ is analytically isomorphic to a curve germ of the form $x(t) = t^p,\ y(t) = t^m$, or there exists some $\lambda > m$ with $\lambda + p = \min(\Lambda \setminus \Gamma^*)$, such that γ is a-equivalent to the curve germ

$$x(t) = t^p, \quad y(t) = t^m + t^\lambda + \sum_{\substack{i>\lambda \\ i\notin\Lambda-p}} a_i t^i,\ a_i \in \mathbb{C}. \tag{2.1}$$

We give a special name to λ: We call it the *Zariski invariant* of the curve γ, as it was proved by Zariski in [25] to be an analytic invariant of the curve γ. We will often use this form and refer to it simply as the normal form. Note that here we use *the* normal form as we find the coefficients, a_i, are unique up to some action of $\mathbb{C}^*$.

In the case that γ is a-equivalent to a curve of the form $x = t^p,\ y = t^m$, we call it the *Quasi-Homogeneous (QH)* curve and we find that $\Lambda = \Gamma^*$, and vice-versa. In fact we have that $\mathcal{O}d\mathcal{O}$ contains only exact differential one-forms if and only if γ is a-equivalent to a QH curve of the form $x = t^p,\ y = t^m$ if and only if $\Lambda = \Gamma^*$ (see III.3 in [26], [13]). Let us now consider the case where $\Gamma^* \subsetneq \Lambda$. In this case there is a minimum element in $\Lambda \setminus \Gamma^*$. It is given to us by a non-exact differential one-form, often called the Zariski one-form.

The Zariski invariant and Zariski one-form of a curve germ:

Suppose that we are in the case where our curve γ has $\Lambda \setminus \Gamma^* \neq \emptyset$. We would like to find $\min(\Lambda \setminus \Gamma^*)$, which we will denote as λ_1, the first generator of Λ as a Γ-semimodule. From above it must come from a non-exact differential one-form. We are therefore looking for some one-form $\omega = Adx - Bdy$ with $A, B \in \mathcal{O}$, such that $v(\omega) = \lambda_1$.

By the way that valuations work, we are looking for A and B with the smallest possible valuations such that $v(Adx) = v(Bdy)$. We note that if γ is in normal form, then $v(xdy) = v(ydx) = m + p$. We must have that $m + p$ is the minimum such valuation where two distinct one-forms agree. Indeed if say $v(Adx) < m + p$, then it either has valuation m or it has valuation a multiple of p, as $v(Adx) \in \Gamma$. Only dy has valuation m up to higher valued terms. Similarly any multiple of p that is less than m must be of the form $x^i dx$ up to a constant, and higher valued terms.

Therefore we set $A = px$ and $B = my$ and denote $\omega_1 := mydx - pxdy$. We call this the *Zariski one-form*. In the case where γ is already in its normal form we immediately have that $v(\omega_1) = \lambda + p \notin \Lambda$. Therefore our we have $\lambda_1 = \lambda + p = \min(\Lambda \setminus \Gamma^*)$, the Zariski invariant shifted up by the multiplicity p of our curve germ. In general it is rather difficult to find the next minimal generators of Λ, especially for general Puiseux Characteristic.

We can do quite a bit more if we restrict ourselves to the *coprime* case, where we simply have $\Gamma = \langle p, m \rangle$. In this case we will present a procedure, which bears some resemblance to the bead game called Mancala, that provides us with all possible value sets for a pair of coprime positive integers. There is still much work to do in the case where our semigroup is generated by more than two elements. Recent work by Abreu and Hernendes help shed some light on the subject in their work in [1]. We will dedicate some effort to this case later on as well.

Next let us consider another analytic invariant of a plane curve germ that is related to the value set, called the Tjurina number.

Tjurina Number of an Analytic Curve Germ: Earlier we mentioned the Milnor number, μ, that in deformation theory is the dimension of the Milnor Algebra of a curve $f = 0$. The algebra is used to construct versal deformations of a singular plane curve germ (see Section 2 of [4]), and in the case of semigroups we recall that the Milnor number is the conductor $c(\Gamma)$.

We have also an analytic invariant that relates to the Tjurina algebra of the curve germ γ given by $f = 0$. We denote the *Tjurina number* by τ and it is defined to be

$$\tau := \dim_{\mathbb{C}} \left(\frac{\mathbb{C}[[X, Y]]}{(f, f_x, f_y)} \right).$$

The Tjurina algebra is related to constructing miniversal deformations of γ and the relation is given by Tjurina and presented as Corollary II.1.17 in [12].

In general we have as stated in [2] that $\mu/\tau < 4/3$ for any singular analytic plane curve germ. Though at first none of this seems to relate to the value set, Hefez and Hernandes note in [15] that due to a proof by Berger in [7] we have the formula

$$\mu - \tau = |\Lambda \setminus \Gamma^*|. \tag{2.2}$$

Therefore if we know the value set of the curve germ, and the conductor of its semigroup, then we can compute the Tjurina number quite easily. Later in this work we will use the recursive formula given in [17] to demonstrate how one can compute the minimal Tjurina number, $\tau_{\min}$, of the topological class of curve germs with $\Gamma = \langle p, m \rangle$, and compare them to results in [2].

We finish this section on value sets with some particular examples of all possible value sets for plane curve germs with a fixed coprime Γ, and the corresponding normal forms that produce these value sets. All of these forms can be found in [14], which give all possible normal forms up to and including multiplicity 4.

Example 8. Let γ be a curve germ with $\Gamma = \langle 2, 2k+1 \rangle$ for any $k \geq 1$. Since we automatically have that the conductor of Γ is $2k$ and that $\min(\Lambda \setminus \Gamma^*) > 2k+1$ we must have that $\Lambda \setminus \Gamma^* = \emptyset$. Note further that by Zariski in III.3 in [26] we have no choice but to conclude the well known fact that γ is a-equivalent to (t^2, t^{2k+1}).

Example 9. Let γ be a curve germ with $\Gamma = \langle 3, 3k+1 \rangle$ for some $k \geq 1$. We then claim that the Zariski invariant of γ completely determines the Analytic type of the curve. That is, there are no moduli, and only discrete data for curves of multiplicity 3.

To see this, let us suppose that γ is not quasi-homogeneous and that the Zariski invariant of γ is $\lambda \neq \infty$. With $\Gamma = \langle 3, 3k+1 \rangle$, and $\lambda > 3k+1$, we must have that $\lambda \equiv 2 \mod 3$, since once we get to $3k+1 \in \Gamma$ we have all multiples of 3 and all integers congruent to 1 mod 3 greater than $3k$. Now $\lambda + 3 = \min(\Lambda \setminus \Gamma^*)$ and so as soon as we get to $\lambda + 3$ in the value set Λ we have also all integers congruent to 2 mod 3. It follows that $c(\Lambda) = \lambda + 1$, since if $\lambda = 3(k+r)+2, r \geq 0$, then we must already have $3(k+r+1), 3(k+r+1)+1 \in \Lambda$ but $\lambda \notin \Lambda$.

Observing the normal form given by (2.1) for multiplicity 3, we find that there is no room for any other nonzero coefficients above λ, since $(\Lambda \setminus \Gamma^*) \cap (\mathbb{N}_0 + \lambda + 3) = \emptyset$. Therefore γ is a-equivalent to the curve $(t^3, t^{3k+1} + t^{3(k+r)+2})$. Note that the conductor of Γ is $\mu = 6k$. The number of gaps in Γ is thus $3k$. The number of gaps between 0 and $3k+1$ is $2k$. Therefore there are $3k - 2k = k$ gaps above $3k+1$. This implies there are $k-1$ choices for the Zariski invariant for γ not QH. This is because the normal form theorem in [13] shows we cannot have $\lambda = \mu - 1$, the Frobenius number.

Adding back in the QH case for γ this gives us k possible Λ for $\Gamma = \langle 3, 3k+1 \rangle$, which implies that for our $\lambda = 3(k+r)+2$ we have $0 \leq r \leq k-2$. This gives a complete description of possible value sets for curve with $\Gamma = \langle 3, 3k+1 \rangle$, and their possible normal forms. A similar process can be done for $\Gamma = \langle 3, 3k+2 \rangle$.

Example 10. Let us now assume that γ has semigroup $\Gamma = \langle 4, 9 \rangle$, which has conductor $\mu = 24$. From Table 1 in Section 4 of [14], Hefez and Hernandes give us the possible normal forms for γ as follows:

Normal Form	$\Lambda \setminus \Gamma^*$	λ
(t^4, t^9)	$\emptyset$	∞
$(t^4, t^9 + t^{10} + at^{11}),\ a \neq 19/18$	$\{14, 19, 23\}$	10
$(t^4, t^9 + t^{10} + \frac{19}{18}t^{11} + at^{15})$	$\{14, 23\}$	10
$(t^4, t^9 + t^{11})$	$\{15, 19, 23\}$	11
$(t^4, t^9 + t^{15})$	$\{19, 23\}$	15
$(t^4, t^9 + t^{19})$	$\{23\}$	19

Notice in the table that we have a modulus in two of our normal forms. This is the lowest multiplicity and smallest generators in general for Γ for which moduli appear. We can also observe that the second row gives us the generic Λ type, which specializes to any of the other types.

The reader is highly encouraged to do some calculations with the forms in rows 2 and 3 to observe the effects of coefficients on our value sets.

This concludes our examples and our section on value sets of plane branches. We end our preliminaries with a discrete invariant of Legendrian curve germs that we will explore in detail later.

2.2.3 Semigroups of Legendrian Curves

Given a Legendrian curve germ, we can consider the regular functions on the curve similarly to the way we did for plane curve germs. These regular functions have a pullback into $\mathbb{C}[[t]]$. This gives us a valuation on the regular functions on a Legendrian curve germ.

In Section 2.1.5 we found that any singular Legendrian curve germ can be thought of as the prolongation of some singular plane curve germ that can be parameterized as $\gamma(t) = (x(t), y(t))$ with $v(x^2) < v(y)$.

Let $\mathcal{O}$ be the regular functions on γ. The regular functions on our Legendrian curve γ^1 are then given by

$$\mathcal{O}[[y']] = \mathcal{O} + y'\mathcal{O} + (y')^2\mathcal{O} + \dots,$$

and the Legendrian semigroup is given by $v(\mathcal{O}[[y'(t)]])$. In most cases we will assume that we already have our γ and it has semigroup Γ. In this case we will denote the Legendrian semigroup of γ^1 as Γ^1.

It is clear from the equation above that the Delorme module of γ is contained in the regular functions on the Legendrian curve germ γ^1. This immediately implies that the Delorme semimodule is contained in the Legendrian semigroup. Later we will show that there is an analogous notion to the Zariski invariant for Legendrian curves and their semigroups, which is closely related to the containment of the Delorme module.

In [6] they give a definition for the generic projection of a Legendrian curve germ, and also give an algorithm to compute what they call the generic Legendrian semigroup associated to a pair of coprime integers p and m with $2p < m$.

Later in this work we will show that there are certain directions related to the distribution Δ_1 that one can use to project a contact curve to a plane curve and always find the same set of discrete analytic invariants. These invariants will also be contact invariants of the contact curve in the sense that they are invariants under contact transformations followed by projections. Even further we can show that there is some analytic jet information that is invariant under contact transformation and projections in these directions that are essentially transverse to the contact distribution Δ_1.

Let us finish this section with an example on computing some Legendrian semigroups.

Example 11. Let us consider plane curve germs with $\Gamma = \langle 4, 9\rangle$. The table in Example 22 gives all possible normal forms for these plane curve germs. Any singular Legendrian curve germ with $RRVTT$ as its code word will be contact equivalent to the lift of one the curve germs in the table. Let us assume that our plane curve germ γ is already in normal form, so that

$$\gamma^1 : x(t) = t^4, \quad y(t) = t^9 + h.o.t., \quad y'(t) = \frac{9}{4}t^5 + h.o.t.$$

Since $n \cdot v(y') \in \Gamma^1$ for all $n \geq 0$, we have that $\langle 4, 5\rangle \in \Gamma^1$. Note that $\langle 4, 5\rangle$ has conductor $3 \cdot 4 = 12$. Also because we now have y' at our disposal, we can consider the fact that $v(xy') = v(y)$. Let us consider the possible valuations of $v(y - \frac{4}{9}xy')$.

In the case that $y(t) = t^9$ or $y(t) = t^9 + t^\lambda$, $\lambda \geq 15$, we have that $v(y - \frac{4}{9}xy') \geq 15$, which is greater than the conductor of our Legendrian semigroup, as it is greater than the conductor of $\langle 4, 5\rangle$. Thus in this case our Legendrian semigroup is $\Gamma^1 = \langle 4, 5\rangle$.

In the case that $y(t) = t^9 + t^{11}$, we have that $v(y - \frac{4}{9}xy') = 11$, and so the Legendrian semigroup is given by $\Gamma^1 = \langle 4, 5, 11\rangle$, which is not the semigroup of any plane curve germ.

Finally in the case that $y(t) = t^9 + t^{10} + at^{11} + h.o.t.$ where we put no restrictions on a, we find that there are two possibilities depending on the value of a. One can verify that if $a = \frac{10}{9}$ then $\Gamma^1 = \langle 4, 5\rangle$, and otherwise $\Gamma^1 = \langle 4, 5, 11\rangle$. These are all of the possible normal forms, and therefore all of the possible Legendrian semigroups for the code word $RRVTT$.

This concludes our example and our preliminaries. In our next chapter we will formally assign sets of jets of singular plane curve germs to points in the monster, and from there we will show how value sets of plane curve germs can also be assigned to certain regular points in the Monster tower.

Chapter 3

Value Sets of Points in the Monster Tower

The goal of this chapter is to assign value sets of plane curve germs to points in the Monster Tower. We will see that the value set of the point $u \in M(n)$ is related to the critical code word of the point, and also the level n at which the point u sits. In order to do this we will need to define a particular set of plane curve germs that we can assign to our point $u \in M(n)$. With the help of Montgomery and Zhitomirskii in Chapter 4 of [19] we will show that this particular set of plane curve germs assigned to u all have analytically equivalent r-jets for some fixed $r > 0$ if the point u is indeed a regular point (see Defintion 10). From this we will be able to assign value sets to points $u \in M(n)$ by comparing r and n.

This will ultimately lead to our main result, which gives a recursive formula for the generic value set assigned to generic points $u \in M(n)$ with a given critical RVT code word with only one critical block. We now establish that every equi-singularity class of analytic plane curve germ has a finite positive integer l associated to it, such that if we know the l-jet of a plane curve germ in this class, then we know its value set Λ.

3.1 Λ Jet Identification Number for a Topological Class

Let us review some notation from the preliminaries. We assume that we are working with analytic irreducible plane curve germs $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ with Puiseux Characteristic $(p; m, \beta_2, \ldots, \beta_g)$, and associated semigroup $\Gamma = \langle p, m, \nu_2, \ldots, \nu_g \rangle$. We denote the conductor of the semigroup Γ as μ for the Milnor Number. We also assume that Λ is the associated value set of differential one-forms (equiv. the Γ-semimodule) of the curve γ. We denote its conductor as $\mathrm{c}(\Lambda)$. From the main result in [13], any curve germ in the topological class with the above Puiseux Characteristic is analytically equivalent to a curve germ parameterized by:

$$x = t^p, \quad y = t^m + t^\lambda + \sum_{\substack{i \notin \Lambda - p \\ i > \lambda}} a_i t^i, \ a_i \in \mathbb{C}. \tag{3.1}$$

We will denote the r-jet of a curve germ γ at the origin as $j^r(\gamma)$. Let us now define what we mean by a Λ jet identification number:

Definition 16. Let $(p; m, \beta_2, \dots, \beta_g)$ be a Puiseux Characteristic of a plane curve germ, with $2 < p < m$. Then the Λ *jet identification number for* $(p; m, \beta_2, \dots, \beta_g)$ is the minimum positive integer l such that if any two plane curve germs γ_1 and γ_2, both with Puiseux Characteristic $(p; m, \beta_2, \dots, \beta_g)$, have different value sets $\Lambda_1 \neq \Lambda_2$, then $j^l(\gamma_1) \neq j^l(\gamma_2)$.

We now show that in general this number l is finite. This finite aspect of l is mainly a result of Zariski's short parameterization in III.1.2 of [26], of which the normal form of Hefez and Hernandes is a refinement.

Proposition 6. *The Λ jet identification number l for any Puiseux Characteristic $(p; m, \beta_2, \dots, \beta_g)$ is bounded by the conductor μ of the semigroup Γ associated to $(p; m, \beta_2, \dots, \beta_g)$.*

Proof. Suppose that two analytic plane curve germs γ_1 and γ_2, both with Puiseux Characteristic $(p; m, \beta_2, \dots, \beta_g)$, have value sets Λ_1 and Λ_2, respectively, such that $\Lambda_1 \neq \Lambda_2$. Now suppose for contradiction that we further have $j^\mu(\gamma_1) = j^\mu(\gamma_2)$. Let us assume we have coordinates such that $x = t^p$ for both γ_1 and γ_2. Then we can write

$$\gamma_1 : \ x(t) = t^p, \ \ y(t) = t^m + \sum_{i=m+1}^{\mu} a_i t^i + \sum_{i > \mu} a_i t^i \tag{3.2}$$

$$\gamma_2 : \ x(t) = t^p, \ \ y(t) = t^m + \sum_{i=m+1}^{\mu} a_i t^i + \sum_{i > \mu} b_i t^i. \tag{3.3}$$

Since every integer greater than equal to μ is a valuation of some polynomial in x and y, we see that by possibly two different analytic isomorphisms of the plane we have that both curve germs γ_1 and γ_2 are analytically equivalent to the curve germ

$$\gamma : \ x(t) = t^p, \ \ y(t) = t^m + \sum_{i=m+1}^{\mu} a_i t^i.$$

This curve germ has a unique value set, Λ, associated to it. Since the value set Λ is an analytic invariant, it must be that γ_1 and γ_2 both have the same value set. This contradicts our original assumption that they were not equal. Thus we must have that μ is an upper bound on l the Λ jet identification number. □

Let us define what it means to associate a value set to a jet of particular size.

Definition 17. Let $(p; m, \beta_2, \ldots, \beta_g)$ a Puiseux Characteristic of an analytic plane curve germ, and l the Λ jet identification of $(p; m, \beta_2, \ldots, \beta_g)$. Then for any γ with P.C. characteristic $(p; m, \beta_2, \ldots, \beta_g)$, the associated value set Λ to $j^k(\gamma)$, $k \geq l$, is the value set Λ of the curve γ.

Note any $\tilde{\gamma}$ such that $j^k(\tilde{\gamma}) = j^k(\gamma)$ will have the same Λ, since $k \geq l$ and so this is a good definition. If k is large enough to determine the topological type of any curve germ with the same k-jet (see Section 3.2.2 below), then we could also consider t he *set* of value sets "belonging" to the k-jet $j^k(\gamma)$ for some γ in the given topological class. By belonging we simply mean all possible value sets associated to all possible curves γ with the the given k-jet.

In the case where our Puiseux Characteristic is of the form $(p; m)$, equivalently our RVT code word only has one critical block, we can compute the precise value of l.

3.1.1 Determining the Λ Jet Identification Number for $(p; m)$

We now determine the value for the Λ jet identification number for a given set of coprime integers p, m. A formula for a general Puiseux characteristic will require more knowledge on value sets and how they possibly build from values sets of semiroots of the curve (see [1]). This is saved for a later work. For now we present the following theorem.

Theorem 6. *The Λ jet identification number for the coprime integers $2 < p < m$ is the number*

$$l = \mu - p - 1.$$

Proof. First we show that l it must at least be $\mu - p - 1$. For this we will take the two curve germs

$$\gamma_{QH} \colon x = t^p, \quad y = t^m$$
$$\gamma_{Fr} \colon x = t^p, \quad y = t^m + t^{\mu-p-1}$$

We note that $\Lambda_{QH} \neq \Lambda_{Fr}$, since the valuation of the one-form $pxdy - mydx$ differs for each curve. These two curves are already in Zariski short form (ref Z), and so we must have that $v(pxdy - mydx)$ is the minimum possible valuation of any element in $\Lambda \setminus \Gamma$. Here we have $v_{QH}(pxdy - mydx) = \infty$, but on the other hand $v_{Fr}(pxdy - mydx) = \mu - 1$, the Frobenius number. By the above comment we must have that $\Lambda_{QH} \setminus \Gamma_{QH} = \emptyset$ whereas $\Lambda_{Fr} \setminus \Gamma_{Fr} = \{\mu - 1\}$. Thus $\Lambda_{QH} \neq \Lambda_{Fr}$.

Now consider that $j^i(\gamma_{QH}) = j^i(\gamma_{Fr})$ as long as $i < \mu - p - 1$. On the other hand it is clear that $j^{\mu-p-1}(\gamma_{QH}) \neq j^{\mu-p-1}(\gamma_{Fr})$. This shows that we must at least compare the $\mu - p - 1$-jets of any two curve germs to see a distinction in their jets if they have differing Λ. Hence $l \geq \mu - p - 1$.

We next want to show that l is at most $\mu - p - 1$. Let us consider two curve germs γ_1 and γ_2, with Γ-semimodules Λ_1 and Λ_2, respectively, such that $\Lambda_1 \neq \Lambda_2$. We show that $j^{\mu-p-1}(\gamma_1) \neq j^{\mu-p-1}(\gamma_2)$. Let us assume towards contradiction that instead we have $j^{\mu-p-1}(\gamma_1) = j^{\mu-p-1}(\gamma_2)$. We may assume (WLOG) that γ_1 is in the reduced normal form of Hefez and Hernandes (see Section 2.2.2). Otherwise we could use an analytic isomorphism to take γ_1 to its reduced normal form, and the jets of the two transformed curve germs would remain equal (i.e. if Φ is the isomorphism, then $j^{\mu-p-1}(\Phi \circ \gamma_1) = j^{\mu-p-1}(\Phi \circ \gamma_2)$).

Therefore, avoiding the quasi-homogenous curve for now, we have the parameterization:

$$\gamma_1\colon\ x = t^p, \quad y = t^m + t^{\lambda_1} + \sum_{\substack{i \notin \Lambda_1 - p \\ i > \lambda_1}} a_i t^i.$$

Note here that $i \leq \mu - p - 1$, since $\Gamma^* - p \subseteq \Lambda_1 - p$, and $c(\Gamma - p) = \mu - p$. Hence the last possible gap in $\Lambda_1 - p$ is $\mu - p - 1$. We now use the assumption that $j^{\mu-p-1}(\gamma_1) = j^{\mu-p-1}(\gamma_2)$. If this is the case we must have

$$\gamma_2\colon\ x = t^p + at^{\mu-p} + h.o.t., \quad y = t^m + t^{\lambda_1} + \sum_{\substack{i \notin \Lambda_1 - p \\ i > \lambda_1}} a_i t^i + h.o.t.$$

That is, γ_2 has the same parameterization as γ_1 up to powers of t greater than $\mu - p - 1$.

We now show that those higher order terms are inconsequential in determining the value set of the curve. Let us first note that if $\lambda_1 = \mu - p - 1$ we are in the case above with $\gamma_1 = \gamma_{Fr}$. It is clear for γ_2 that the Zariski form will then give us a valuation of $\mu - 1$. There are no other gaps left between $\mu - 1$ and $\mu - p - 1$ in Λ, so the terms greater than or equal to $\mu - p$ can no longer play a part in calculating the value set Λ_2, as their valuations would be beyond the conductor. Thus we already have our contradiction in that $\Lambda_1 = \Lambda_2 = \Gamma \cup \{\mu - 1\}$.

Now suppose that $\lambda_1 < \mu - p - 1$. If this is the case, then again we have that γ_2 must share the same Zariski invariant as γ_1, namely, λ_1. This implies $\min(\Lambda_2 \setminus \Gamma) = \lambda_1 + p < \mu - 1$. This automatically implies that $\mu - 1 \in \Lambda_2$, since if $\lambda_1 + p \in \Lambda_2 \setminus \Gamma$, then there exists $a, b > 0$ such that $\lambda_1 + p = pm - am - bp$. We also have that $\mu - 1 = pm - m - p$, and so $\lambda_1 + p + (a-1)m + (b-1)p = \mu - 1$. Since $(a-1)m + (b-1)p \in \Gamma$, $\mu - 1 \in \Lambda_2$. As the above paragraph shows, if $\mu - 1 \in \Lambda_2$ then there are no gaps in Λ_2 that are greater than $\mu - p$. It follows that it is impossible for the terms in $x(t)$ and $y(t)$ in the above parameterization for γ_2 that are greater than $\mu - p - 1$ to alter the value set Λ_2, as these terms can only produce valuations greater than $\mu - p - 1$. Hence we must again have that Λ_2 is completely determined by terms in $x(t)$ and $y(t)$ that are less than $\mu - p$. But these terms are the same in both γ_1 and γ_2 by the equality of their jets. It must then be that $\Lambda_2 = \Lambda_1$, a contradiction.

Finally we look at the case where $\gamma_1 = \gamma_{QH}$. If this is the case then γ_2 has a Zariski one-form $\omega = pxdy - mydx$ with valuation greater than $\mu - 1$. This is enough to show that γ_2 is analytically equivalent to the quasi-homogeneous curve germ. Indeed suppose that α is a one-form on the curve γ_2 such that $v(\alpha) \in \Lambda_2 \setminus \Gamma$. Then it is well known (ref) that we can write $\alpha = Adx - Bdy$ with $A, B \in \mathcal{O}_{\gamma_2}$ the set of regular functions on the curve γ_2. Note here that we must have $v(Adx) = v(Bdy)$, which implies

$$v(A) + p = v(B) + m. \tag{3.4}$$

We consider that $v(A)$ is equal to the lowest valued monomial in x and y in terms of valuations of powers of t. We also consider that since p, m are coprime, we must have that this monomial uniquely provides this valuation, and there are no other monomials of this order (as we assume $v(\alpha)$ produced something less than the conductor of Γ).

Thus $v(A) = v(x^j y^k) = jp + km$. Similarly for B we must have $v(B) = v(x^r y^s) = rp + sm$. By (1) we have

$$(j+1)p + km = rp + (s+1)m.$$

Since we are assuming $v(\alpha) < \mu$, we must have $v(A), v(B) < \mu$ and so with p, m coprime, the above equation implies $r = j+1$ and $k = s+1$. This gives $A = ax^j y^{s+1} + h.o.t.$ and $B = bx^{j+1}y^s$ so that $Adx - Bdy = ax^j y^{s+1}dx - bx^{j+1}y^s dy + h.o.t.$ or

$$Adx - Bdy = x^j y^s(aydx - bxdy) + h.o.t.$$

It is not hard to see that $a = cm$ and $b = cp$ for some c in order for $v(\alpha)$ to satisfy the conditions of being a gap of Γ, when explicitly dealing with the parametrization of γ_2. If not, there could not be a cancellation of powers of t, and $v(\alpha)$ would be an element of Γ. Thus we can write

$$Adx - Bdy = x^j y^s c\omega + h.o.t.$$

where ω is the standard Zariski one-form for the curve γ_2. This shows that the Zariski one-form is indeed the minimal possible valuation outside of Γ in the case of γ_2, but in this case with $\gamma_1 = \gamma_{QH}$ we have that $v(\omega) > \mu - 1$ and so it must be that γ_2 is equivalent to the quasi-homogenous curve, which is the case if and only if $\Lambda_2 = \Gamma^* = \Lambda_1$, again a contradiction.

This shows that is is sufficient to let $l = \mu - p - 1$ in order to see a distinction in l-jets of two curves with different value sets. From the first part of the proof we also see that it is necessary, and thus the Λ jet identification number for the coprime integers p, m is $l = \mu - p - 1$.

□

We give an example of computing the number l, and the two curves mentioned in the proof above.

Example 12. Suppose we have $(4; 9)$ as our Puiseux characteristic. Then $\gamma_{QH}(t) = (t^4, t^9)$ and $\gamma_F = (t^4, t^9 + t^{19})$. We have that $\mu = (4-1)(9-1) = 24$, and so $l = \mu - p - 1 = 24 - 4 - 1 = 19$. A quick check using the Zariski one-form shows that $\min(\Lambda_F \setminus \Gamma^*) = 23$ for γ_F.

Now that we have proved the theorem, we would like to attempt to make a similar statement about points in the Monster Tower. That is, at what level, if any, do we see two curves with different value sets prolonging through different points? To do this we will need to develop quite a bit of machinery. Fortunately much of this has already been done to some extent by Montgomery and Zhitomirskii in Chapter 4 of [19]. The author does not consider the next section his original work, but rather an extension of that found in [19]. None-the-less it is necessary for the results in this chapter to push their work down to plane curve germs by restricting our notion of equivalence to analytic isomorphisms of the plane and their lifts into the Monster Tower.

In particular we will need to develop the idea of an analytic jet identification number for points in the Monster Tower in regards to equivalence of plane curve germs rather than contact equivalence of Legendrian curves as is done in [19]. We are thus leaving the somewhat more natural notion of equivalence of points in the Monster Tower behind as we restrict to analytic equivalence. However with this restriction we will see that we can obtain a finer stratification of the Monster Tower beyond that of RVT code words. We now formally develop the connection between points in the Monster Tower, and jets of analytic plane curve germs.

3.2 The Analytic Jet Identification Number of a Regular Point in the Monster Tower

We turn our attention back to points in the Monster Tower. Following along with the ideas in Chapter 4 of [19], we devote the following section to defining exactly what the analytic jet identification number is for a regular point $u \in M(n)$.

3.2.1 Defining The Analytic Jet Identification Number of a Point in the Monster Tower

In order to define the analytic jet Identification number of a regular point in the Monster Tower, we will first need to associate a set of analytic plane curve germs to the given point. These curves should all have a couple of specific properties. We should obviously impose that these plane curve germs prolong though our given point. We also would want them comparable in some other way, and it would be most appropriate to have these curve germs all regularized at or before the level at which our point sits in the Monster Tower. Let us define this set. First we define the following set of integral curve germs at the level in the Monster Tower of our given point.

Definition 18. Let $u \in M(n)$. We define the set

$$\mathrm{Int}^{\mathrm{Reg}}(u) := \{\gamma \colon (\mathbb{C}, 0) \to (M(n), u) \mid \gamma \text{ is regular at } u\}.$$

Recall from Definition 11 that for γ to be regular in $M(n)$ it must be integral and immersed, with $\gamma'(0)$ in a regular direction. We now associate to our point the desired set of plane curve germs mentioned above.

Definition 19. Let $u \in M(n)$. We define the set

$$\mathrm{Pl}(u) := \{\gamma_n \mid \gamma \in \mathrm{Int}^{\mathrm{Reg}}(u)\}.$$

In other words $\mathrm{Pl}(u)$ is the set of all plane curve germs that are projections of regular curves through u down to the base $\mathbb{C}^2$. We again emphasize that in this work we do not consider immersed curves in the fiber of $M(1)$ to be regular curves, and so to be regular the projection, γ_n, cannot be constant at the base.

With these definitions in hand, we are now ready to define what we mean by the analytic jet identification number of a point $u \in M(n)$.

Definition 20. Let $u \in M(n)$ and suppose there exists an r such that the following two properties hold:

1. If $\gamma \in \mathrm{Pl}(u)$, then for any $\tilde{\gamma}$ a plane curve germ such that $j^r(\gamma) = j^r(\tilde{\gamma})$ we necessarily have $\tilde{\gamma} \in \mathrm{Pl}(u)$.
2. The set $j^r(\mathrm{Pl}(u))$ consists of only one r-jet up to reparameterization.

Then r is called the *analytic jet identification number* of the point u, and the single r-jet in 2 is called the *associated r-jet* to u.

We would now like to develop the notion that points in the Monster are analytically equivalent if and only if they have the same analytic jet identification number r, and have analytically equivalent associated r-jets. Let us define what it means to have analytic equivalence of jets.

Definition 21. Let γ_1 and γ_2 be analytic plane curve germs. We say that $j^k(\gamma_1)$ is *analytically equivalent* to $j^k(\gamma_2)$ if there exists an analytic isomorphism Φ such that $j^k(\Phi \circ \gamma_1) = j^k(\gamma_2)$ up to reparameterization.

We now state a proposition that mirrors closely the results of Theorem 4.14 in [19] and the fact that analytic isomorphisms lift to contactomorphisms.

Proposition 7. *Two points u and $\tilde{u}$ are analytically equivalent in $M(n)$, $n > 0$, if and only if their exists two curve germs $\gamma \in Pl(u)$ and $\tilde{\gamma} \in Pl(\tilde{u})$ that are analytically equivalent.*

Proof. First suppose two points $u, \tilde{u}$ in the Monster are analytically equivalent. Then by definition there exists an analytic isomorphism germ Φ of the plane such that $\Phi^n(u) = \tilde{u}$. Take any $\gamma \in \mathrm{Pl}(u)$ and consider the curve $\Phi \circ \gamma$. This curve germ is certainly analytic to γ and we claim it is in $\mathrm{Pl}(\tilde{u})$. Indeed we see that by results from Montgomery Zhitomirskii (see 2.6, 2.7, 2.12, 4.14 in [19]) and the fact that Φ^1 is a germ of a contactomorphism, that prolongation and lifts of contactomorphisms commute (see 2.9 in [19]). This implies

$$(\Phi \circ \gamma)^n(o) = (\Phi^n \circ \gamma^n)(0) = \Phi^n(u) = \tilde{u}.$$

Since Φ^n is a local symmetry of $M(n)$, is follows that $\Phi \circ \gamma$ is regular at $\tilde{u}$ as well. Thus $(\Phi^n \circ \gamma^n)_n \in \mathrm{Pl}(\tilde{u})$. We further have that projections commute with local symmetries and inverse to prolongation down to level 1. Since we are insisting that Φ is an analytic isomorphism of the plane, we can further project all the way down to the base to find that $(\Phi^n \circ \gamma^n)_n = \Phi \circ \gamma$. We thus now shown the forward direction of the proposition.

Now suppose that two curve germs $\gamma \in \mathrm{Pl}(u)$ and $\tilde{\gamma} \in \mathrm{Pl}(\tilde{u})$ are analytically equivalent. Then there again exists a Φ such that $\Phi \circ \gamma = \tilde{\gamma}$. From above we have

$$\Phi^n(u) = (\Phi^n \circ \gamma^n)(0) = (\Phi \circ \gamma)^n(0) = \tilde{\gamma}^n(0) = \tilde{u}.$$

Hence, u and $\tilde{u}$ are analytically equivalent. □

Next we give results that will help us complete the picture of how regular points in the Monster Tower correspond to jets of curve germs in the plane.

Proposition 8. *If $u \in M(n)$ has a jet identification number, then this number is unique.*

Proof. The proof is nearly identical to proposition 4.21 in [19], which applies almost entirely to this situation. □

Proposition 9. *If u has a jet identification number r and $\tilde{u}$ is analytically equivalent to u, then $\tilde{u}$ also has jet identification number r.*

Proof. This proof is again nearly identical to the proof given in proposition 4.22 in [19]. One merely needs to replace the word contactomorphism with analytic isomorphism throughout the proof. □

We next give the main theorem of this section, which gives the complete picture of points and jets of curve germs.

Theorem 7. *Let u and $\tilde{u}$ be points in the Monster $M(n)$, $n > 0$, with defined jet identification numbers r and $\tilde{r}$ respectively. Then u and $\tilde{u}$ are analytically equivalent if and only if $r = \tilde{r}$ and any two curve germs $\gamma \in Pl(u)$ and $\tilde{\gamma} \in Pl(\tilde{u})$ have analytically equivalent r-jets.*

Proof. Assume first that u and $\tilde{u}$ are analytically equivalent. Then Proposition 9 gives that $r = \tilde{r}$. By the proof of the forward direction of Proposition 7, we have that there exists a Φ such that for any $\gamma \in \mathrm{Pl}(u)$, $\Phi \circ \gamma \in \mathrm{Pl}(\tilde{u})$. Now take any $\tilde{\gamma} \in \mathrm{Pl}(\tilde{u})$. By the definition of jet identification number, $j^r\mathrm{Pl}(\tilde{u})$ consists of only one r-jet up to reparameterization. Therefore we must have $j^r(\Phi \circ \gamma) = j^r(\tilde{\gamma})$ up to reparametrization. This is the definition of having analytically equivalent r-jets for plane curve germs (see Def. 20). This completes the forward direction of the theorem.

The other direction is left to the reader as it is a straightforward application of the above propositions and part 1 of Definition 20. $\square$

This ends our section on jet identification number. We will return to this concepts at the end of this section to give a precise value for r given the RVT class of the point u. We will also therefore give a complete description of the points in $M(n)$ that have a jet identification number. For this we will need the concept of parameterization number.

3.2.2 Parametrization Number of a Point in the Monster

We start with the definition of a well-parameterized curve, and it's order of good parameterization.

Definition 22. An analytic curve germ $\gamma(t) = (x(t), y(t))$ is a *poorly-parameterized* if there exists analytic germs $\phi\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ and $\mu\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ such that $\gamma(t) = (\mu \circ \phi)(t)$. Otherwise γ is called *well-parameterized*.

Definition 23. The *order of good parameterization* of a well-parameterized analytic curve germ $\gamma(t) = (x(t), y(t))$ is the number d such that if any other curve germ $\tilde{\gamma}(t)$ has $j^d(\tilde{\gamma}) = j^d(\gamma)$, then $\tilde{\gamma}(t)$ is also well-parameterized.

It is now that we will explicitly start working with RVT code words and Puiseux Characterisitics of plane curve germs. We have that a topological class of curve germs is closed under analytic transformations (as they are germs of homeomorphisms). We next consider prolonging these curve germs up into the Monster Tower $M(n)$, and evaluating these curve germs at $t = 0$ to give us a set of points $S(n) \subset M(n)$. We note that $S(n)$ is then closed under lifted analytic isomorphisms. That is, there is a locus of points in the Monster Tower $M(n)$ that corresponds to a certain topological class of curves in the plane. We recall from Proposition 4 that this topological class has a unique critical RVT code word, i.e. a word that ends in a V or T.

The length of the word corresponds to the *regularization level* in the Monster Tower: any curve germ γ in the topological class corresponding to α will have the property that the point $\gamma^n(0)$ is critical, γ^n is regular at $\gamma^n(0)$, and hence at $\gamma^{n+1}(0)$ is a regular point, where n is the length of α. Let us denote the locus in $M(n)$ corresponding to α as (α). Also denote $\mathrm{Pl}(\alpha)$ as the set of plane curve germs $\mathrm{Pl}(u)$ for all $u \in (\alpha)$. We have the following proposition.

Proposition 10. *Let α be a critical RVT code word, and $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ an analytic plane curve germ. Then $\gamma \in Pl(\alpha)$ if and only if α is the RVT code word of γ (i.e. γ is in the topological class corresponding to the RVT code word α).*

Proof. Suppose γ has RVT code word α. Then $\gamma^r(0) \in (\alpha) \subset M(n)$. Moreover, γ is regular at $\gamma^n(0)$. Thus $\gamma^n \in \mathrm{Int}^{\mathrm{Reg}}(\gamma^r(0))$. Finally we have from previous results that $(\gamma^n)_n = \gamma$, and so $\gamma \in \mathrm{Pl}(\gamma^n(0))$ and $\gamma^n(0) \in (\alpha)$, so $\gamma \in \mathrm{Pl}(\alpha)$.

Now suppose $\gamma \in \mathrm{Pl}(\alpha)$. Then by definition $\gamma^n(0) \in (\alpha)$, and γ^n is regular at $\gamma^n(0)$. It follows immediately by the way an RVT code is assigned to a curve germ that γ has critical code word α. □

We recall from Proposition 5 that any a-equivalent point $\tilde{u}$ to the point $u \in (\alpha)$ has the same RVT code word of u, namely α. It is also clear that because the curves in $\mathrm{Pl}(\alpha)$ are regular at level n (n again the length of α), the points on the locus (αR^q) in $M(n+q)$ have the same property as the locus (α) in that $\gamma \in \mathrm{Pl}(\alpha R^q)$ has topological type corresponding to the RVT code word α. We now need to say what we mean when we assign a Puiseux Characteristic to an arbitrary code word α.

Definition 24. Let α be an RVT code word of any type. Write $\alpha = \beta R^q$ with β critical, and $q \geq 0$. Then the Puiseux Characteristic *associated* to α is the Puiseux Characteristic of the critical code word β.

Next we have an important result.

Proposition 11. *Every curve germ in the same topological class has the same order of good parametrization. This order of good parameterization is equal to the last entry in the Puiseux Characteristic of the curve.*

Proof. We leave the details of the proof to the reader noting that any curve in a given topological class is analytically equivalent to curve in reduced normal form of Hefez and Hernandes (see main result in [13]), or even simply Zariski's short form in III.1 of [26]. Either of these forms will consist of a certain restricted set of exponents, including the essential ones with nonzero coefficients (see essential exponents in **??**). From here it is easy to see that the order of good parameterization is the same for any curve germs sharing the same topological type, and it is equal to the last entry of the Puiseux Characteristic. □

We are now ready to assign a parameterization number to any point in the Monster Tower for $n > 0$.

Definition 25. Let $u \in M(n)$, $n > 0$. Then the *parameterization number* of u is the order of good parameterization of any plane curve germ in $\mathrm{Pl}(u)$.

Note that Proposition 11 guarantees that this is a good definition. It is also worthwhile to note that every point in the Monster Tower above the base has a parameterization number associated to it. This is not the case for the jet identification number, as we will see, and so not the case for the Λ identification number as well. One could sum this up as: every point has an associated semigroup Γ in the Monster Tower, but not every point has an associated value set (equiv. semimodule) Λ. We finish this section on parameterization number with one final result:

Corollary 1. *Let $u \in M(n)$, $n > 0$. Let α be the associated RVT code word to u, and $(\beta_0; \beta_1, \beta_2, \ldots, \beta_g)$ be the Puiseux Characteristic of α. Then the parameterization number of u is β_g.*

Proof. This follows immediately from the definition of parameterization number, and Propositions 10, and 11. □

This ends the section on parameterization number. In the next section we compute the analytic jet identification number using the parameterization number.

3.2.3 Computing the Analytic Jet Identification Number of a Point in the Monster

We now wish to give a complete description of the points in the Monster Tower that have a jet identification number, and what that number is.

Theorem 8. *Let $u \in M(n)$, $n > 0$ and α the associated RVT code word. Then u has a jet identification number if and only if α is not critical (ends in an R). That is, if and only if u is a regular point.*

We will need some other results before we can prove this theorem. The following is a crucial result from [19], which is presented in chapter 4 of their work and proved in chapter 8.

Theorem 9. (Theorem B in Chapter 4 of [19]) *Let c^* be a plane curve germ with order of good parameterization d and regularization level r in the Monster, with $r \geq 3$. Let c be another plane curve germ. Let $q \geq 1$. Then the r-step prolongations $(c^*)^r$ and $(c)^r$ have the same q-jet up to reparameterization if and only if the curves c^* and c have the same $(d + q - 1)$-jets up to reparameterization.*

The next theorem gives us the existence of jet identification numbers for regular points.

Theorem 10. *Let $u \in M(n)$, $n > 0$, be regular with associated code word $\alpha = \beta R^q, q \geq 1$, and β critical. Let $(\beta_0; \beta_1, \beta_2, \ldots, \beta_g)$ be the associated Puiseux Characteristic of α. Then the analytic jet identification number of u is*

$$l = \beta_g + q - 1.$$

Proof. The result of the theorem comes from examining Proposition 11 above, and Theorem B in section 4.10 in [19]. □

We end this section with the proof that critical points do not have jet identification numbers.

Proposition 12. *Let $u \in M(n)$, $n > 1$. If u is critical (i.e. if its associated code word ends in a V or T), then u does not have a jet identification number.*

Proof. Assume for contradiction that l is the jet identification number of u. Since u is a point in the Monster, we know that u must have a parameterization number d. Fix a curve germ $\gamma \in \mathrm{Pl}(u)$.

First assume that $l < d$. Then we can find a curve $\tilde{\gamma}$ such that $j^l(\tilde{\gamma}) = j^l(\gamma)$ with $\tilde{(}\gamma)$ having order of good parameterization less than d, which is the order of good parameterization of γ. To see this more clearly, one can introduce coordinates so that

$$\gamma\colon\ x = t^p, \quad y = t^m + t^\lambda + \sum_{\substack{i \notin \Lambda - p \\ i > \lambda}} a_i t^i.$$

Truncating γ below the t^d term in our parameterization of y above and then reducing terms will give us the desired $\tilde{\gamma}$ we seek. In this case we have that $\tilde{\gamma}$ will have an order of good parameterization less than d, and so we must have by Proposition 11 that $\tilde{\gamma} \notin \mathrm{Pl}(u)$. However this is impossible if $j^l(\tilde{\gamma}) = j^l(\gamma)$ by definition of jet identification number. Thus we cannot have $l < d$.

Now suppose $l \geq d$. If this is the case then take any regular one-step prolongation of u, and call this $\hat{u}$. We have many choices in what regular direction we wish to prolong u, as there is essentially a whole $\mathbb{C}$ available (there are only two special directions that we must avoid in an entire $\mathbb{C}P^1$). We will show that no matter what "choice" we intend to make for $\hat{u}$ we will always have $\gamma^{n+1}(0) = \hat{u}$ for our (arbitrary) fixed $\gamma \in \mathrm{Pl}(u)$. Let α be the associated critical code word to u, so that $\hat{u} \in (\alpha R) \subset M(n+1)$. Theorem 3 and the following remark (see theorem 4.10 in M&Z) guarantee that $\hat{u}$ will have the same parameterization number as u itself. By Theorem 10, $\hat{u}$ has jet-identification number d as well.

Now take a curve $\hat{\gamma} \in \mathrm{Pl}(\hat{u})$. Since $\hat{u}$ is a regular point, $\hat{\gamma} \in \mathrm{Pl}(u)$ as it is a regular prolongation of a curve in the level below, and is itself regularized. By definition of jet identification number $j^l(\hat{\gamma}) = j^l(\gamma)$ up to reparameterization. Since $l \geq d$ it is automatic that $j^d(\hat{\gamma}) = j^d(\gamma)$. Since d is the jet-identification number of $\hat{u}$ we again, by definition, have that $\gamma \in \mathrm{Pl}(\hat{u})$. If this is the case, then $\gamma^{n+1}(0) = \hat{u}$. However, γ is a fixed germ, and there are nearly a $\mathbb{C}$'s worth of possible choices for $\hat{u}$. Thus we have our contradiction, and therefore we u has not jet identification number as a critical point. □

We now have all the necessary ingredients to show that Theorem 8 is indeed valid. In a later section, we will use the coordinates on the Monster Tower for a given code word α as we did in Section 2.1.4. With this set of coordinates we will give some description of points on the locus (α) in relation to the curves in $\mathrm{Pl}(\alpha)$. This ends our section on analytic jet identification number. We will now determine which points at which levels in the Monster Tower have to them a single associated value set, given their associated semigroup, equivalently their RVT code word.

3.3 Λ Identification Level of a Critical Word in the Monster

In this section we will tie the previous two sections together to associate a level in the Monster tower to a critical RVT code word where the points on a special locus at this level all have associated value sets. These points will necessarily have jet-identification numbers, and we will use these and the results of the previous sections to show that there is a well-defined minimum level, and an easily determined locus of points. To establish that this level exists and is well-defined, we will first need a lemma:

Lemma 1. *Let α be a critical code word. Then there exists a $q_0 \geq 1$ such that for all $q \geq q_0$, and for any $u \in (\alpha R^q)$, all curve germs in Pl(u) share the same value set Λ.*

Proof. We need to show that this q_0 exists. In order to do so, let us consider that if $q \geq 1$ then any $u \in (\alpha R^q)$ has a jet identification number. This number is based on q. Indeed the jet identification number for any $u \in (\alpha R^q)$ is given by $r = d + q - 1$ where d is the parameterization number of (α). Clearly if we continue to add R's at the end of our code word, we will continue to increase the jet identification number of our locus. On the other hand the Λ jet identification number l for α is a fixed number entirely base on the RVT code word of α.

We now note that if u has jet identification number r, then $j^r(\text{Pl}(u))$ consists of only one jet. If r is greater than or equal to l it follows that for any $\gamma \in \text{Pl}(u)$, $j^r(\gamma)$ has an associated value set. That is, r is large enough that for any $\gamma \in \text{Pl}(u)$, $j^r(\gamma)$ completely determines the value set of the curve germ γ.

We now can show that all curves in $\text{Pl}(u)$ must share the same value set for sufficiently large r and hence sufficiently large q. Indeed, suppose we have that q is large enough so that $r \geq l$. Take any two curves $\gamma, \tilde{\gamma} \in \text{Pl}(u)$. Then we must have by definition of jet identification number that $j^r(\gamma) = j^r(\tilde{\gamma})$. Since $r \geq l$ it follows that the r-jets of γ and $\tilde{\gamma}$ have associated value sets, and that they must too be equal by equality of jets. In other words, the r-jets of γ and $\tilde{\gamma}$ completely determine their value sets, and they are equal. By transitivity, we must have that $\text{Pl}(u)$ consists of only one value set Λ. □

Now that we have proved the lemma we have the ability to define the Λ level in the monster.

Definition 26. Let α be a critical code word with length k. Then the Λ *identification level* of α (or Λ level for short) in the Monster tower is the level $k + q_0$ where $q_0 \geq 1$ is the minimum q_0 such that for any $u \in (\alpha R^{q_0})$, all curve germs in $\text{Pl}(u)$ share the same value set.

It is clear that at the Λ level in the Monster, points have associated l-jets that have themselves associated value sets. What should also be clear is that if we are to take regular prolongations of these points in the Monster, these points will themselves have associated jets with well-defined associated value sets. We note that from Section 3.1.1 we can give a precise value for the Λ level in the monster if α has associated Puiseux Character $(p; m)$.

Theorem 11. *Let α be a critical code word with only one critical block corresponding to the Puiseux Characteristic $(p;m)$. Let k be the regularization level of α in the Monster. Then the Λ level in the Monster tower for α is $L = k + \mu - p - m$.*

Proof. This is comes immediately from the results in Theorem 6 and Corollary 1 with the fact that the parameterization number $d = m$ in this case, and the Λ jet identification number $l = \mu - p - 1$. □

In the general case, instead of equality, we obtain an upper bound of $k + \mu - p - \beta_g$. We will discuss this more in Section 3.5 where we make some attempts to understand the Λ jet identification number and Λ identification level for general α. For now we simply make the above assertion without proof, but note that it is not a difficult result to obtain per the comments and results of Section 3.1.

At this point we have given our attention mainly to critical words, or equivalently Puiseux Characteristics and their loci in the Monster Tower. Now we wish to consider individual points in the Monster tower. To do so we will also want to study individual plane curve germs and their various jets. A closer examination of parameterizations of these plane curve germs allows us to see that for a given curve there may be jets of lower order than the Λ jet identification number for the topological class of the given curve which still allow us to determine the given curve's value set. If this is the case, one logically concludes that some points in the Monster too must have this property. Our next section is devoted to this idea.

3.4 Λ Level for Points in the Monster Tower

In this section we will define a Λ level for certain points similar to that for an entire critical locus. We will first need to establish that individual curve germs have Λ jet identification numbers.

3.4.1 Λ Jet identification Number for Plane Curve Germs

We now return to the plane, and consider fixing a certain curve germ $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$. It is possible that this γ has a value set that can be determined from a lower order jet than the Λ jet identification number of its topological class. For example, curve germs in the class $(3; 3k+1)$ have value sets completely determined by their Zariski invariant λ. In this case, we would only need the λ-jet of the curve to fully determine its value set. Therefore individual curves may have smaller Λ jet identification numbers than their topological class. In fact this is often the case, and so we wish to formally define this notion.

Definition 27. Let $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ be a plane curve germ. Then the Λ *jet identification number* for γ is the minimum positive integer I such that for any curve germ $\tilde{\gamma}\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ with $j^I(\tilde{\gamma}) = j^I(\gamma)$ we have $\Lambda_\gamma = \Lambda_{\tilde{\gamma}}$.

It is clear that $I \leq l$, the Λ jet identification number of the topological class of γ. Therefore this number is finite, and must be well-defined by the existence of l.

Example 13. Let $\Gamma = \langle 4, 9 \rangle$ which gives code word $\alpha = RRVTT$. Let $\gamma : x(t) = t^4,\ \ y(t) = t^9 + t^{10}$. Then the Λ jet identification number of γ is $I = 11$. Indeed consider the curve $\tilde{\gamma} : x(t) = t^4,\ \ y(t) = t^9 + t^{10} + \frac{19}{18} t^{11}$. We have $j^{10}(\tilde{\gamma}) = j^{10}(\gamma)$, yet $j^{11}(\tilde{\gamma}) \neq j^{11}(\gamma)$. Thus $I \geq 11$. To show that it must be 11 we note that there are only two value sets for the class α that include the value 14: (1) the value set $\Lambda_\gamma \setminus \Gamma = \{14, 19, 23\}$; (2) the value set $\Lambda_{\tilde{\gamma}} \setminus \Gamma = \{14, 23\}$. If some curve γ' has $j^{11}(\gamma') = j^{11}(\gamma)$ then we must have $\gamma' : x(t) = t^4 + at^{12} + h.o.t.,\ \ y(t) = t^9 + t^{10} + bt^{12} + h.o.t.$ A quick computation shows that $\Lambda_{\gamma'} = \Lambda_\gamma$.

On the other hand the Λ jet identification number for α is $l = \mu - p - 1 = 24 - 4 - 1 = 19$.

We now give a proposition that in a way is an example which illustrates to some extent the structure of the Λ jet identification number of curves with rather high minimum values for their Zariski invariants.

Proposition 13. *Let γ be a plane curve germ with semigroup $\Gamma = \langle p, m \rangle$. Suppose that γ has min$(\Lambda \setminus \Gamma) = \lambda + p = pm - m - ap$ with $p < ap < m$. Then the Λ jet identification number for γ is $I = \lambda$.*

Proof. If γ has $pm - m - ap \in \Lambda$ then we automatically must have that $pm - m - (a-i)p \in \Lambda$ for $0 \leq i \leq a$. If γ also has that $\min(\Lambda \setminus \Gamma) = \lambda + p = pm - m - ap$, then we must have that $\Lambda = \Gamma \dot{\cup} \{pm - m - (a-i)p\}_{0 \leq i < a}$, since there are no other gaps in Γ greater than $\lambda + p$ other than those mentioned above.

There are coordinates for which γ is parameterized by

$$x(t) = t^p \quad y(t) = t^m + t^\lambda + h.o.t.$$

It is rather clear that we must have $I \geq \lambda$ by similar arguments to those in the example above. We now wish to show that it must be equal to λ. Indeed suppose that γ' is a curve germ such that $j^\lambda(\gamma') = j^\lambda(\gamma)$. Then we must have

$$\gamma' : x(t) = t^p + bt^{\lambda+1} + h.o.t., \quad y(t) = t^m + t^\lambda + h.o.t.$$

A quick check shows that this curve has Zariski invariant λ and so must have, by the same argument above for γ, a value set equal to γ. Thus it must be that the Λ jet identification number for the curve γ is $I = \lambda$. □

We can go to the other extreme and consider the generic value set Λ_{gen} of any topological class. In this case we would like to determine if the curve given has generic value set. We now make a claim towards the generic Λ jet identification number of a plane curve.

Proposition 14. *Let γ be a plane curve germ with semigroup $\Gamma = \langle p, m \rangle$. Suppose γ has generic value set Λ_{gen}, then the Λ jet identification number I for γ is bounded by the inequality $m < I < m + p$.*

Proof. The proof of this proposition follows immediately from Proposition 4 of Delorme in [10], which gives the existence of a generic polynomial relying on the first sequential $p - 1$ coefficients greater than m of the short form parameterization of $y(t)$ for the curve germ γ.

This implies the generic form is completely determined by these coefficients. That is to say, with the knowledge of these coefficients, one could match the generators of Λ for γ to those of Λ_{gen} provided by Delorme's algorithm in [10]. It follows that any other curve with these coefficients would necessarily give us the same result, and therefore we need to know at most the $m + p - 1$-jet of γ to determine the value set of any other curve with the same $m + p - 1$ jet as γ.

To show the lower bound, we see that we need to at least know the $m + p + 1$ jet of some other curve germ γ' to know that it has the minimal possible value for the Zariski invariant of curves in the topological class of Γ. This is a necessary condition of any curve with generic value set Λ_{gen}. $\square$

From the above proposition we see that it is enough to know the $m + p - 1$-jet of any curve germ γ with $\Gamma = \langle p, m \rangle$ to determine whether or not it is of the generic type. If γ does not have an $m + p - 1$-jet equal to some other curve with generic value set, then it cannot have the generic value set.

This concludes our section on Λ jet identification number of a plane curve germ. We now consider that from this definition, we can give a pointwise definition of Λ level in the Monster Tower.

3.4.2 Λ Level of a Point in the Monster Tower

We would now like to assign a minimal level to a point in the Monster tower for which the projection of that point down to that level has a well-defined associated value set. This will generally involve the jet identification number of a point and jet identification number of the projection of the point down to a point with the same critical RVT code word. We should not expect that all points in the Monster will have this property.

We recall that if $u \in M(n)$ then we denote the k-step projection of u by $\pi_k(u) \in M(n-k)$. We now define the Λ level for a point in the Monster.

Definition 28. Let $u \in M(n)$ be regular, and suppose there is a number $s \geq 0$ such that all curve germs in $\text{Pl}(\pi_k(u))$ share the same value set for $0 \leq k \leq s$, but not all curve germs in $\text{Pl}(\pi_{s+1}(u))$ share the same value set. Then we say that u has Λ *identification level* $n - s$.

We assume u is regular, and therefore has a jet identification number. However, the Λ level in the Monster Tower may not exist for u. That is, it is quite possible that $\mathrm{Pl}(u)$ contains curve germs with differing value sets. Then by definition u cannot be assigned a Λ level in the Monster Tower. On the other hand if there exists a Λ level for u, then there is a well-defined value set we may associate to the point u, namely the one belonging to all curves in $\mathrm{Pl}(u)$.

Definition 29. Let u have a Λ identification level in the Monster Tower. Then the *value set associated to* u is the value set shared by all the curves in the set $\mathrm{Pl}(u)$.

We now characterize the Λ level for a point in the Monster using jet identification numbers and Λ jet identification numbers. We will need a lemma and a proposition.

Lemma 2. *Suppose $u \in M(n)$ is a regular point with jet identification number r. Suppose further that $\gamma \in Pl(u)$ with Λ jet identification number I such that $r \geq I$. Then every $\gamma' \in Pl(u)$ has Λ jet identification number I.*

Proof. Suppose that $r \geq I$. Now suppose there is some $\gamma' \in \mathrm{Pl}(u)$ with Λ jet identification number I'. We have $j^r(\gamma') = j^r(\gamma)$ by definition of jet identification number of u and so since $j^I(\gamma) = j^I(\gamma')$, by definition of Λ jet identification number we must have that γ and γ' share the same value set.

Now let μ be any plane curve germ such that $j^I(\mu) = j^I(\gamma')$. Then by above we have $j^I(\mu) = j^I(\gamma) = j^I(\gamma')$ and so by definition of I, we must have that μ has the same value set as γ, hence the same value set as γ'. We must have that $I' \leq I$ by the minimality condition of Λ jet identification number. Supposing strict inequality $I' < I$ contradicts the minimality of I for γ leaving the only possibility to be that $I = I'$. $\square$

Now we can prove a proposition in regard to the definition of Λ level in the Monster.

Proposition 15. *Let $u \in M(n)$ be a regular point with jet identification number r. Then u has a Λ level in the Monster if and only if for every $\gamma \in Pl(u)$ with Λ jet identification number I we have $r \geq I$.*

Proof. Suppose first that u has a Λ level in the Monster tower. Now let $\gamma \in \mathrm{Pl}(u)$, with Λ jet identification number I and take any plane curve germ γ' such that $j^r(\gamma') = j^r(\gamma)$. By definition of jet identification number, we necessarily have that $\gamma' \in \mathrm{Pl}(u)$. By definition of Λ level in the Monster, we must have that $\mathrm{Pl}(u)$ has curve germs all sharing the same value set. Thus $\Lambda_{\gamma'} = \Lambda_\gamma$. By minimality of I, we must have that $r \geq I$.

Now suppose that $r \geq I$ for all $\gamma \in \mathrm{Pl}(u)$. We need to show that $\mathrm{Pl}(u)$ has curve germs all sharing the same value set. Suppose $\gamma_1, \gamma_2 \in \mathrm{Pl}(u)$. By the lemma we have that γ_1 and γ_2 both have Λ jet identification number I. Then we have by definition of jet identification number of points in the Monster that $j^r(\gamma_1) = j^r(\gamma_2)$. Since $r \geq I$ we have $j^I(\gamma_1) = j^I(\gamma_2)$ and so they must share the same value set by definition of Λ jet identification number for plane curves. It follows that for $s = 0$, $\mathrm{Pl}(\pi_s(u))$ consists of plane curve germs all sharing the same value set. This is sufficient for there to exist a maximal s, so that u has a Λ level in the Monster. $\square$

Proposition 15 shows that we need the jet identification number of the point to equal or exceed the Λ jet identification number of the curves in the set $\mathrm{Pl}(u)$. Equipped with this proposition, we now use Section 3.4.1 to give some results about the Λ level in the Monster tower for certain points. First let us monetarily consider the consequences of Proposition 8, and assume that $u \in M(n)$ is some regular point, such that the jet identification number r of u is precisely equal to the Λ jet identification number I of the curves in $\mathrm{Pl}(u)$, that is $r = I$ for u.

Let us assume that u has associated word αR^q where α is critical and of length k. We then recall that if d is the parameterization number of u, then the jet identification number for u is $r = d + q - 1$. Setting $r = I$ we have $I = d + q - 1 \implies q = I - d + 1$, so that the level of such a u is given by

$$L = I - d + k + 1. \tag{3.5}$$

With Proposition 15 we can now use this as our definition for the Λ level of a point in the Monster Tower. The following is essentially a corollary of Proposition 13.

Proposition 16. *Suppose $u \in M(n)$ is a regular point with jet identification number r, and associated $\Gamma = \langle p, m \rangle$. Suppose further that we find that some $\gamma \in Pl(u)$ has Zariski invariant $\lambda = pm - m - (a+1)p$ for $p < ap < m$. If $r \geq \lambda$, then u has Λ level in the Monster $L = \lambda - m + k + 1$ or*

$$L = pm - 2m - (a+1)p + k + 1,$$

where k is the regularization level of Γ, *that is the length of the critical word corresponding to* Γ.

Proof. The proof is given by combining Propositions 13 and 15, and equation 3.5. $\square$

The proposition above can be considered a upper bound in some ways when wanting to compute the Λ level of a point in the Monster with a single Puiseux pair.

We end this section with a theorem in regards to determining which points have the generic value set associated to them. In doing so we give a lower bound for the level in the Monster Tower at which we can determine if a point on a given RVT locus has this generic value set associated to it. Let us first define what we mean when we say a generic point of an RVT code word locus (αR^q).

Definition 30. Let $u \in (\alpha R^q)$, where α is a critical RVT code word, and $q > 0$ as before (so u is a regular point). Then u is called a *generic point* of the locus (αR^q) if and only if there exists some $\gamma \in \mathrm{Pl}(u)$ with the generic value set.

We note that this definition is indeed good, as these points will form an open and dense set in the following sense. If u has $\gamma \in \mathrm{Pl}(u)$ with $\Lambda(\gamma) = \Lambda_{\mathrm{gen}}$, and it is regular, then u has a generic r-jet, where r is the analytic jet identification number of u. If we take all $u \in (\alpha R^q)$ with these generic r-jets, then we will clearly have formed an open and dense set.

Now we present the final theorem of the section.

Theorem 12. *Let $u \in (\alpha R^q)$ where α is a critical word with only one critical block corresponding to the semigroup $\Gamma = \langle p, m\rangle$. Assume that $q \geq p$. Then u has associated to it the generic value set Λ_{gen} if and only if it is generic. That u is generic if and only if every $\gamma \in Pl(u)$ has the same value set given by Λ_{gen}.*

Proof. Suppose first that u is a generic point. Then there exists a $\gamma \in \mathrm{Pl}(u)$ such that $\Lambda(\gamma) = \Lambda_{\mathrm{gen}}$. By Proposition 14 we have that γ must have a Λ jet identification number $I < m + p$. By Theorem 10 u has jet identification number $r = m + q - 1$. By assumption we have that $q \geq p$ and so $r \geq m + p - 1 \geq I$. Finally by Proposition 15 we have that u has a Λ level in the Monster Tower, and that $\mathrm{Pl}(u)$ has curves all sharing the same value set. This value set is clearly that of γ which was assumed to be the generic type. Thus u is, by definition, a generic point.

The reverse direction is trivial, since all $\gamma \in \mathrm{Pl}(u)$ have generic value set, hence u is generic by definition. □

This concludes our section on the Λ level of a point for a regular point in the Monster. We now look forward to the next section, where we make some characterizations about the Λ level in the Mobster for a critical word that has more than one isolated critical block. That is a word corresponding to a case where the Puiseux characteristic is given by more than a single pair of coprime numbers.

3.5 Λ Level for Code Words with More Than One Critical Block

In the previous sections we have mostly dealt with two generator Puiseux Characteristics corresponding to RVT code words with one critical block. In this section when referring to longer words what we mean is RVT code words with more than one critical block. We will heavily reference the work of Abreu and Hernandes in [1] throughout, and use their results to develop some of our own about the Monster tower. Mainly we will be using the concept of semiroots of plane curve germs. Let us explore semiroots and how they relate to points in the Monster.

3.5.1 Semiroots and Projections of Points in The Monster Tower

We begin with definitions, starting with some notation to help simplify our writing.

Definition 31. Let $(\beta_0; \beta_1, \ldots, \beta_g)$, with $\beta_0 = p, \beta_1 = m$, be a Puiseux Characteristic. Define $e_0 := p,\ n_0 := 1$, and

$$e_i := \gcd(e_{i-1}, \beta_i), \qquad n_i = \frac{e_{i-1}}{e_i} \text{ for } 1 \leq i \leq g.$$

Now we define a semiroot of a plane curve germ.

Definition 32. Let $\gamma(t) = (x(t), y(t))$ be an irreducible plane curve germ with Puiseux Characteristic $(\beta_0; \beta_1, \beta_2, \ldots, \beta_g)$, and semigroup $\Gamma = \langle \nu_0, \nu_1, \ldots, \nu_g \rangle$. Then for $0 \leq k \leq g$ a *k-semiroot* of γ is a plane curve germ $\gamma_k(t)$ given by the equation $f_k(X, Y) = 0$, such that $f_k \in \mathbb{C}\{X\}[Y]$ is irreducible, and monic of degree p/e_k with $v(f_k(x(t), y(t))) = \nu_{k+1}$, the $(k+1)$th generator of Γ (here we set $v_{g+1} := \infty$).

We find that given these conditions, one immediately has that γ_k has Puiseux Characteristic given by $(\beta_0; \beta_1, \beta_2, \ldots, \beta_k)/e_k$ (see Section 2 of [1]). We also note that if we find an irreducible curve germ μ such that

$$j^{\beta_{k+1}-1}(\mu(t^{e_k})) = j^{\beta_{k+1}-1}(\gamma), \tag{3.6}$$

then μ is a k-semitroot of γ (see Corollary 5.3 in [22]). Furthermore we can easily construct a k-semiroot μ such that equation (3.6) holds. This is done by truncation and reduction of terms below β_{k+1}, which will always give a semiroot of the curve, and obviously has the property above.

To formalize notation, denote Γ_k and Λ_k as the semigroup and set of valuations of one-forms on the curve germ γ_k, respectively. Also denote $v_k := v_{\gamma_k}$, the valuation with respect to the pullback of $\gamma_k(t)$ into $\mathbb{C}\{t\}$.

We have a definition for a special collection of all levels of semiroot.

Definition 33. A set $\{\gamma_k : \gamma_k \text{ is a k semiroot for } \gamma,\ 0 \leq k \leq g\}$ is called a *complete system of semiroots of γ.*

Though at first definition 32 may appear mysterious, it is rather clear once an example is provided. In general a semiroot of a curve is not difficult to find if one is given a normal or short form of a curve. Let us illustrate this with a simple example:

Example 14. Let us consider the curve

$$\gamma\colon x(t) = t^4, \quad y(t) = t^6 + t^7.$$

This curve is in the topological class $(4; 6, 7)$, which has corresponding $\Gamma = \langle 4, 6, 13 \rangle$. A 1-semiroot of this curve is given by truncating the parameterization for $y(t)$ before the t^7 term, and reducing all common powers of t in x and y. That is we first truncate to get (t^4, t^6), then reduce to get $\gamma_1(t) = (t^2, t^3)$. This curve is well known to be irreducible.

Furthermore we see that γ_1 is defined by the equation $Y^2 - X^3 = 0$. A quick computation shows that

$$(y(t))^2 - (x(t))^3 = 2t^{13} + t^{14},$$

which clearly has valuation 13.

Now that we have illustrated what a semiroot is, we now would like to see how they relate to points in the Monster tower. Suppose that $u \in M(n)$ lies on a locus (αR^q), $q \geq 1$, with α a critical code word consisting of more than one isolated critical block. Assume also that $(\beta_0; \beta_1, \beta_2, \ldots, \beta_g)$ is the Puiseux Characteristic of u. Now consider the projection $\pi_k(u)$ down to some level $M(n-k)$. We consider if there are any relations between the curves in $\mathrm{Pl}(u)$ and $\mathrm{Pl}(\pi_k(u))$. We claim that the curves in $\mathrm{Pl}(\pi_k(u))$ will contain semiroots of the curves in $\mathrm{Pl}(u)$ and which semiroot will depend on the k and the RVT code word α. The assertion is obvious if $k < q$ using the g-semiroot of any curve in $\mathrm{Pl}(u)$, that is the curve germ itself.

Let us rewrite

$$\alpha = R^{q_1} C_1 R^{q_2} C_2 \ldots R^{q_g} C_g R^q,$$

where C_i are entirely critical words, and the $q_i \geq 1$ for all $i \in \{1, \ldots, g\}$. Let us also denote $|C_i|$ for the length of C_i. Suppose now that

$$q + |C_g| < k < q + q_g + |C_g|.$$

Then $\pi_k(u) \in (R^{q_1} C_1 R^{q_2} C_2 \ldots R^{q_{g-1}} C_{g-1} R^{q'})$ for some $0 < q' < q_g$. We claim that for any $\gamma \in \mathrm{Pl}(u)$ there exists a $g-1$-semiroot γ_{g-1} of γ, such that $\gamma_{g-1} \in \mathrm{Pl}(\pi_k(u))$. Note that if $\gamma \in \mathrm{Pl}(u)$ then γ has Puiseux Characteristic $(\beta_0; \beta_1, \beta_2, \ldots, \beta_g)$. Therefore any $g-1$-semiroot of γ must have Puiseux Characteristic $(\beta_0; \beta_1, \beta_2, \ldots, \beta_{g-1})/e_{g-1}$. Furthermore we can construct a γ_{g-1} such that $j^{\beta_g - 1}(\gamma_{g-1}(t^{e_{g-1}})) = j^{\beta_g - 1}(\gamma)$. This implies $\gamma_{g-1}^{n-k}(0) = \pi_k(u)$. Since γ_{g-1} has Puiseux Characteristic given above, we also have that γ_{g-1}^{n-k} is regularized for all k in the range above, and so we have that $\gamma_{g-1} \in \mathrm{Pl}(\pi_k(u))$. All together this gives us the following proposition.

Proposition 17. *Let $u \in M(n)$ be a regular point with length g Puiseux Characteristic. Take any γ in Pl(u). Then for any $k \geq 0$ such that $\pi_k(u)$ is regular, there exists a $0 \leq j \leq g$ and a j-semiroot γ_j such that $\gamma_j \in$ Pl($\pi_k(u)$). That is Pl($\pi_k(u)$) contains a semiroot of γ.*

With this proposition, we have connected projections of regular points to regular points with semiroots of curves in their plane set. With this idea in mind we now introduce a result of Abreu and Hernandes that will allow us to say more about Λ levels in the Monster for certain words.

3.5.2 A Result of Abreu and Hernandes

To sufficiently summarize their results, we need two functions defined on the value set Λ_k by Abreu and Hernandes in [1]. A similar idea can be found in Section 4 of [11].

Definition 34. Define the function

$$\begin{aligned}\Theta_k \colon \Lambda_k &\to \Gamma_k \cup \{\infty\}\\ \delta_k &\mapsto \max\{\nu_k(B) : \delta_k = \nu_k(Adx - Bdy)\}\end{aligned}$$

Further define

$$\begin{aligned}\rho_k \colon \Lambda_k \setminus \Gamma_k &\to \Lambda \setminus \Gamma\\ \delta_k &\mapsto \begin{cases} e_k\delta_k & \text{if } e_k(\delta_k - \Theta_k(\delta_k)) < \beta_{k+1}\\ \beta_{k+1} + e_k\Theta_k(\delta_k) & \text{if } e_k(\delta_k - \Theta_k(\delta_k)) > \beta_{k+1}\end{cases}\end{aligned}$$

We now present one of the main theorems of Abreu and Hernandes, which will allow us to make statements about the Λ level of certain points in the Monster.

Theorem 13. (Theorem 5.2 in [1]) *For any branch C_f in the topological class determined by $\Gamma_f = \langle \nu_0, \ldots, \nu_g\rangle$ with $n_g = 2$ we have that*

$$\Lambda_f \setminus \Gamma_f = \rho_{g-1}(\Lambda_{g-1} \setminus \Gamma_{g-1}) \dot{\cup} \{\nu_g - 2\delta : \delta \in \mathbb{N}^* \setminus \Lambda_{g-1}\} \dot{\cup} \{\nu_g + 2\delta : \delta \in \mathbb{N} \setminus \Gamma_{g-1}\}, \quad (3.7)$$

and $\sharp(\Lambda_f \setminus \Gamma_f) = \mu_{g-1}$. In particular, we have that $\tau_f = \mu_f - \mu_{g-1}$ where μ_f and τ_f are the Milnor and Tjurina of C_f respectively.

In summary, this theorem allows us to relate value sets of semiroots of curves to the value set of the the original curve if $n_g = 2$. Notice that the equality sign in (3.7) allows us to precisely know the value set of the original curve given the value set of the $g-1$-semiroot. We will now apply this knowledge to characterize the Λ jet identification number for a curve germ, and hence a Λ level in the Monster for a regular point with the special property that the corresponding Puiseux Characteristic of the point has $n_g = 2$.

3.5.3 The Λ Level of Points With Critical Word $\alpha R^q V$, $q > 0$.

We would now like to determine the Λ level of the code word $\alpha R^q V$ with $q > 0$ and no restrictions on α. If we are given an single point $u \in (\alpha R^q V R^s)$, $s > 0$ we would also like to determine the Λ level of u as well, and even further show that there is a maximal projection of u down to a certain level that will still provide enough information for us to determine the value set of u itself. We start with the Λ level of the code word $\alpha R^q V$.

Proposition 18. *Let α be any RVT code word (including the empty word). Then the code word $\alpha R^q V$ with $q > 0$ has Λ level in the Monster $L = |\alpha R^q V| + 1$, that is the length of the word plus 1.*

Proof. We are tasked with proving that $q_0 = 1$ in the definition of Λ level for a critical word. That is, we must show that for any $u \in (\alpha R^q V R)$ all curve germs in $\mathrm{Pl}(u)$ share the same value set. We note that any curve in $\mathrm{Pl}(u)$ has critical word $\alpha R^q V$ and so by Theorem 13, any curve germ in $\mathrm{Pl}(u)$ has its value set completely determined by any $g-1$-semiroot.

It then suffices to show that any two curves in $\mathrm{Pl}(u)$ share a $g-1$-semiroot, and therefore must share the same value set. Since u is a regular point, it has a jet identification number. This number is rather easy to calculate, as it is simply the last entry in the Puiseux Characteristic, say β_g. Now let us take any two curves $\gamma, \tilde{\gamma} \in \mathrm{Pl}(u)$. We claim that since $j^{\beta_g}(\gamma) = j^{\beta_g}(\tilde{\gamma})$, they must share a common $g-1$-semiroot. Indeed for any $g-1$-semiroot γ_{g-1} of γ we have

$$j^{\beta_g - 1}(\gamma_{g-1}(t^{e_{g-1}})) = j^{\beta_g - 1}(\gamma(t)) = j^{\beta_g - 1}(\tilde{\gamma}(t)).$$

This implies γ_{g-1} is a $g-1$-semiroot of $\tilde{\gamma}$ (see equation 3.6). Hence all curves in $\mathrm{Pl}(u)$ must share a common $g-1$-semiroot, which completely determines the value set of each curve. $\square$

The proposition implies that any point $u \in (\alpha R^q V R^s)$ for any $q, s > 0$ has Λ level at most $|\alpha R^q V R|$. From Proposition 17, we have that $\gamma_{g-1} \in \mathrm{Pl}(\pi_k(u))$ for $s+1 < k < q+s+1$, whenever $\gamma \in \mathrm{Pl}(u)$. If $\mathrm{Pl}(\pi_k(u))$ consists of curve germs all sharing the same value set, then we can determine the value set associated to u by knowing the value set associated to $\pi_k(u)$. What is remarkable here is the bound on k, as the statement is obvious for $0 < k < s$.

Suppose now that α is critical. One can see immediately that if q is large enough, then $|\alpha R^q|$ may exceed the Λ level in the Monster of the critical word α. If this is the case, then any point at the Λ level of α has an associated value set. This includes any point that is a projection of a point on the locus $(\alpha R^q V R^s)$. Therefore, if q is large enough for $|\alpha R^q|$ to exceed the Λ level in the Monster for α, then any $u \in (\alpha R^q V R^s)$ has that if $\pi_k(u)$ is its projection down to the Λ level of α, then $\mathrm{Pl}(\pi_k(u))$ has only one associated value set. It must be that some $g-1$-semiroot of u is contained in $\mathrm{Pl}(\pi_k(u))$. These last two paragraphs are summarized by the following theorem:

Theorem 14. *Suppose $u \in (\alpha R^q V R^s)$ with $q, s > 0$, and α a critical RVT code word. Further suppose that $|\alpha R^q|$ exceeds the Λ level of the word α. Then the value set associated to u is completely determined by the value set of the projection of u down to the Λ level of α.*

This ends our section on longer words. In the next chapter we will develop some results about plane curve germs RVT code words with more than one critical block, in hopes to enrich our knowledge about points with the same RVT code word. For now we turn to studying the locus of points in the Monster Tower corresponding to a given RVT code word, and use the previous sections to develop some structure of this locus.

3.6 Hypersurfaces of RVT Loci in the Monster Tower

With the results developed so far in this chapter we can begin to discuss the geometric structure of points of a given RVT code word locus $(\alpha) \subset M(n)$. Let us start with a result that will help guide us to a more robust understanding of the points lying in (α)

Proposition 19. *Let W be a critical code word with a single critical block corresponding to $\Gamma = \langle p, m\rangle$, with $p > 2$. Also suppose that $\Gamma \neq \langle 3, 4\rangle$ or $\langle 3, 5\rangle$. If $m \not\equiv -1$ (mod p) and $m \neq p+1$, there is a hypersurface $\Sigma \subset (WRR)$ defined by the condition that a plane curve germ $\gamma \in Pl(\Sigma)$, if and only if $p + m + 1 \notin \Lambda$, the value set of γ. If $m \equiv -1$ (mod p) or $m = p + 1$, then there is a hypersurface of points $\Sigma \subset (WRRR)$ defined by the condition that $\gamma \in Pl(\Sigma)$ if and only if $p + m + 2 \notin \Lambda$.*

Proof. Suppose that we are in the first case, where $m \not\equiv -1$ (mod p) and $m \neq p + 1$. Then we have that $m + p + 1 \notin \Gamma$. Now take any $u \in (WRR)$ and consider that the analytic jet identification number of the point u is given by $m + 1$ (see Theorem 10). Therefore every $\gamma \in \mathrm{Pl}(u)$ has the same $m + 1$ jet up to reparameterization.

From 7, any two points in (WRR) are a-equivalent if and only if they have analytically equivalent $m+1$-jets. Since $\gamma \in \mathrm{Pl}(u)$, we must have that γ has code word W, and therefore is analytically equivalent to a curve germ of the form $\tilde{\gamma} = (t^p, t^m + at^{m+1} + h.o.t.)$. Therefore there is some $\tilde{u} \in (WRR)$ such that u is a-equivalent to $\tilde{u}$ and $\tilde{\gamma} \in \mathrm{Pl}(\tilde{u})$. Note further that if $\tilde{\gamma} \in \mathrm{Pl}(\tilde{u})$, then so must $(t^p, t^m + at^{m+1})$ as these two curve germs clearly have the same $m + 1$-jet.

It follows that any point $u \in (WRR)$ has a curve germ $\gamma \in \mathrm{Pl}(u)$ that is a-equivalent to the curve $\tilde{\gamma} = (t^p, t^m + at^{m+1})$. A quick check shows that if $a \neq 0$, then $m + p + 1 \in \Lambda$, and if $a = 0$ then $m + p + 1 \notin \Lambda$. It is also clear that for a general nonzero a, the curve $(t^p, t^m + at^{m+1})$ specializes to the quasi-homogeneous curve (t^p, t^m), i.e. the case where $a = 0$.

Finally we have that $(t^p, t^m + at^{m+1})$, with $a \neq 0$, is a-equivalent to the curve $(t^p, t^m + t^{m+1})$ by the normal form theorem of Hefez and Hernandes in [13]. Thus all points $u \in (WRR)$ that have a $\gamma \in \mathrm{Pl}$ with $m+p+1 \in \Lambda$ must form an open and dense set, with closure all of (WRR). We also have that $a = 0$ is a closed condition. Therefore $\gamma \in \mathrm{Pl}(\Sigma)$ if and only if $a = 0$ for some analytically equivalent plane curve germ, if and only if $p+m+1 \notin \Lambda$.

If on the other hand we have that $m \equiv -1$ (mod p) or $m = p+1$, then $m+p+1 \in \Gamma$ and so we must go one step higher to $(WRRR)$. At this point the proof is nearly identical to that given above, but instead we work with $(t^p, t^m + at^{m+2})$. This completes our proof. □

We can continue inductively up the Monster Tower and consider the word WR^q for $q > 2$ or 3 depending on the congruency of m modulo p. Note that if $u \in (WR^q)$ for $q > 2$, such that $\pi_{q-2}(u) \in \Sigma$, then u itself sits on a hypersurface of (WR^q) defined by $m+p+1 \notin \Lambda$ for any $\gamma \in \mathrm{Pl}(u)$. This is due to the fact that we always have $\mathrm{Pl}(u) \subseteq \mathrm{Pl}(\pi(u))$ if $q > 2$. Let us call this set of points Σ^{q-2}. Similarly for the case where we need to consider $q > 3$.

In general it becomes increasingly difficult to state a clean proposition about the structure as we go up higher in the Tower. One can imagine that along these hypersurfaces, we find other "hypersurfaces", or codimension 1 sets of points that are again determined by whether or not the next possible minimum value of $\Lambda \setminus \Gamma^*$ is in fact in Λ or not. Let us illustrate this with an example.

Example 15. Let us consider the case where $\Gamma = \langle 4, 9\rangle$. This corresponds to the word $W = RRVTT$. We then have that $\Sigma \subset (RRVTTRR)$ is characterized by $\gamma \in \mathrm{Pl}(\Sigma)$ if and only if $14 \notin \Lambda$. Now suppose we have that $u \in (RRVTTRRR)$ such that $\pi(u) \in \Sigma$. We therefore have that for any $\gamma \in \mathrm{Pl}(u)$, $m + p + 1 \notin \Gamma$.

From here we ask if $p + m + 2 = 15 \in \Lambda$. If so then γ must be a-equivalent to $(t^4, t^9 + at^11 + h.o.t.)$. A similar argument to the above proposition holds and we can see there is an open and dense set $S \subset \Sigma^1$ defined by $\gamma \in \mathrm{Pl}(S)$ if and only if $15 \in \Lambda$. It follows that the complement of this set, i.e. the set defined by $14, 15 \notin \Lambda$ must have codimension 1.

We can also consider the case where we in fact have that $p + m + 1 \in \Lambda$ for our set $\mathrm{Pl}(u)$. If this is the case, we may encounter more structure above depending on the code word W. Let us illustrate this with another example

Example 16. Again take the word $W = RRVTT$ and consider a point $u \in (RRVTTRRR)$ such that $\pi(u) \notin \Sigma$. Thus $14 \in \Lambda$ and we have that any $\gamma \in \mathrm{Pl}(u)$ has jet a-equivalent to a jet of the form $(t^4, t^9 + t^{10} + at^{11})$. From the table of Hefez and Hernandez in [14] we have that if $a = 19/18$, then $19 \notin \Lambda$, and otherwise it is. It is clear that the other curve germs with $a \neq 19/18$ then specialize to $(t^4, t^9 + t^{10} + \frac{19}{18}t^{11})$.

By similar argument, we must have a closed set of points that are defined by the condition that u is in this set if and only if all $19 \notin \Lambda$ for all $\gamma \in \mathrm{Pl}(u)$

In general one would need to algorithmically find these different sets of points, and they are very dependent on the p and m that we are given. One can also say something similar about hypersurfaces and the structure of points in the case where Γ is not coprime. This requires the use of semiroots and a bit more work, of which some small steps have been taken near the end of the following chapter.

Overall we find that in the case where there are two critical blocks, and $n_2 > 2$ (see Definition 31), there is a similar hypersurface of points on (WRR). The defining quality of this hypersurface is a bit more involved to state, and depends on the essential exponents of the curve. We conjecture that this is the case in general for an arbitrary number of critical blocks with $n_g > 2$.

In the following section we give our main result of the paper, which involves the generic set of points on the locus (WR^{p-1+q}) for W critical, with one critical block, and $q \geq 0$.

3.7 The Generic Points of an RVT Locus

This section is devoted to stating the main result of this work. We would like to give a complete characterization of the generic points of the locus of a given critical code word extended by a sufficient number of R's. We recall Theorem 12, which allows us to consider the generic points of WR^{p-1+q} where W is critical, with one critical block, and $q \geq 0$. We now state our main theorem.

Theorem 15. *Let W be a critical RVT code word with one critical block and corresponding semigroup $\Gamma = \langle p, m \rangle$. Let $u \in (WR^{p-1+q})$ where $q \geq 0$. Then u is a generic point of (WR^{p-1+q}) if and only if every plane curve germ $\gamma \in Pl(\pi_q(u))$ has generic value set Λ_{gen} given by the recursive formula in Theorem 18.*

Proof. This theorem is a more precise restatement of Theorem 12, and a result of Theorem 18, which we prove in the next chapter. Indeed we have that since $\pi_q(u) \in (WR^{p-1})$, then Theorem 12 applies. In that case we only need to find one $\gamma \in \mathrm{Pl}(\pi_q(u))$ with generic value set that is given by the recursive formula in 18. □

To see examples of how one computes recursively the generic value set of a generic point of the Locus (WR^{p-1+q}), see the following chapter, particularly Section 4.1.4.

One can ask if there is a precise minimum bound on the level in the Monster Tower we need to go to for the result of Theorem 12 to hold. Delorme in [10] gives us an upper bound of $p-1$ R's after our last critical letter in W. We speculate that it is in fact less than this, and that there is a precise number of R's that one needs to add to W for the results of Theorem 12 to hold.

This exact number is sum of the number of "jumps" one must take in the recursive formula in Theorem 18, that is the difference between the u_i and the g_i in the theorem. We will leave this result for later work.

This concludes our section and our chapter on value sets of points in the Monster Tower. We look forward to our next chapter, where we give a procedure for obtaining all value sets associated to any coprime Γ. We will also, of course, recursively compute the minimal generators of the generic value set of and given corpime Γ. Lastly we will explore some results for value sets of plane curve germs that have a Γ that is not coprime.

Chapter 4

Value Sets of Analytic Curve Germs

In this chapter we wish first to provide rules for what we will call a coordinated Mancala game. Every outcome of this game will correspond to a value set of an analytic plane curve germ with a two generator semigroup. We will then give a recursive formula for the minimal generators of the value set that corresponds to the minimal Mancala game. We will show through Delorme's algorithm in [10] that this minimal coordinated game corresponds to the generic value set of a topological class of plane curve germs with a 2 generator semigroup.

From there we will begin to explore value sets of plane curve germs with semigroups that have 3 or more minimal generators. In this part of the section we will partially describe the equi-singular classes of plane branches that have only one value set associated to them using what is known as semiroots of the curve (see [1]). Finally, we will attempt to formulate some sort of converse to the latter, where we show that certain equi-singular classes must have more than one values set associated to them.

We start by defining how to play the coordinated Manacala game for what we call the coprime case, and prove that every game leads to a value set of a curve that has a two generator semigroup.

4.1 Coordinated Mancala for the Two Generator Semigroup

In the classic Mancala game, played in many countries around the world, there is a rectangular board with small bins carved into it. These bins usually appear in two columns and multiple rows. In each of these bins one places a number of beads or stones, and the game goes on by players picking up beads in bins and distributing them throughout the other bins. There are, of course, rules attached as to how one may pick up the beads and distribute them.

In order to compute a value set Λ of a singular plane curve germ with semigroup Γ, we will give a set of rules for what we call a coordinated Mancala game. This game can be played for a general equi-singularity class represented by a Γ with any number of minimal generators. However, if we take a Γ generated by more than 2 elements, we find that not every Mancala game leads to a value set of a plane curve germ. On the other hand if we restrict ourselves to $\Gamma = \langle p, m\rangle$, then we can prove, thanks to Delorme in [11] and Almirón in [3], that a Γ-semimodule belongs to some plane curve germ if and only if there is a coordinated Mancala game that produces this semimodule.

The author would like to acknowledge Gary Kennedy in this work, who has been instrumental in helping develop the rules of the coordinated Mancala game. This work could not have been completed without his input and creativity.

4.1.1 The Rules of the Game:

We now present the game for the two generator semigroup $\Gamma =< p, m >$.

Each turn of the game begins with a $\mu \times k$ matrix whose entries are 0 and 1. The matrix represents the present state of the game. 1s correspond to filled bins and 0s to empty bins in Mancala. The turn either produces a $\mu \times (k+1)$ matrix of this same type or calls the game over.

The rows of the matrices are labelled by the integers $i = 0, 1, \ldots, \mu - 1$. The columns are to be identified with subsets of the integers $\{0, 1, \ldots, \mu - 1\}$. The column C_i can be thought of as a map $\{0, 1, \ldots, \mu - 1\} \to \{0, 1\}$ and as such it represents the characteristic function of its corresponding subset of $\{0, 1, \ldots, \mu - 1\}$.

Using the identification of columns C_j with subsets, write

$$\min(C_j) = \lambda_j.$$

Thus λ_j is the smallest filled bin of column C_j. When the game is over, the λ_j will be the generators of the desired semi-module.

Setting up the game board: We start off with two columns C_{-1} and C_0 having associated minima $\lambda_{-1} = p, \lambda_0 = m$. C_{-1} corresponds to the subset $(p+\Gamma) \cap [0, \mu-1]$ and C_0 to the subset $(m+\Gamma) \cap [0, \mu-1]$. The notation $[a, b]$ means the closed interval $a \leq t \leq b$ so this act of intersecting a set of integers with $[0, \mu - 1]$ is just a way of truncating it by deleting from it all integers bigger than the conductor μ. So the initial matrix for our game board is a $\mu \times 2$ matrix of 1s and 0s. These two columns are placed next to each other, rows aligned, to form the initial game board. See Example 17 and figure 4.1

Taking a Turn: After the *ith* turn we have $i + 2$ columns $C_{-1}, C_0, \ldots, C_{i-1}$. We continue to use the symbol C_j for its associated subset of $\{0, \ldots, \mu - 1\}$, the subset whose characteristic function is the column vector C_j. A single turn involves three steps. The first of these steps concerns finding collisions. By a "collision" of two columns we mean an element lying in the intersection of their corresponding subsets, so a bin having two beans in it.

1. Find the *minimal collision*, u_{i-1}, amongst the columns:

$$u_{i-1} := \min(\bigcup_{j \neq k}(C_j \cap C_k)).$$

   This integer u_{i-1} labels the smallest row having two beans in its bin. See figure 4.1 where this is row 13, i.e $u_0 = 13$, for the first move. We will see later on that the minimal collision occurs with the last column C_{i-1} so that $u_{i-1} = \min(C_j \cap C_{i-1})$ for some $j < i-1$. We assume this fact for now.

2. Remove the bead in the bin u_{i-1} from column C_{i-1}, along with all beads in those of its bins labelled by integers in $\Gamma + u_{i-1}$. In other words, replace C_{i-1} by $C_{i-1} \setminus (u_{i-1} + \Gamma)$. This will form our new column relabeled as C_{i-1}.

3. Choose any value $\lambda_i > u_{i-1}$ which is not in any of the columns. That is, select some integer $[u_{i-1}, \mu - 1] \setminus (\cup_{-1 \leq j \leq i-1} C_j)$. Form the new column:

$$C_i = (\Gamma + \lambda_i) \cap [0, \mu - 1].$$

   There is an alternate choice here: set $\lambda_i = \infty$. This signifies that the game is over. We stop the game with the columns $C_{-1}, C_0, \ldots, C_{i-1}$, this C_{i-1} being our just-modified last column from the previous step.

Game Over. It could be that there are no collisions amongst the $i + 1$ columns we have at the beginning of our turn. In this case we declare the game over and stop with the columns we started with. It could also be that there are no λ_i available left to choose besides $\lambda_i = \infty$. (In other words, we might have that $[u_{i-1} + 1, \mu - 1] \cap (\cup_{-1 \leq j \leq i-1} C_j)$ is empty.) In that case we have to choose $\lambda_i = \infty$ and declare the game over as per the alternate choice is step 3 above.

This completes the description of the game and its rules.

The game stops in at most $min(p - 2, \mu - 1 - (p + m))$ moves, since the λ_i are strictly increasing integers in $[p + m, \mu - 1]$ and they must represent distinct classes modulo p.

Move 1 is always the same.

Move 1: The first collision occurs at $u_0 = m + p$. Note that $m + p \in C_{-1} \cap C_0$. We leave the reader to verify that there are no collisions between C_0 and C_{-1} smaller than $m+p$.

We now choose any $\lambda_1 > u_0$ such that $\lambda_1 \notin \Gamma^*$ since $C_{-1} \cup C_0 = \Gamma^* \cap [0, \mu-1]$. Alternatively, one may call (i.e. end) the game here by declaring that $\lambda_1 = \infty$. This completes the first move.

Let us now proceed with an example to illustrate how this coordinated Mancala game works.

Example 17. Take $\Gamma = \langle 4, 9\rangle$. We recall that $\mu = 24$ in this case, and so we have 2 columns with 23 bins each for move 0. In column C_{-1} we have a bead in each bin with value in $(\Gamma+4) \cap [1, \dots, F]$, and in C_0 we have $(\Gamma+9) \cap [1, \dots, F]$ (see figure 4.1). This implies our $u_0 = 13$. We then choose a λ_1. We have that the set of gaps in Γ^* that are greater than 13 are given by $\{14, 15, 19, 23\}$. Let us play this game by choosing $\lambda_1 := 14$. We now have C_{-1} is unchanged $C_0 = (\Gamma+9) \setminus (\Gamma+13) \cap [1, \dots, F]$ and $C_1 := (\Gamma+14) \cap [1, \dots, F]$.

We have now completed move 1 and are ready for move 2. We find that

$$u_1 := \min(C_0 \cap C_1) = 18.$$

The only row of bins that does not have a bead in it above 18 in our three columns is the row of bins labeled 19. Let us choose 19 as our λ_2. This will complete the game, as there are now no gaps above 18. We then have $C_1 = (\Gamma+14) \setminus (\Gamma+18) \cap [1, \dots, F]$ and $C_2 = (\Gamma+19) \cap [1, \dots, F]$. This concludes our coordinated Mancala game for this example. Note that we could have also chosen to throw the bead in bin 19 in our right most column to ∞. Note further that any choice of λ_1 other than 14 would have immediately ended the game, since if we choose 15 instead, then 19 and 23 are automatically in C_2 by the coordinated move. Below is a figure that shows the columns generated after each move.

4.1.2 Coordinated Mancala in Relation to Plane Curve Germs:

Now that we have described how to play the game, we would like to relate it to a plane curve germ $(x(t), y(t))$ that has a two generator semigroup $\Gamma = \langle p, m\rangle$. As before we define $v(dx) := v(x) = p$ and $v(dy) := v(y) = m$. Then we note that columns C_{-1} and C_0 are precisely the valuations of monomial one-forms on the curve given by $v(x^i y^j dx)$ and $v(x^k y^l dy)$ respectively.

The nth move produces a new rightmost column C_n with the $\lambda_n = \min(C_n)$. If we can show there is a one-form ω_n on the plane curve germ such that $v(\omega_n) = \lambda_n$, then our new column C_n is precisely the set of valuations given by $v(x^i y^j \omega_n)$ up to the conductor of Γ. Note that only needing monomials is specific to the fact that $\Gamma = \langle p, m\rangle$ so that every integer in Γ is given by $v(x^i y^j)$ for some $i, j \geq 0$.

The question now becomes, for any choice of λ_n, is there a plane curve germ that has on it the one-form ω_n? The answer is in fact, yes! To show this is the case, we will need to define what an increasing Γ-semimodule is. This will require a bit of notation that was first presented by Delorme in [10] and later in [11].

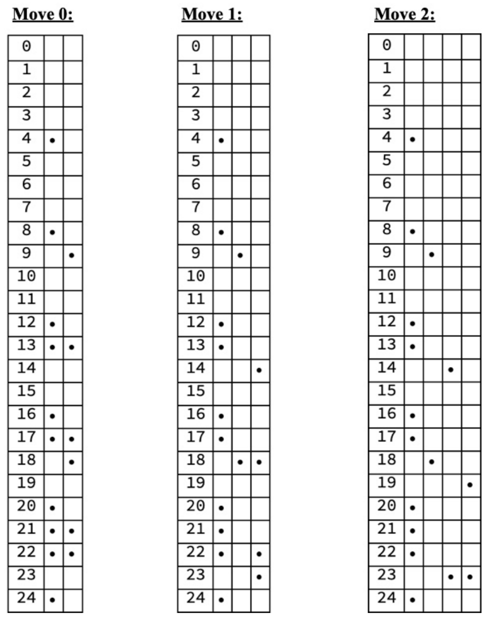

Figure 4.1: The columns generated by the game played in Example 17.

Definition 35. Let $\lambda_{-1} < \lambda_0 < \lambda_1 < \cdots < \lambda_N$ be the minimal generators of a Γ-semimodule Λ. Define the set

$$E_i := \bigcup_{-1 \le j \le i} (\Gamma + \lambda_j) \quad \text{for } -1 \le i \le N.$$

We also define

$$u_i := \min\left((\Gamma + \lambda_i) \cap E_{i-1}\right) \quad \text{for } 0 \le i \le N.$$

We have in general that $\Lambda = E_n$ and $\lambda_i \notin E_{i-1}$, for $0 \le i \le N$. In the case of the Mancala game we are interested in setting $\lambda_{-1} = p,\ \lambda_0 = m$, and this gives $u_0 = m + p$. We will eventually show that these u_i are precisely what we called the minimum collisions in the game. We define what it means to be an increasing Γ-semimodule, which will help us attach a curve to a coordinated Mancala game.

Definition 36. Let $\lambda_{-1} < \lambda_0 < \lambda_1 < \cdots < \lambda_N$ be the minimal generators of a Γ-semimodule Λ, and the E_i and u_i be given as above. Then we call Λ an *increasing* Γ-semimodule if $\lambda_{i-1} < u_{i-1} < \lambda_i$ for all $1 \le i \le N$.

As it turns out, for every increasing Γ-semimodule Λ for the two generator case for which $\lambda_{-1} = p$ and $\lambda_0 = m$ there exists a corresponding plane curve germ with value set given precisely by Λ. This result can be found as the main result in [3] and is summarized as follows.

Theorem 16. (P. Almirón [3]) *Let $\Gamma = \langle p, m \rangle$, and Λ a Γ-semimodule minimally generated by $p < m < \lambda_1 < \cdots < \lambda_N$. Then there exists a plane curve germ with semigroup Γ and value set Λ if and only Λ is an increasing Γ-semimodule.* □

There is a notion of a Mancala game that is not coordinated. If we play this game we often find that no plane curve germ can achieve the value set of the resulting game. This is due to the fact that we do not achieve an increasing semimodule. We therefore do not go any further in to detail about a general Mancala game in this work that is not coordinated.

On the other hand given a semigroup $\Gamma = \langle p, m \rangle$ and taking Theorem 16 into account, if we show that playing every coordinated Mancala game in the coprime case gives us every possible increasing Γ-semimodule such that $\lambda_{-1} = p$, and $\lambda_0 = m$, then we have effectively shown that every value set Λ associated to an equi-singularity class given by Γ is the result of some coordinated Mancala game. To do this, we will need a couple of lemmas provided to us by Delorme in [11].

Lemma 3. (Delorme, Lemma 10 in [11]) *Let r and q be two elements of $\mathbb{Z}$ such that $|r - q| \notin \Gamma$. Set*

$$u = \inf\{(\Gamma + r) \cap (\Gamma + q)\} \quad \textit{and} \quad \bar{u} = u + c - mp.$$

Furthermore set $v = r + q + mp - u$ and $\bar{v} = v + c - mp$. Then the following relations hold:

1. $(\Gamma + r) \cap (\Gamma + q) = (\Gamma + u) \cup (\Gamma + v)$,
2. $(\Gamma + r) \cup (\Gamma + q) = (\Gamma + u - mp) \cap (\Gamma + v - mp)$,
3. $\mathbb{N} + \bar{v} \subset (\Gamma + r) \cup (\Gamma + q)$,
4. $(\mathbb{N} + \bar{u}) \cap ((\Gamma + r) \cup (\Gamma + q)) = (\Gamma + v - mp) \cap (\Gamma + u)$.

We can calculate u and v from r and q as follows: if $|r - q| \notin \Gamma$ then there exists one, and only one, pair $(\alpha, \beta) \in]0, p[\times]0, m[$ such that $q - r = \alpha m - \beta p$. Then u and v are, up to order, the two numbers $r + \alpha m$ and $q + (p - \alpha)m$. □

The other lemma of Delorme that we now state and later make use of will give us the ability to greatly simplify our arguments.

Lemma 4. (Delorme, Lemma 12 in [11]) *For any integer $i \in \{0, \ldots, N - 1\}$, with $\bar{u}_i = u_i + c - mp$, there exists a number $c_i \in \mathbb{Z}$ such that*

$$(\mathbb{N} + \bar{u}_i) \cap E_i = (\mathbb{N} + \bar{u}_i) \cap (\Gamma + c_i).$$

□

Delorme provides a recursive formula for c_i. Define $c_0 = 0$. Then

$$c_{i+1} = c_i + \lambda_{i+1} - u_{i+1}. \tag{4.1}$$

We are now in a position to prove that every increasing Γ-semimodule is the result of a coordinated Mancala game, and therefore any value set of any plane curve germ with semigroup Γ is the result of a coordinated Mancala game. We state and prove the theorem.

Theorem 17. *Let $\Gamma = \langle p, m \rangle$ be a semigroup and Λ a Γ-semimodule minimally generated by $p < m < \lambda_1 < \cdots < \lambda_N$. Then Λ is the value set of some plane curve germ in the equi-singularity class corresponding to Γ if and only if Λ is the result of some coordinated Mancala game.*

Proof. Let us form a recursive proof based on the move of the game we are on. For any set of nonnegative integers S, let us denote $S_k := S \cap [0, \ldots, k]$, It is clear in move 0 that we have $C_{-1} \cup C_0 = (E_0)_F = ((\Gamma + p) \cup (\Gamma + m))_F = \Gamma^*_F$.

Move 1: For move 1 we have $m < u_0 < \lambda_1$, and the minimum collision $u_0 = p + m$, as we have already noted before. Furthermore we must show that our minimum collision for move 1 is indeed the u_1 given in Definition 35, so that $\lambda_1 < u_1$.

By rules of the game, in move 1 we are required to form the column C_1 by choosing a $\lambda_1 > u_0$, or send $\lambda_1 \to \infty$. If $\lambda_1 < \infty$, we claim that in general that once we have completed step 2 for move 1 (see Step 2 in Section 4.1.1), we will have $C_{-1} \cap C_0 = \emptyset$. To see why this is the case, consider that by Lemma 3 with $r = p$, $q = m$ we have the equality

$$(\Gamma + p) \cap (\Gamma + m) = (\Gamma + m + p) \cup (\Gamma + mp) \tag{4.2}$$

After step 2 for move 1, C_0 has beads removed from all bins with values in $\Gamma + m + p$. Furthermore $mp > \mu > F$, so our claim that $C_{-1} \cap C_0 = \emptyset$ holds. Note all these elements that we removed from C_0 remain present in C_{-1} by Equation 4.2 above.

We are therefore looking for a minimum collision between C_1 and C_i where $i \in \{-1, 0\}$. By definition and by 4.2 we have $C_1 = (\Gamma + \lambda_1)_F$, and therefore we are looking for $\min(C_1 \cap C_i) = \min((\Gamma + \lambda_1) \cap \Gamma^*) = u_1$, as in Definition 35. Since $\lambda_1 \notin C_i$, and $\min(\Gamma + \lambda_1) = \lambda_1$ we must have that $\lambda_1 < u_1$ as desired. Finally we note that $C_{-1} \cup C_0 \cup C_1 = (E_1)_F$. We are now in position to make our general recursive move.

Move i, $1 < i \leq N$: Suppose that we have made $i - 1$ moves in our Mancala game, and we are on move i. Further suppose as in move 1 that at each move j we have formed our new rightmost column by removing beads from the previous rightmost column, so that

$$C_j = (\Gamma + \lambda_j)_F \quad \lambda_j \notin E_{j-1}.$$

Now suppose we are on move i so that $C_{i-1} = (\Gamma + \lambda_{i-1})_F$. Furthermore we may assume that the inequality $\lambda_{j-1} < u_{j-1} < \lambda_j$ for all $1 \leq j \leq i - 1$ holds, and that the $u_j =$

$\min((\Gamma + \lambda_j) \cap E_{j-1})$ are indeed the minimal collisions defined in the Mancala game, and simultaneously in Delorme's definition. Finally we conclude by our recursion that

$$\bigcup_{k=-1}^{j} C_k = (E_j)_F, \ \forall\, 0 \leq j \leq i-1$$

and that $\min(C_j \cap C_k) > c(E_{i-1})$, or empty, whenever $-1 \leq j < k < i+1$. Here $c(E_{i-1})$ is the conductor of the Γ-semimodule E_{i-1}. Thus we may assume we are looking for a collision between C_{i-1} and C_j, with $-1 \leq j < i-1$. We then have for this particular j that

$$u_{i-1} := \min((\Gamma + \lambda_{i-1}) \cap E_{i-2}) = \min(C_{i-1} \cap \Big(\bigcup_{k=-1}^{i-2} C_k\Big)) = \min(C_{i-1} \cap C_j).$$

We are ready to make the ith move, and note that we are allowed to choose any $\lambda_i > u_{i-1}$, such that $\lambda_i \notin \cup_{k=-1}^{i-1} C_k$ equivalently $\lambda_i \notin E_{i-1}$, provided it exists. Assuming that it does, let us suppose that we arbitrarily choose such a λ_i.

We show that any choice of our λ_i still gives us the increasing semimodule condition. That is, once we have made the ith move, and look amongst the new columns for the minimal collision of beads, we will find that our minimal collision is greater than λ_i and is again given by u_i as in definition 35. Hence we will maintain the property that $\lambda_i < u_i < \lambda_{i+1}$, provided $i < N$.

Even before we have chosen our $\lambda_i > u_{i-1}$ such that $\lambda_i \notin E_{i-1}$, we have recursively that $c_{i-1} = c_{i-2} + \lambda_{i-1} - u_{i-1}$ from Lemma 4. We also have by definition of the rules of the game that the beads in C_j will remain untouched for all $-1 \leq j \leq i-2$. Thus amongst these columns we can be sure that $\min(C_j \cap C_k) > c(E_i)$ for all $1 \leq j < k \leq i$. Our concern is with column C_{i-1} We would like to show that once we have completed Step 2 of move i, that is we have removed the beads in C_{i-1} that are also in $\Gamma + u_{i-1}$, we have $\min(C_{i-1} \cap C_j) > c(E_i)$, for $-1 \leq j < i-1$. If this is so, then we can be sure that our minimal collision of beads involves $C_i = (\Gamma + \lambda_i)_F$, or is beyond the conductor $c(E_i)$, which would conclude our game.

Let us make use of Delorme's lemmas to see why this the case. Suppose we have not yet removed the beads from C_{i-1}. We note that since $\lambda_{i-1} > u_{i-2} > \bar{u}_{i-2}$, then by our recursion so far and Lemmas 1, 3 and 4

$$C_{i-1} \cap \Big(\bigcup_{k=-1}^{i-2} C_k\Big) = ((\Gamma + \lambda_{i-1}) \cap E_{i-2})_F \tag{4.3}$$

$$= ((\Gamma + \lambda_{i-1}) \cap (\Gamma + c_{i-2}))_F \tag{4.4}$$

$$= ((\Gamma + u_{i-1}) \cup (\Gamma + c_{i-1} + mp))_F\,. \tag{4.5}$$

Therefore all bead collisions between C_{i-1} and the other columns are given by the union above. Finally by Lemma 3.3 we have that

$$(\mathbb{N}_0 + c_{i-1} + mp)_F \subset \bigcup_{k=-1}^{i-1} C_k = (E_{i-1})_F,$$

so that $c(E_{i-1}) \leq c_{i-1} + mp$.

Let us now take Step 2 of move i, and remove all of the beads in C_{i-1} that are also in $\Gamma + u_{i-1}$. From the equalities above, we see that by removing these beads in C_{i-1}, we are removing all collisions of beads between C_{i-1} and all other columns C_j, $j < i-1$, that of value less than $c(E_{i-1})$, as we had wanted to show. We therefore have that if there are any minimal collisions of beads that are less that $c(E_i)$, they must occur between the rightmost column C_i and some other column C_j, $j < i$. This also implies $u_i = \min((\Gamma + \lambda_i) \cap E_{i-1})$ is our minimum collision of beads for Step 1 of move $i+1$.

This shows recursively that any coordinated Mancala game will provide us with an increasing Γ-semimodule in the coprime case. By the main result of Almirón in [3] (see also Proposition 20 below) it must be that there exists a plane curve germ with value set generated by the resulting minimum beads in each column of our game. It is rather clear that every increasing value set can be achieved by playing all the Mancala games in the coprime case. Therefore, since every plane curve germ produces some increasing value set whenever its semigroup Γ is coprime, then we must have that one of the games results in this value set. This proves both directions of the theorem, and concludes our proof. $\square$

Now that we have proved our theorem we may ask: given a result of a Mancala game, how does one find a plane curve germ that has this given value set? Similarly we could ask: given a plane curve germ, how does one use the Mancala game to determine the value set. The latter question is a bit more simple than the former. Given an analytic plane curve germ, one can assume coordinates that give us a normal form for its parameterization. From there we can use the Zariski one-form to get the first generator, λ_1.

After this we can use the columns to indicate which monomial one-forms we should try multiplying the Zariski one-form by to get our minimum collision of beads, u_1. From there it is a matter of computation to determine how far to throw the u_1 bead and all of the other beads in $\Gamma + u_1$, to achieve the appropriate λ_2. The process continues, and we will always use our newly found one-form ω_i with $v(\omega_i) = \lambda_i$ at each step according to our game.

Let us show how one can achieve a curve germ given an outcome of Coordinated Mancala game.

4.1.3 A plane curve germ for each Mancala game

For the sake of self containment, we give a proof that will also provide an algorithm to obtain a plane curve germ with value set Λ generated by the minimal beads in each column of a given coordinated Mancala game. Note that Almirón has proved the existence of such a plane curve germ in [3], and so here we present a proof and algorithm that, for the author, is more related to the ideas in the coordinated Mancala game. To complete our proof we will need a lemma of Delorme:

Lemma 5. (Delorme Lemma 12 part (b) in [11]) *Let γ be a plane curve germ with semigroup Γ and value set Λ generated by $\lambda_{-1} < \lambda_0 < \lambda_1 < \cdots < \lambda_N$, with $v(\omega_j) = \lambda_j$. Then there is a relation*

$$\omega_{l+1} = \sum_{-1 \le j \le l} F_{j,l}\omega_j$$

where the $F_{j,l} \in \mathcal{O}$, and $u_l = v(F_{l,l}) + \lambda_l = \inf\{v(F_{j,l}) + \lambda_j\}$.

This lemma helps us understand the structure of one-forms that have valuations that minimally generate Λ. We see that if we can find such a γ then it must adhere to the results in the lemma. We now state and prove our proposition.

Proposition 20. *Let the minimum values in each column of a completed coordinated Mancala game be given by $p < m < \lambda_1 < \cdots < \lambda_N$, which, as minimal generators produce an increasing Γ-semimodule Λ, with $\Gamma = \langle p, m \rangle$. Then there exists a plane curve germ γ that has value set Λ.*

Proof. We will prove this by recursively generating one forms ω_l with $v(\omega_l) = \lambda_l$. We will use undetermined coefficients on the parameterization of $y(t)$, and at each step in the recursion we will determine a specific subset of these coefficients. At each step only higher order coefficients compared to the previous step will need to be determined. Each recursive step reflects a move in the coordinated Mancala game. This proof will also provide a clear algorithm for obtaining such a curve germ. We start with the plane curve germ

$$\gamma : \; x = t^p, \;\; y = t^m + \sum_{i>m} a_i t^i,$$

where the a_i are yet to be determined. We see that regardless of the a_i, $v(dx) = p$ and $v(dy) = m$. Therefore we set $\omega_{-1} = dx$ and $\omega_0 = dy$. We note that this gives $u_0 = m + p$, reflecting the 0th move in the coordinated game.

Forming ω_1: Step 1 of the recursion involves the Zariski one-form $\omega_1 = pxdy - mydx$. We want to choose the a_i so that $v(\omega_1) = \lambda_1$. We use the Zariski one-form because xdy and ydx have the minimum possible collision of valuations of one-forms. This follows along nicely with lemma 5. Note that pulling back ω_1 gives

$$\omega_1 = \sum_{i>m} pa_i(i-m)t^{i+p-1}dt$$

so that $v(\omega_1)$ is $i+p$ where i is the smallest integer such that $a_i \neq 0$. In order to achieve λ_1 in its value set $y(t)$ must have the form $y(t) = t^m + at^{\lambda_1 - p} + \sum_{i>\lambda_1-p} a_i t^i, a \neq 0$ A scaling of y and t converts $a \neq 0$ to 1 so that

$$y(t) = t^m + a_{\lambda_1-p}t^{\lambda_1-p} + \sum_{i>\lambda_1-p} a_i t^i.$$

with

$$\omega_1 = p(\lambda_1 - p - m)a_{\lambda_1-p}t^{\lambda_1-1}dt + \sum_{i>\lambda_1-p} pa_i(i-m)t^{i+p-1}dt,$$

We have determined $\lambda_1 - (m+p) = \lambda_1 - u_0$ coefficients in forming ω_1. Namely we set $a_{m+1} = a_{m+2} = \ldots = a_{\lambda_1-p-1} = 0$ and we must insist that $a_{\lambda_1-p} \neq 0$. Let us fix $a_{\lambda-p} \neq 0$. Note that we have determined precisely $j_1 := \lambda_1 - u_0$ coefficients for $y(t)$.

Note also that we can give a weight to each coefficient based off of its index. We will use the weighting

$$w(a_i) := i - m, \text{ for all } i \geq m.$$

We see that the coefficients of ω_1 are homogeneously weighted.

Now let us move on to forming ω_2, as this will illustrate the recursive method by which we choose the a_i in a less trivial way than for ω_1.

Forming ω_2: Suppose now that $N > 1$, i.e. that λ_1 does not generate Λ over Γ, here is how we will construct ω_2 with $v(\omega_2) = \lambda_2$. We note that since $\lambda_1 \notin \Gamma$, then $\lambda_1 = pm - \alpha_1 p - \beta_1 m$ for some $1 \leq \alpha_1 < m-1$ and $1 \leq \beta_1 < p-1$. We have from our coordinated Mancala game that $u_1 \in \Gamma$ as it must be a collision between valuations of monomial one-forms, and monomials times ω_1. We thus require some minimal $\alpha \in \Gamma$ such that $\lambda_1 + \alpha \in \Gamma$. This is the same as determining $\min\{\alpha_1 p, \beta_1 m\}$.

Let us first assume that $\min\{\alpha_1 p, \beta_1 m\} = a_1 p$. We then have $u_1 = pm - \beta_1 m$, and

$$v(x^{\alpha_1}\omega_1) = pm - \beta_1 m = v(d(y^{p-\beta_1})).$$

We have that

$$\begin{aligned} y^{p-\beta_1} = & t^{m(p-\beta_1)} + (p-\beta_1)t^{m(p-\beta_1-1)} \sum_{i \geq \lambda_1 - p} a_i t^i \\ & + \binom{p-\beta_1}{2} t^{m(p-\beta_1-2)} \Big(\sum_{i \geq \lambda_1-p} a_i t^i \Big)^2 + \ldots \\ & \cdots + \Big(\sum_{i \geq \lambda_1 - p} a_i t^i \Big)^{p-\beta_1}. \end{aligned}$$

We see that the second term in $y^{p-\beta_1}$ consists of only linear coefficients a_i, while the rest of the terms consist of nonlinear coefficients. Therefore, once we have collected all coefficients of the same power of t, we will always have a linear term of maximal index. The rest of the terms will be nonlinear with weights adding up to the weight of the maximally indexed term.

This property of course holds when we take $d(y^{p-\beta_1}) = (p - \beta_1)y^{p-\beta_1-1}\omega_0$. We have that the first nonzero term in $x^{\alpha_1}\omega_1$ is $p(\lambda_1 - p - m)a_{\lambda_1-p}t^{u_1-1}dt$, and the first nonzero term for $d(y^{p-\beta_1})$ is $u_1 t^{u_1-1}dt$. Therefore it is clear that if we multiply $d(y^{p-\beta_1})$ by $p(\lambda_1 - p - m)a_{\lambda_1-p}/u_1$, we will resolve the collision between the two one-forms. Notice here that we have adjusted the weighting of each coefficient in $(p - \beta_1)y^{p-\beta_1-1}\omega_0$ by the weight of a_{λ_1-p}. It is because of this that homogeneous weighting of the coefficients remains in tact after we take the difference

$$x^{\alpha_1}\omega_1 - (p(\lambda_1 - p - m)a_{\lambda_1-p}/u_1)d(y^{p-\beta_1}). \tag{4.6}$$

This implies that we have not accidentally cancelled out any of our undetermined coefficients, and that they still must appear in the same way as before. Namely we have that each undetermined coefficient must show up in a power of t and there is a minimal power of t for which this occurs. Because of the weighting of the coefficients, it must show up *linearly* in this minimal power of t. We ultimately have this property: If a_{i_k} first appears in any coefficient for say t^k in our difference in (4.6), then it will do so linearly. Furthermore we must then have that a_{i_k+1} appears linearly in t^{k+1}, and this is the first appearance for a_{i_k+1} as well.

Examining ω_1 we see that $a_{\lambda-p+1}$ must appear linearly on the term $t^{u_1}dt$, which has valuation u_1+1. From our above paragraph we can see that there must exist some value of $a_{\lambda-p+1}$ that will allow us to maintain the valuation of $u_1 + 1$ and a value of $a_{\lambda-p+1}$ which will force the valuation to be greater than $u_1 + 1$. If we choose the latter value, we will then have that $a_{\lambda-p+2}$ appears linearly in the coefficient of $t^{u_1+1}dt$, and we may make the same two choices.

This can continue until we have reached λ_2. We also see that this can continue until we have reached the conductor of our semimodule generated by our Mancala game. This is important, as we do need to consider being able to form a curve that has this property (i.e. the case where $N = 1$). Once we have chosen successively (and successfully) the appropriate number of a_i, which will be precisely $j_2 = \lambda_2 - u_1$, we can write that

$$\omega_2 := x^{\alpha_1}\omega_1 - (p(\lambda_1 - p - m)a_{\lambda_1-p}/u_1)d(y^{p-\beta_1}).$$

Delorme's Lemma shows us that if we have achieved the valuation λ_2 using the above difference, then we must have found our generating one-form, which makes it appropriate to label it ω_2. Indeed we have this whole time been working with a curve in the correct equisingularity class regardless of the coefficients. This implies whatever coefficients we choose, we will still achieve an increasing semimodule, and it is clear that we are achieving the one generated by the Mancala game up to move 2.

In the case where $\min\{\alpha_1 p, \beta_1 m\} = \beta_1 m$, we would need the forms $y^{\beta_1}\omega_1$ and $d(x^{m-\alpha_1})$ instead. This argument requires a bit less, since $x = t^p$ and so we are just subtracting a single term. We have also shown above that y^k for any $k > 0$ has a particular form, so that the same arguments hold in terms of preserving the homogeneous weighting of the coefficients. Therefore similar results hold, and we are able to form our ω_2 using a similar difference.

This concludes our work on how to construct ω_2. We now move on to the general recursion, and assure the reader that there are no new ideas that are needed to construct our desired one-forms.

General Recursive Step: For each $1 \leq n \leq N$, define

$$j_n := \lambda_n - u_{n-1}, \text{ and } J_n = \sum_{k=1}^{n} j_k.$$

Suppose that we have recursively formed ω_j for $-1 \leq j \leq n-1$, and therefore have determined J_{n-1} of the a_i. We have by recursion that

$$\omega_j = \sum_{k=m+p}^{J_{n-1}+m+p} d_{jk} t^k dt + \sum_{k > J_{n-1}+m+p} u_{jk} t^k dt,$$

where the d_{jk} consist only of determined coefficients, and the u_{jk} have undetermined coefficients. Here all of the $d_{jk} = 0$ for $i < \lambda_j - 1$. If we denote the determined coefficients of $y(t)$ as d_i, then we can write at step n that

$$y(t) = t^m + \sum_{i=m+1}^{J_{n-1}+m} d_i t^i + \sum_{i > J_{n-1}+m} a_i t^i.$$

We have recursively that

$$u_{jk} = r_{jk} a_{i_k} + P_{jk}(d_{m+1}, \ldots d_{J_{n-1}+m+1}, a_{J_{n-1}+m+2}, \ldots a_{i_k - 1}),$$

Where $r_{jk} \neq 0$ a constant, P_{jk} is a weighted homogeneous polynomial in the determined and undetermined coefficients with indices less that i_k, and $i_k - m$ is the homogeneous weight of all terms in u_{jk}. Furthermore $a_{i_{k+1}} = a_{i_k+1}$.

Let us now observe a few more consequences of our recursion. We may assume by recursion that so far Delorme's lemma 5 holds for all of our ω_j. Since $v(\omega_{n-1}) = \lambda_{n-1}$ we know already that there must exist a monomial $x^r y^s$ such that $v(x^r y^s \omega_{n-1}) = u_{n-1}$, and $x^r y^s$ has minimal valuation in the sense of lemma 5. It follows as well that because u_{n-1} is the minimal collision at move $n-1$, there must exist a ω_j such that

$$v(F_j \omega_j) = v(x^r y^s \omega_{n-1}),$$

where $F_j \in \mathcal{O}$, and can usually be considered a monomial. Therefore there is some linear combination of the two such that

$$v(ax^r y^s \omega_{n-1} - bF_j\omega_j) > u_{n-1}.$$

This linear combination must maintain the property that each coefficient has weighted homogeneous terms, and therefore one of them must be linear and minimal. The first nonzero term in $ax^r y^s \omega_{n-1} - bF_j\omega_j$ must therefore have the $a_{J_{n-1}+m+1}$ show up linearly, and all other terms are comprised of determined coefficients. The next coefficient of $ax^r y^s \omega_{n-1} - bF_j\omega_j$ will have $a_{J_{n-1}+m+2}$ show up linearly, and all other terms will be comprised of either determined coefficients, or lower $a_{J_{n-1}+m+1}$.

This argument continues on for each coefficient, and so it is possible to give this one-form any valuation we so choose greater than u_{n-1} by making a certain choice of the a_i. We start with $a_{J_{n-1}+m+1}$ and determine the higher indices recursively until we have chosen exactly j_n of them, and so that we can now call

$$\omega_n = ax^r y^s \omega_{n-1} - bF_j\omega_j,$$

with $v(\omega_n) = \lambda_n$. We continue inductively until we reach λ_N and from there we choose the a_i up until $v(\omega_{n+1}) > c(\Lambda)$, and set all other $a_i = 0$. This will guarantee that we have found a plane curve γ with the correct value set, completing the proof. $\square$

We note that there are other methods of finding a plane curve germ with that has a certain given Λ. We note that there are inevitably some free choices to make in terms of the a_i, especially if the a_i have $i \in \Gamma - p$. It is likely the case that in our actual computations we would start out with only p undetermined coefficients for $y(t)$ and then add more if needed. We could also remove many of the terms in each ω_j at each step n if the terms have powers that are already in the Γ-semimodule generated by the λ_j for $j \leq n$. Then there are quite a few less coefficients to determine, as setting them to zero is just as good as leaving them as free coefficients.

Now that we have proved our proposition, and in doing so provided an algorithm to obtain the curve, let us see an example of how this works.

Example 18. Example of Λ to a curve

This concludes our section on coordinated Mancala for the coprime case. We now devote a section to an important coordinated Mancala game for a coprime semigroup that results in the generic value set of an equis-singularity class of plane curve germs. In this section we will be able to give a recursive formula for the minimal generators of this value set, or Mancala game. This new formula developed in part by Lee McEwan (see [17]) will help complete the proof of the main result of this work.

4.1.4 The Minimal Mancala Game and the Generic Value Set for coprime Γ.

There is one particular game of interest to us that we will call the *minimal* coordinated Mancala game. Let us first define what this means.

Definition 37. For a pair of coprime integers p, m we say the *minimal* coordinated Mancala game is the game played by choosing the $\lambda_i = g_i$ where

$$g_i := \min((\mathbb{N}_0 + u_{i-1}) \cup E_{i-1}^c).$$

That is $g_i - u_{i-1}$ is the smallest it can possibly be so that $g_i \notin E_{i-1}$.

When we compare this to the generic algorithm in Delorme's 1974 article [10], we see immediately that this minimal Mancala game must produce the generic value set, denoted Λ_{gen}, of a topological class of plane curve germs with Puiseux Characteristic $(p; m)$. We would like to know the outcome of the minimal Mancala game for a general pair of coprime integers $p < m$. The first and second move of the minimal game is not that difficult to compute. In general we have that if $m \not\equiv -1 \pmod{p}$ and $m \neq p + 1$, then

$$g_1 = p + m + 1. \tag{4.7}$$

Indeed $u_0 = m + p$, and we must have $p + m + 1 \notin \Gamma$. Otherwise there exist $a, b \geq 0$ such that $(a-1)p + (b-1)m = 1$. One of these coefficients must be negative, so either $a = 0$ or $b = 0$. In the first case $m = p + 1$ and in the second case $m = (a-1)p - 1$. In the excluded cases we have that $g_1 = p + m + 2$.

The second move must come from choosing a $g_2 > u_1$ in a minimal way. This is not so difficult since we must have $u_1 \in \Gamma$. We then just need to look at gaps in Γ that are also not in $\Gamma + g_1$. We illustrate this with an example:

Example 19. Consider $\Gamma = \langle 10, 23 \rangle$. Since $23 = 2 \cdot 10 + 3$, we see by the last remark that $u_0 = m + p = 33$ and $g_1 = u_0 + 1 = 34$. To find $u_1 = g_1 + \gamma$, write out the elements of Γ and observe that $\gamma = 2m = 46$ is the smallest element of Γ that satisfies the condition $g_1 + \gamma \in E_0 = \Gamma^*$, whence $u_1 = 80$. Then to find $g_2 \in (\mathbb{N} + u_1) \setminus E_1$, write out the first few elements of $\Gamma + g_1$ to check that $u_1 + 1$ is *not* in E_1, so $g_2 = u_1 + 1 = 81$. Find $u_2 = g_2 + \gamma_2$ by seeing that $g_2 + m = 104 = g_1 + 7p \in E_1$ via direct inspection of E_1, and no element γ smaller than m exists in Γ^* such that $g_2 + \gamma \in E_1$. One can continue in this way, but the checking becomes harder as the sets E_i become more complicated.

In general the minimal Mancala game or equivalently the Delorme algorithm is rather difficult to compute, as it is not clear how one immediately determines both the g_i and the u_i at each step. In order to develop a recursive formula, and simplify the computations in the minimal game, we make use of the c_i in Lemma 4. Let us continue the minimal game from the above example.

Example 20. (continued from Example 19.) For $\Gamma = \langle 10, 23 \rangle$ we found $u_0 = 33$, $g_1 = 34$, $u_1 = 80$, and $g_2 = 81$. In particular, $u_1 = g_1 + 2m$, so by Delorme's formula $c_1 = c_0 + (g_1 - u_1) = -2m = -46$. To find the smallest $\gamma \in \Gamma$ such that $u_2 = g_2 + \gamma \in E_1$, we can instead test:

$$\text{For which } \gamma \text{ is } \ g_2 + \gamma \in (\Gamma + c_1)? \ \text{– that is, } \ g_2 + \gamma - c_1 \in \Gamma? \tag{4.8}$$

Since $g_2 - c_1 + \gamma = 127 + \gamma$, we see that $\gamma = m$, because it verifies the condition in (4.8) and the only smaller values of γ are p and $2p$, which both fail condition (4.8). Thus $u_2 = g_2 + m = 104$. Then to compute $g_3 = u_2 + r$ we replace the algorithm's requirement that r be the minimal positive integer such that $u_2 + r \notin E_2$, with the condition $u_2 - c_2 + r \notin \Gamma$. Since $u_2 - c_2 = u_2 + m = 127$, and $128 \notin \Gamma$, we find that $r = 1$ and $g_3 = u_2 + 1 = 105$. One more cycle of calculation completes the algorithm: We now have $c_2 = c_1 + (g_2 - u_2) = -3m = -69$, and we seek γ_3 such that $g_3 - c_2 + \gamma_3 \in \Gamma$. Thus γ_3 is the smallest element of Γ such that $174 + \gamma_3 \in \Gamma$. Since $8m = 184 = 174 + p$, we have $\gamma_3 = p$. So $u_3 = g_3 + \gamma_3 = 115$. To find g_4, we seek r such that $u_3 + r - c_3 \notin \Gamma$. Since $u_3 - c_3 = 115 + 79 = 194$, we check that $194 + r \in \Gamma$ for $r = 1, 2$ and $194 + 3 \notin \Gamma$. Thus $g_4 = u_3 + 3 = 118$. The algorithm now ceases: $g_4 - c_3 + \gamma_4 = 118 + 79 + \gamma_4 = 197 + \gamma_4$ is in Γ if $\gamma_4 = p = 10$. But then $u_4 = g_4 + \gamma_4 = 128$ and $u_4 - c_4 = 217$ is greater than $\mu = 198$, so no $r > 0$ exists satisfying $u_4 - c_4 + r \notin \Gamma$.

We see that as we choose larger p, and m our work becomes more and more difficult. It is now that we present a recursive formula to explicitly determine these g_i. We will need some definitions and preliminaries first.

Explicit calculation of the generators: preliminaries

We introduce the ingredients of our main result, which presents explicit formulas for the generators of Λ_{gen}. Central to our calculation is the data provided by the Euclidean algorithm applied to m and p. Let s be the number of steps in the Euclidean algorithm for m and p, define $p_0 = p$, and

$$\begin{aligned} m &= k_0 p_0 + p_1 \\ p_0 &= k_1 p_1 + p_2 \\ p_1 &= k_2 p_2 + p_3 \\ &\vdots \\ p_{s-2} &= k_{s-1} p_{s-1} + 1 \end{aligned} \tag{4.9}$$

where $1 < p_i < p_{i-1}$ for $1 \leq i \leq s-1$. In accordance with the above, set $p_s = 1$. The number s is called the *level* of the semigroup. The numbers p_i are the *divisors* and k_i are the *quotients* for the semigroup. Sometimes we refer to p_{s-1} as the *final divisor*. It is natural to define $k_s = p_{s-1}$ and $p_{s+1} = 0$, so we may conveniently write $p_{s-1} = k_s p_s + p_{s+1}$.

From this sequence we derive related numbers: Let

$$\begin{pmatrix} A_0 & A_1 \\ B_0 & B_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & k_0 \end{pmatrix} \tag{4.10}$$

and for $1 \le i < s$ define

$$\begin{pmatrix} A_{i+1} \\ B_{i+1} \end{pmatrix} = \begin{pmatrix} A_{i-1} & A_i \\ B_{i-1} & B_i \end{pmatrix} \begin{pmatrix} 1 \\ k_i \end{pmatrix}. \tag{4.11}$$

Then by induction we have

$$p_i = (-1)^i B_i p + (-1)^{i-1} A_i m, \quad 1 \le i \le s. \tag{4.12}$$

In particular for $i = s$ we have

$$1 = (-1)^s B_s p + (-1)^{s-1} A_s m. \tag{4.13}$$

The following simple induction will also be useful.

Lemma 6. *For* $1 \le j \le s$,

$$p = A_j p_{j-1} + A_{j-1} p_j \tag{4.14}$$

$$m = B_j p_{j-1} + B_{j-1} p_j \tag{4.15}$$

□

The case $i = s$ in the last lemma is particularly useful:

$$p = A_s p_{s-1} + A_{s-1} \text{ and } \quad m = B_s p_{s-1} + B_{s-1}. \tag{4.16}$$

Recalling that the final divisor p_{s-1} is also k_s, the last equations can be seen as an extension of (4.11) if we take $A_{s+1} = p$ and $B_{s+1} = m$. The equations (4.12), (4.11), and (4.16) are central to the proof of the main theorem.

Following Delorme, we can represent generators in a standard form: Any element of $\Lambda \setminus \Gamma^*$ can be written uniquely as

$$g = pm - am - bp \tag{4.17}$$

where $0 < a < p$ and $0 < b < m$.

In addition to the numbers already defined, we set $n_s := p_{s-1}$, and for $1 \le l \le s$, denote $N_l = \sum_{j=l}^{s} n_j$, and $N_{s+1} = n_{s+1} = 0$, where we recursively define

$$n_l = \begin{cases} 0, & \text{if } 2 \mid N_{l+1} \text{ and } n_{l+1} \neq 0, \\ k_l, & \text{if } 2 \nmid N_{l+1} \text{ or } n_{l+1} = 0. \end{cases} \tag{4.18}$$

Note that n_s is never zero, and it is impossible for two consecutive values of n_l to be zero.

Explicit calculation of the generators: Main Result

As always Γ is the semigroup generated by coprime integers p and m. The case where $p = 2$ is trivial, and henceforth we assume $p > 2$.

Theorem 18. *For $1 \leq i \leq N_1 - 1$ let*

$$u_i = g_i + \gamma_i, \text{ and}$$
$$g_{i+1} = u_i + p_j$$

where $j \in [1, s]$ is determined by $N_{j+1} \leq i \leq N_j - 1$ and

$$\gamma_i = \begin{cases} (B_j - 1)p & \text{if} \quad 2 \nmid j \text{ and } 2 \nmid i, \\ p & \text{if} \quad 2 \nmid j \text{ and } 2 \mid i, \\ (A_j - 1)m & \text{if} \quad 2 \mid j \text{ and } 2 \nmid i, \\ m & \text{if} \quad 2 \mid j \text{ and } 2 \mid i. \end{cases} \tag{4.19}$$

Then the generic Γ-semimodule Λ_{gen} is generated by $g_{-1} = p$, $g_0 = m$, $g_1 = p + m + 1$, and by

$$g_{i+1} = g_i + \gamma_i + p_j \tag{4.20}$$

for $1 \leq i \leq N_1 - 1$.

Our notation agrees with that used in Delorme's algorithm, and the generators produced by recursion (4.20) are minimal except in circumstances we now explain. Some generators produced by Theorem 18 are not minimal in two situations: (a) The first inequality constraining i in the statement does not build in the stopping condition: "Stop when $u_i - c_i \geq \mu$". (b) It is possible for particular γ_i defined in the theorem to be zero. This happens when $k_0 = 1$ or $k_1 = 1$. In Section 4.1.4 we identify non-minimal generators produced by the recursion in this situation. Outside of this case, the correct stopping point for i and thus the precise identification of the set G of minimal generators is given in Corollary 3. The correct identification of G and $|G|$ when $k_0 = 1$ or $k_1 = 1$ is given in Section 4.1.4.

A general formula for the conductor of Λ_{gen} is given in Corollary 2.

In Section 4.1.4 we present worked-out examples, and in Section 4.1.4 relate our calculation to that of the (minimal) Tjurina number.

Proof of Theorem 18. First we recapitulate the algorithm of Delorme, relying heavily on Lemma 4. Each generator is calculated in two steps: (1) From the last generator g_i, a "collision" $u_i = \gamma_i + g_i$ is computed. It is the smallest value of the form $\gamma + g_i$, $\gamma \in \Gamma$, which belongs to E_{i-1}, the set generated by the previous generators $\{g_j\}_{j \leq i-1}$ under the action of Γ. (2) The next generator, g_{i+1}, is found by taking the "minimal jump" from u_i;

that is, $g_{i+1} = u_i + r_i$ where r_i is the smallest positive integer r such that $u_i + r$ is not in E_i. The calculations of u_i and g_{i+1} are simplified by Lemma 4, which allows us to use $\Gamma + c_{i'}$ in the role of $E_{i'}$ ($i' = i-1$ or i), where $c_{i'} = c_{i'-1} + g_{i'} - u_{i'}$ and $c_0 = 0$. This reduces calculating r_i to finding the least $r > 0$ satisfying $u_i - c_i + r \notin \Gamma$, and similarly for γ_i finding the least $\gamma \in \Gamma$ such that $\gamma + g_i - c_{i-1} \in \Gamma$.

We will need two levels of induction: on the index of the generator i, and on the level j of the Euclidean Algorithm (the latter begins at $j = s$ and decreases). The proof will focus on the numbers g_i, u_i and c_i. It follows from the recursive property of c_i (see Lemma 4) and the definition of γ_i that $c_i = -\sum_{a=1}^{i} \gamma_a$. We first prove the base case ($j = s$) for $1 \le i \le k_s - 1$.

Base Case $j = s$: We assume $2 \nmid s$, since the calculations for $s =$ (even) are exactly parallel (we will however provide the equivalent intermediate expressions). By definition $c_0 = 0$, and so $g_1 - c_0 = g_1$. Thus we seek the smallest $\gamma_1 \in \Gamma$ such that $g_1 + \gamma_1 \in E_0 = \Gamma^*$. This is simplified by expressing $g_1 - c_0$ in standard form (4.17). First note that by (4.13) and $2 \nmid s$

$$1 = A_s m - B_s p. \tag{4.21}$$

Because g_1 is given by $m + p + 1$ (see (4.7)), and $p = k_s A_s + A_{s-1}$ by (4.16), we obtain

$$g_1 - c_0 = pm - ((k_s - 1)A_s + A_{s-1} - 1)m - (B_s - 1)p. \tag{4.22}$$

Now compare the terms of (4.22). The inequality $k_s = p_{s-1} > 1$ ensures

$$((k_s - 1)A_s + A_{s-1} - 1)m \ge A_s m > B_s p > (B_s - 1)p,$$

and so the minimal element in Γ to add to g_1 must be $\gamma_1 = (B_s - 1)p$, as any smaller element would result in negative coefficients for both p and m. The algorithm sets $u_1 = g_1 + \gamma_1$, so we have $u_1 = pm - ((k_s - 1)A_s + A_{s-1} - 1)m$ and $c_1 = -(B_s - 1)p$. Therefore

$$u_1 - c_1 = pm - ((k_s - 1)A_s + A_{s-1} - 1)m + (B_s - 1)p \in \Gamma. \tag{4.23}$$

We continue now to show that $r_1 = 1$. Note that $1 = p_s$, as expected in the base case. By (4.21) we have

$$u_1 - c_1 + 1 = pm - ((k_s - 2)A_s + A_{s-1} - 1)m - p.$$

If the coefficient of the middle term is zero, then $u_1 - c_1 > \mu$ and the algorithm stops with g_1. Otherwise $u_1 - c_1 + 1$ is not in Γ, and so $r_1 = 1$. Thus $g_2 = u_1 + 1$ and we have

$$g_2 - c_1 = u_1 - c_1 + 1 = pm - ((k_s - 2)A_s + A_{s-1} - 1)m - p.$$

This establishes the first step of induction for the base case.

Remark 1. The structure of the calculation that emerges has the following form:

1. Begin the step by expressing $g_i - c_{i-1}$ in standard form (4.17):

$$g_i - c_{i-1} = pm - a_i m - b_i m.$$

2. Calculate γ_i as the smaller of the two terms $a_i m$ and $b_i p$.

3. Because $u_i = g_i + \gamma_i$ and $c_i = c_{i-1} - \gamma_i$, it is simplest to think of $u_i - c_i$ as obtained from $g_i - c_{i-1}$ by reversing the sign of the smaller term:

$$u_i - c_i = \begin{cases} pm + a_i m - b_i p, & \gamma_i = a_i m < b_i p, \\ pm - a_i m + b_i p, & \gamma_i = b_i p < a_i m. \end{cases}$$

4. Find the smallest $r_i > 0$ such that $u_i - c_i + r_i \notin \Gamma$. We show below that $r_i = p_j$ for level j, and compute the interval of values i belonging to this level. Then $g_{i+1} = u_i + r_i$ and the process repeats.

Induction step for base case. Suppose $1 \leq i < k_s$, and inductively assume

$$g_i - c_{i-1} = pm - ((k_s - i)A_s + A_{s-1} - 1)m - \begin{cases} (B_s - 1)p & \text{if} \quad 2 \nmid i, \\ p & \text{if} \quad 2 \mid i. \end{cases} \tag{4.24}$$

If $2 \mid i$ in (4.24) then clearly $\gamma_i = p$. Thus

$$u_i - c_i = pm - ((k_s - i)A_s + A_{s-1} - 1)m + p,$$

and, provided $i < k_s$, we check that $r_i = 1$:

$$u_i - c_i + 1 = pm - ((k_s - i - 1)A_s + A_{s-1} - 1)m - (B_s - 1)p \notin \Gamma \tag{4.25}$$

since both coefficients are negative (here we assume $A_{s-1} > 1$; see Section 4.1.4). Now suppose $2 \nmid i$. Then since $k_s - i > 0$ we have $((k_s - i)A_s + A_{s-1} - 1)m > (B_s - 1)p$. Hence $\gamma_i = (B_s - 1)p$. A computation as above shows that $r_i = 1$ again. Thus we have $r_i = 1$,

$$\gamma_i = \begin{cases} (B_s - 1)p & \text{if} \quad 2 \nmid i, \\ p & \text{if} \quad 2 \mid i \end{cases}$$

and $g_{i+1} - c_i$ has the same form as equation (4.24). Thus the induction step is proved. In particular, for $i = k_s - 1$ we find if $2 \nmid s$ then

$$g_{N_s} - c_{N_s - 1} = pm - (A_{s-1} - 1)m - \begin{cases} (B_s - 1)p & \text{if} \quad 2 \nmid N_s, \\ p & \text{if} \quad 2 \mid N_s. \end{cases} \tag{4.26}$$

The case $2 \mid s$ is exactly analogous with roles reversed: we now have $B_s p - A_s m = 1$, which provides $B_s p > A_s m$, and so $\gamma_1 = (A_s - 1)m$, etc. Thus if $2 \mid s$ then

$$g_{N_s} - c_{N_s - 1} = pm - (B_{s-1} - 1)p - \begin{cases} (A_s - 1)m & \text{if} \quad 2 \nmid N_s, \\ m & \text{if} \quad 2 \mid N_s. \end{cases} \tag{4.27}$$

This establishes the case where $j = s$ and $1 \leq i \leq N_s - 1$. Note that in this case we need not prove $r_i = 1$ is the *minimal* jump, since r_i must be positive. Lastly, if $A_{s-1} = 1$, the generator g_{N_s} belongs to $\Gamma + c_{N_s - 1}$, and so is not minimal; see Section 4.1.4.

Induction on level $j < s$. We assume the theorem formulas for $j + 1$ and prove them for j. We illustrate the pattern of moving from an odd level to an even level, hence assume $2 \mid j$. The other case will be clear with obvious reversal of roles. We will also first assume $n_{j+1} \neq 0$.

Assuming $2 \mid j$, and $n_{j+1} \neq 0$, we have by induction (compare (4.26))

$$g_{N_{j+1}} - c_{N_{j+1} - 1} = pm - (A_j - 1)m - \begin{cases} (B_{j+1} - 1)p & \text{if } 2 \nmid N_{j+1} \\ p & \text{if } 2 \mid N_{j+1} \end{cases} \tag{4.28}$$

Here for reference is the equivalent induction statement assuming $2 \nmid j$ (compare (4.27)):

$$g_{N_{j+1}} - c_{N_{j+1} - 1} = pm - (B_j - 1)p - \begin{cases} (A_{j+1} - 1)m & \text{if } 2 \nmid N_{j+1} \\ m & \text{if } 2 \mid N_{j+1} \end{cases} \tag{4.29}$$

We first consider the case where $2 \nmid N_{j+1}$. Since $2 \mid j$, by (4.12) we have

$$p_j = B_j p - A_j m. \tag{4.30}$$

This implies $(A_j - 1)m < (B_{j+1} - 1)p$, since we always have $p < m$ and $B_j < B_{j+1}$. Thus (see Remark 1) we have $\gamma_{N_{j+1}} = (A_j - 1)m$.

It follows that

$$u_{N_{j+1}} - c_{N_{j+1}} = pm + (A_j - 1)m - (B_{j+1} - 1)p. \tag{4.31}$$

We claim $r_{N_{j+1}}$ is p_j. Indeed by (4.30)

$$u_{N_{j+1}} - c_{N_{j+1}} + p_j = pm - m - (B_{j+1} - B_j - 1)p \notin \Gamma \tag{4.32}$$

since $B_{j+1} > B_j + 1$. In order to show that p_j is minimal, we first relate any positive number r to the divisors p_i in the Euclidean algorithm (4.9).

Definition 38. Suppose $0 < r < p_j$. Set $r^1 = r$, and for $a > 0$ define $\{r^{1+a}, \alpha_{j+a}\}$ by the conditions $r^a = \alpha_{j+a} p_{j+a} + r^{1+a}$ and $0 \leq r^{1+a} < p_{j+a}$. Let $h = \min\{j + a | r^{1+a} = 0\}$.

Returning to the proof, suppose $r < p_j$. Then with $\{\alpha_i\}$ as in Definition 38 we have $r = \sum_{i=j+1}^{h} \alpha_i p_i$. By (4.12) we may write $p_i = (-1)^{i+1}(A_i m - B_i p)$, and

$$\begin{aligned} r &= \sum_{i=j+1}^{h} (-1)^{i+1} \alpha_i (A_i m - B_i p), \\ &= \left(\sum_{i=j+1}^{h} (-1)^{i+1} \alpha_i A_i \right) m + \left(\sum_{i=j+1}^{h} (-1)^{i} \alpha_i B_i \right) p. \end{aligned} \tag{4.33}$$

We will apply the following lemma to the analysis of r.

Lemma 7. *Let $x = \sum_{i=u}^{h} (-1)^{i+1} \alpha_i A_i$ and $y = \sum_{i=u}^{h} (-1)^{i} \alpha_i B_i$, where $u \le h \le s$ and $0 \le \alpha_i \le k_i$. Assume $\alpha_h > 0$.*

1. *$x > 0 \Leftrightarrow 2 \nmid h$, and $y > 0 \Leftrightarrow 2 \mid h$.*
2. *If in addition $\alpha_i = k_i \Rightarrow \alpha_{i+1} = 0$ then*
   - *(a) $x > 0 \Rightarrow x \ge A_u$, and $y > 0 \Rightarrow y \ge B_u$.*
   - *(b) $x < 0 \Rightarrow x > A_s - p$, and $y < 0 \Rightarrow y > B_s - m$.*

We defer the proof of Lemma 7 to the end of this section, and proceed with the proof of the main theorem. Write $r = xm + yp$, where x and y are the sums in (4.33). By the equations (4.9) for p_i, and since $r < p_j$, the recursion in Definition 38 ensures $\alpha_i \le k_i$ and $\alpha_i < k_i$ unless $\alpha_{i+1} = 0$. So x and y satisfy Part 2 of Lemma 7, with $u = j + 1$.

Now consider $u_{N_{j+1}} - c_{N_{j+1}} + r$ with $r < p_j$. From (4.31)

$$u_{N_{j+1}} - c_{N_{j+1}} + r = pm + (A_j + x - 1)m - (B_{j+1} - y - 1)p. \tag{4.34}$$

Either $x > 0$ and $y < 0$, or the reverse (since $r < p$). Suppose first $x > 0$ and $y < 0$. Then

$$u_{N_{j+1}} - c_{N_{j+1}} + r = (A_j + x - 1)m + (m - B_{j+1} + y + 1)p$$

Then by Lemma 7 we have $y > B_s - m$, and both coefficients are positive. Now suppose $x < 0$ and $y > 0$. Then

$$u_{N_{j+1}} - c_{N_{j+1}} + r = (p + A_j + x - 1)m + (y - B_{j+1} + 1)p$$

where Lemma 7 gives $y \ge B_{j+1}$ and $x > A_s - p$. Thus once again both coefficients are positive, so in either case adding r to $u_{N_{j+1}} - c_{N_{j+1}}$ results in an element of Γ. Therefore with the starting assumption $2 \nmid N_{j+1}$, we see $r = p_j$ is the minimal jump needed to escape Γ. Therefore $r_{N_{j+1}} = p_j$ and (compare (4.32))

$$g_{N_{j+1}+1} - c_{N_{j+1}} = pm - m - (B_{j+1} - B_j - 1)p. \tag{4.35}$$

The pattern is now set for k_j steps. For $0 \le i \le k_j - 1$ we have $r_{N_{j+1}+i} = p_j$,

$$\gamma_{N_{j+1}+i} = \begin{cases} (A_j - 1)m & \text{if} \quad 2 \nmid (N_{j+1} + i), \\ \quad m & \text{if} \quad 2 \mid (N_{j+1} + i) \end{cases}$$

and $g_{N_{j+1}+i+1} = u_{N_{j+1}+i} + \gamma_{N_{j+1}+i}$, so

$$g_{N_{j+1}+1+i} - c_{N_{j+1}+i} = pm - (B_{j+1} - iB_j - 1)p - \begin{cases} (A_j - 1)m & \text{if} \quad 2 \nmid (N_{j+1} + i) \\ m & \text{if} \quad 2 \mid (N_{j+1} + i). \end{cases} \tag{4.36}$$

Now suppose instead we have that $2 \mid N_{j+1}$. Then the induction statement (4.28) is

$$g_{N_{j+1}} - c_{N_{j+1}-1} = pm - (A_j - 1)m - p, \tag{4.37}$$

and we can assume $A_j > 1$ (otherwise the algorithm ends). Hence in this case $\gamma_{N_{j+1}} = p$, and we have

$$u_{N_{j+1}} - c_{N_{j+1}} = pm - (A_j - 1)m + p.$$

First note that $r_{N_{j+1}} \neq p_j$, since $u_{N_{j+1}} - c_{N_{j+1}} + p_j = (p - 2A_j + 1)m + (B_j + 1)p \in \Gamma$. However

$$u_{N_{j+1}} - c_{N_{j+1}} + p_{j-1} = pm - (A_j - A_{j-1} - 1)m - (B_{j-1} - 1)p \notin \Gamma$$

since otherwise either $k_1 = 1$ or $A_j = 2$ (in the latter case $u_{N_{j+1}} - c_{N_{j+1}}$ exceeds μ). One shows that p_{j-1} is the minimal value by analyzing $r < p_{j-1}$ via Lemma 7, exactly as in the last argument. It follows that in this case $r_{N_{j+1}} = p_{j-1}$ and $g_{N_{j+1}+1} = u_{N_{j+1}} + p_{j-1}$, so

$$g_{N_{j+1}+1} - c_{N_{j+1}} = pm - (A_j - A_{j-1} - 1)m - (B_{j-1} - 1)p. \tag{4.38}$$

Because $A_j = k_{j-1}A_{j-1} + A_{j-2}$ we see again by (4.12) that $(B_{j-1} - 1)p$ is the smallest element of Γ which added to (4.38) results in an element of Γ. The previous pattern now repeats: $\gamma_{N_{j+1}+i}$ alternates between $(B_{j-1} - 1)p$ and p, while $r_i = p_{j-1} = A_{j-1}m - B_{j-1}p$ causes the coefficient of m to get closer to zero. Thus for k_{j-1} steps we have

$$g_{N_{j+1}+i+1} - c_{N_{j+1}+i} = pm - (A_j - iA_{j-1} - 1)m - \begin{cases} (B_{j-1} - 1)p & \text{if} \quad 2 \nmid (N_{j+1} + i), \\ p & \text{if} \quad 2 \mid (N_{j+1} + i). \end{cases} \tag{4.39}$$

The assumption ($2 \mid N_{j+1}$) leads to calculations of γ_i and r_i corresponding to level $(j - 1)$ instead of j for an interval of length k_{j-1}, which justifies the definition (4.18) of $n_j = 0$ and $n_{j-1} = k_{j-1}$ in this case.

Now supposing that $n_{j+1} = 0$, then by (4.18) we have $N_{j+1} = N_{j+2}$, and $n_{j+2} \neq 0$. The inductive assumption is given by (4.29), with j replaced by $(j + 1)$:

$$\begin{aligned} g_{N_{j+1}} - c_{N_{j+1}-1} &= g_{N_{j+2}} - c_{N_{j+2}-1} \\ &= pm - (B_{j+1} - 1)p - m. \end{aligned}$$

With the same analysis used in (4.37), we have $\gamma_{N_{j+1}} = m$, and $r_i = p_j$. Thus level $(j+1)$ is empty, and level j proceeds for k_j steps, justifying the definition $n_j = k_j$ in this case. Lastly, the analysis for $2 \nmid j$ is clearly the strict analog of the cases just presented. $\square$

We now prove Lemma 7.

Proof. We prove all the statements for x only, since the arguments for y are strictly analogous.

Proof of 1 ($\Leftarrow$). Suppose $2 \nmid h$ and proceed by induction. First suppose $h = u$. Then

$$x = \alpha_u A_u > 0.$$

Now suppose the implication is true for $h' < h$. Then

$$x = \alpha_h A_h - \alpha_{h-1} A_{h-1} + \sum_{i=u}^{h-2} (-1)^{i+1} \alpha_i A_i$$

(The sum is empty if $h = u + 1$.) By assumption $\alpha_h A_h - \alpha_{h-1} A_{h-1} \geq A_h - \alpha_{h-1} A_{h-1} = (k_{h-1} - \alpha_{h-1}) A_{h-1} + A_{h-2}$. Thus

$$x \geq (k_{h-1} - \alpha_{h-1}) A_{h-1} + \left(A_{h-2} + \sum_{i=1}^{h-2} (-1)^{i+1} \alpha_i A_i \right) \tag{4.40}$$

The first expression is non-negative since $\alpha_{h-1} \leq k_{h-1}$. Adding A_{h-2} to the last sum guarantees the coefficient of the $(h-2)$ term, namely $\alpha_{h-2} + 1$, is positive, so the combined sum is also positive by induction. (If $\alpha_{h-2} + 1$ exceeds k_{h-2}, the violation of the hypothesis is in the positive direction.)

Proof of 1 ($\Rightarrow$). Suppose $2 \mid h$. Apply the previous argument to $-x$, as the only use made of the parity of h is that the highest-index term has positive coefficient.

Proof of 2(a). If $h = u$ and $x > 0$ then $x = \alpha_u A_u \geq A_u$. Suppose $h > u$. Then (4.40) is true and the last sum is positive. Because $\alpha_h > 0$ by assumption, the additional condition ($\alpha_i = k_i \Rightarrow \alpha_{i+1} = 0$) implies that $(k_{h-1} - \alpha_{h-1})$ is positive, so $x > A_{h-1} \geq A_u$.

Proof of 2(b). Assume $x < 0$. First suppose $h = u$. Then $2 \mid u$ and $x = -\alpha_u A_u$. If $u < s$ then

$$x \geq -k_u A_u = -(A_{u+1} - A_{u-1}) > -A_s > -(k_s - 1) A_s - A_{s-1} = A_s - p$$

(see (4.16) and recall $k_s = p_{s-1} > 1$). If instead $u = s$, then by assumption $\alpha_u = \alpha_s \leq k_s - 1$. Then

$$x \geq -(k_s - 1) A_s > -(k_s - 1) A_s - A_{s-1} = A_s - p.$$

Now suppose $h > u$ (and $2 \mid h$). Then

$$x = -\alpha_h A_h - A_{h-1} + \left(A_{h-1} + \sum_{i=u}^{h-1} (-1)^{i+1} \alpha_i A_i \right)$$

The parenthetical term is positive by Part 1. Therefore

$$x > -\alpha_h A_h - A_{h-1} \geq -k_h A_h - A_{h-1} = -A_{h+1}.$$

If $h < s$ then $-A_{h+1} \geq -A_s > -(k_s - 1)A_s - A_{s-1} = A_s - p$. If $h = s$ then $\alpha_h = \alpha_s \leq k_s - 1$ and

$$x > -(k_s - 1)A_s - A_{s-1} = A_s - p.$$

□

Having established Lemma 7, the proof of the main theorem is now complete. We next refine the main result with two corollaries.

Corollary 2. *Let n be the index of the last minimal generator g_n produced by the recursion in Theorem 18. The conductor of Λ_{gen} is given by*

$$c(\Lambda_{gen}) = \mu + c_n.$$

Remark 2. The index n corresponds to the last *minimal* generator produced by Theorem 18, which may occur before the last value of i is reached. It's value is determined in Corollary 3 and Section 4.1.4.

Proof. By Lemma 4 we have

$$(\mathbb{N} + \bar{u}_n) \cap E_n = (\mathbb{N} + \bar{u}_n) \cap (\Gamma + c_n). \tag{4.41}$$

Recall that $E_n = \Lambda_{\text{gen}}$, and notice the obvious fact that the conductor of $(\Gamma + c_n)$ is just $\mu + c_n$. Part (ii) now follows from

Claim: $\bar{u}_n < \mu + c_n$.

For if the Claim is true then (4.41) implies that Λ_{gen} and $\Gamma + c_n$ have the same conductor. But $g_n - c_{n-1}$ is of the form $pm - \alpha m - \beta p$ where $\alpha, \beta > 0$, and $\gamma_n = \min\{\alpha m, \beta p\}$. Clearly $\alpha < p$ and $\beta < m$, so $\alpha m \neq \beta p$. It follows that $u_n - c_n = g_n - c_{n-1} + 2\gamma_n < pm$. From Lemma 4 we have $\bar{u}_n = u_n + \mu - pm$, therefore $\bar{u}_n - c_n = u_n - c_n + \mu - pm < pm + \mu - pm = \mu$. □

Remark 3. Note that $c_n = -\sum_{j=1}^{n} \gamma_j$, so the conductor can be calculated from the recursion of Theorem 18.

Remark 4. Except in the extreme case where either k_0 and/or k_1 equals 1 (discussed in the next section), we have $g_n - c_n = pm - m$, so typically $g_n - (p-1)$ equals the conductor of Λ_{gen}.

Corollary 3. *Suppose $k_0, k_1 \neq 1$, and let g_n be the last minimal generator produced by Theorem 18. Let G be the set of minimal generators for Λ_{gen}. Then $G = \{g_i | -1 \leq i \leq n\}$ with g_i as in (4.20), $|G| = n + 2$ its cardinality, and n is either N_1 or $N_1 - 2$. More precisely, the cardinality of G is given by*

$$|G| = \begin{cases} N_1 + 2 & \text{if } n_1 = 0, \\ N_1 & \text{otherwise.} \end{cases}$$

Proof. Theorem 18 establishes that the recursion (4.20) aligns with Delorme's algorithm. It remains to show when the recursion should end. There are three cases for how the last level, $j = 1$, can begin. If $n_2 \neq 0$ then taking the inductive statement (4.29) with $j = 1$ we have

$$g_{N_2} - c_{N_2-1} = pm - (k_0 - 1)p - \begin{cases} m & \text{if } 2 \mid N_2, \\ (k_1 - 1)m & \text{if } 2 \nmid N_2. \end{cases} \tag{4.42}$$

If however $n_2 = 0$ then we saw that level 2 is empty. In this case the start of level 1 is given by (4.28) with $j = 2$ and $N_3 = N_2$:

$$g_{N_2} - c_{N_2-1} = pm - (k_1 - 1)m - p, \quad \text{where } 2 \mid N_2, \text{ and } n_2 = 0. \tag{4.43}$$

In the first case of (4.42), $n_1 = 0$ by (4.18) and $\gamma_{N_2} = (k_0 - 1)p < m$. Therefore $u_{N_2} - c_{N_2} = pm - m + (k_0 - 1)p$ exceeds the conductor μ, since we assume $k_0 \neq 1$. So g_{N_2} is the final generator. Since $n_1 = 0$ we have $n = N_2 = N_1$ and $|G| = N_1 + 2$.

In the other two cases, we have $n_1 = k_1$. Then $r_{N_2+i} = p_1$, and γ_{N_2+i} alternates between p and $(k_0 - 1)p$. First suppose $k_1 > 2$. After $k_1 - 2$ steps we have

$$g_{N_1-2} - c_{N_1-3} = pm - m - \begin{cases} p & \text{if } \ 2 \mid N_1 \\ (k_0 - 1)p & \text{if } \ 2 \nmid N_1 \end{cases} \tag{4.44}$$

and $g_{N_1-2} - c_{N_1-2} = pm - m$. Thus g_{N_1-2} is the final generator. We have $n = N_1 - 2$ and $|G| = n + 2 = N_1$.

If on the other hand $k_1 = 2$, we have in both cases $g_{N_2} - c_{N_2} = pm - m$. So g_{N_2} is the final generator, and we have $n = N_2$ and $|G| = N_2 + 2$. But now $N_1 = N_2 + k_1 = N_2 + 2$, so again $|G| = N_1$.

□

Remark 5. The recursion of Theorem 18 may stop before $u_i - c_i \geq \mu$ is satisfied. Indeed if $n_1 = 0$ then the final u_i occurs at $i = N_1 - 1$ even though $u_{N_1-1} - c_{N_1-1} < \mu$. In this case the theorem allows the next generator to be defined, namely g_{N_1}. Then Corollary 3 shows that the next output in Delorme's algorithm, *i.e.* $u_{N_1} - c_{N_1}$, does exceed μ, and so the algorithm also stops, and g_{N_1} is the last minimal generator.

Non-minimal generators

In Delorme's algorithm, γ_i is the least element of Γ such that $u_i = g_i + \gamma_i \in E_{i-1}$. Thus non-minimal generators g_i, $i \leq n$, arise in the recursion of Theorem 18 iff $\gamma_i = 0$. This occurs when $A_j = 1$ or $B_j = 1$. We always have $A_1 = 1$ and $B_0 = 1$, but the index for A_j in any γ_i is even, and the index for any B_j is odd. It is possible however to have $A_2 = 1$ or $B_1 = 1$. This is equivalent to the cases (a) $k_1 = 1$ or (b) $k_0 = 1$ respectively. In these cases non-minimal generators may arise in the recursion before we have reached the conductor $c(\Lambda_{\text{gen}})$. Thus

$$G = \{g_i | -1 \leq i \leq n \text{ and } \gamma_i \neq 0\},$$

and the cardinality of G is decreased by the number of times $\gamma_i = 0$.

We summarize the effect on $|G|$ of the various configurations of extreme k_0, k_1. (The list of non-minimal generators in the last column can be empty.)

Constraints		$\|G\|$	Non-minimal generators
$k_0 = 1, k_1 > 1$	$2 \mid N_2$	$N_1 - \left\lfloor \frac{n_1-1}{2} \right\rfloor$	$\{g_{N_2+2j-1} \| 1 \leq j \leq \left\lfloor \frac{n_1-2}{2} \right\rfloor \}$
	$2 \nmid N_2$	$N_1 - \left\lfloor \frac{n_1}{2} \right\rfloor$	$\{g_{N_2+2j} \| 0 \leq j \leq \left\lfloor \frac{n_1-3}{2} \right\rfloor \}$
$k_0 = 1, k_1 = 1$	$n_3 = 0$	$N_1 - \left\lfloor \frac{n_2-1}{2} \right\rfloor$	$\{g_{N_3+2j-1} \| 1 \leq j \leq \left\lfloor \frac{n_2-1}{2} \right\rfloor \}$
	$n_3 \neq 0$	$N_1 - \left\lfloor \frac{n_2}{2} \right\rfloor$	$\{g_{N_3+2j} \| 0 \leq j \leq \left\lfloor \frac{n_2-2}{2} \right\rfloor \}$
$k_0 > 1, k_1 = 1$	$n_1 = 0$	$N_1 - \left\lfloor \frac{n_2-2}{2} \right\rfloor$	If $2 \mid N_3$: $\{g_{N_3+2j-1} \| 1 \leq j \leq \left\lfloor \frac{n_2-1}{2} \right\rfloor \}$,
	$n_1 \neq 0$	$N_1 - \left\lfloor \frac{n_2}{2} \right\rfloor$	otherwise: $\{g_{N_3+2j} \| 0 \leq j \leq \left\lfloor \frac{n_2-2}{2} \right\rfloor \}$

Table 4.1: Non-minimal generators

Remark 6. In the extreme cases treated in this section, the value of the index n of the last minimal generator given by the recursion of Theorem 18 can be deduced from Table 4.1. It is always one more than the index of the last non-minimal generator if that set is non-empty, or $|G| - 2$ otherwise. Equivalently, $n = |G| - 2 +^{\#} \{\text{non-minimal generators}\}$.

Examples

Example 21. Recall Example 19 with semigroup $\Gamma = \langle 10, 23 \rangle$. We compute

$$23 = 2 \cdot 10 + 3$$
$$10 = 3 \cdot 3 + 1$$

Thus the level is $s = 2$, and we easily compute the following table.

i	p_i	k_i	n_i	N_i	A_i	B_i
0	10	2	-	-	0	1
1	3	3	3	6	1	2
2	1	3	3	3	2	7

Here $n_1 \neq 0$, so by Corollary 3 there are $N_1 = 6$ generators, including $p = 10$ and $m = 23$ and $g_1 = 1 + p + m = 34$. Following Theorem 18, we compute each γ_i with $1 \leq i \leq N_1 - 1 = 5$ and corresponding level and jump, and resulting g_i and u_i (while Corollary 3 indicates we should stop at $i = 4$, since $n = N_1 - 2$ in this case).

i	j	γ_i	$r_i = p_j$	g_i	u_i
1	2	$(A_2 - 1)m = 46$	1	34	80
2	2	$m = 23$	1	81	104
3	1	$(B_1 - 1)p = 10$	3	105	115
4	1	$p = 10$	3	118	128
5	1	$(B_1 - 1)p = 10$	3	(131)	(141)

Note that $c_4 = -\sum_{a=1}^{4} \gamma_a = -89$, so $u_4 - c_4 = 217$, which is greater than the conductor of Γ. So the algorithm stops at $i = 4$ and the last displayed generator is redundant, as Corollary 3 implies it should be. Lastly we find the conductor of Λ_{gen}:

$$c(\Lambda_{\text{gen}}) = \mu + c_n = (23 - 1)(10 - 1) - 89 = 109.$$

The next example shows how even large examples can be done easily by hand using the recursion of Theorem 18.

Example 22. Consider $\Gamma = \langle 122, 281 \rangle$. We first compute the numbers $\{p_i, k_i, s\}$:

$$\begin{aligned} 281 &= 2 \cdot 122 + 37 \\ 122 &= 3 \cdot 37 + 11 \\ 37 &= 3 \cdot 11 + 4 \\ 11 &= 2 \cdot 4 + 3 \\ 4 &= 1 \cdot 3 + 1 \end{aligned}$$

so $s = 5$, $\{p_i\} = \{122, 37, 11, 4, 3, 1\}$ and $\{k_i\} = \{2, 3, 3, 2, 1, 3\}$ for $0 \leq i \leq 5$. Next for $1 \leq i \leq 5$ compute via (4.11) the values $\{A_i\} = \{1, 3, 10, 23, 33\}$ and $\{B_i\} = \{2, 7, 23, 53, 76\}$, and by (4.18) find $\{n_i\} = \{3, 3, 0, 1, 3\}$ and $\{N_i\} = \{10, 7, 4, 4, 3\}$. Next compute γ_i and r_i for $1 \leq i \leq 9$ $(= N_1 - 1)$, and obeying the inequalities in the theorem; from these values we immediately calculate the generators, starting with $g_1 = p + m + 1$:

i	j	γ_i	$r_i = p_j$	g_i	u_i
1	5	$(B_5 - 1)p = 9150$	1	404	9554
2	5	$p = 122$	1	9555	9677
3	4	$(A_4 - 1)m = 6182$	3	9678	15860
4	2	$m = 281$	11	15863	16144
5	2	$(A_2 - 1)m = 562$	11	16155	16717
6	2	$m = 281$	11	16728	17009
7	1	$p = 122$	37	17020	17142
8	1	$(B_1 - 1)p = 122$	37	17179	17301
9	1	$p = 122$	37	17338	17460

Notice that $g_9 = g_6 + 5p$. The last generator is therefore redundant (in fact $u_9 - c_9$ exceeds μ). Including $g_{-1} = p$ and $g_0 = m$, we see that Λ_{gen} has 10 $(= N_1)$ generators, including two belonging to Γ.

In order to relate the generators to the Tjurina number, we write them in two ways: numerically and in standard form (4.17). To compute the standard form, apply (4.12) to each p_j and use the recursion of the theorem, *i.e.* $g_1 = p + m + 1$ and $g_{i+1} = g_i + \gamma_i + p_j$.

i	$pm - \alpha_i m - \beta_i p =$	g_i	i	$pm - \alpha_i m - \beta_i p =$	g_i
1	$pm - 88m - 75p =$	404	5	$pm - 25m - 91p =$	16155
2	$pm - 55m - 76p =$	9555	6	$pm - 26m - 84p =$	16728
3	$pm - 22m - 151p =$	9678	7	$pm - 28m - 77p =$	17020
4	$pm - 23m - 98p =$	15863	8	$pm - 27m - 78p =$	17179

We can now easily compute the Tjurina number for Λ_{gen}. Consider the set of integer pairs (α_i, β_i) induced by the generators g_i in standard form, $1 \le i \le 8$. We re-index the pairs so that $\alpha_i < \alpha_{i+1}$. Then by necessity the second coordinate is decreasing:

$$\{(\alpha_i, \beta_i)\} = \{(22, 151), (23, 98), (25, 91), (26, 84), (27, 78), (28, 77), (55, 76), (88, 75)\}$$

These first-quadrant points enclose rectangles of strictly positive values $(\alpha, \beta) \le (\alpha_i, \beta_i)$ which represent elements of $\Lambda_{gen} - \Gamma^*$, and all such elements are represented in this way. Thus the sum of areas of these rectangles counts $|\Lambda_{gen} - \Gamma^*| = \mu - \tau_{\text{gen}}$. Letting $(\alpha_0, \beta_0) = (0, 0)$, the area sum is $\sum_1^8 (\alpha_i - \alpha_{i-1})\beta_i$. Thus

$$\mu - \tau_{\text{gen}} = 22 \cdot 151 + 1 \cdot 98 + 2 \cdot 91 + 1 \cdot 84 + 1 \cdot 78 + 1 \cdot 77 + 27 \cdot 76 + 33 \cdot 75 = 8368.$$

A formula for the minimal Tjurina number

In [2] an explicit formula is given for the minimal (i.e. *generic*) Tjurina number τ_{gen} of any irreducible plane curve germ. We present the output of this formula in the case of a

2-generator semigroup, in terms of the notions defined in this article.

Theorem 19. *Let $\Gamma = \langle p, m\rangle$ as above, $\mu = (p-1)(m-1)$ the conductor of Γ, and denote by $\lfloor x \rfloor$ the floor of x. Then*

$$\tau_{gen} = \mu - \left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{(p-1)^2}{4} \right\rfloor + \left\lfloor \frac{p-1}{2} \right\rfloor + \left\lfloor \frac{p_1}{2} \right\rfloor - \sum_{i=1}^{s-1} \left\lfloor \frac{p_{i-1}}{p_i} \right\rfloor \left\lfloor \frac{p_i^2}{4} \right\rfloor. \tag{4.45}$$

The theorem follows easily from the more general formula of [2] and the fact that the multiplicity sequence of Γ is read off from the Euclidean algorithm. Note that the coefficients in the sum are the same as the numbers k_i, i.e.

$$\left\lfloor \frac{p_{i-1}}{p_i} \right\rfloor = k_i.$$

Example 23. In Example 22 we computed for the generic value set of semigroup $\Gamma = \langle 122, 281\rangle$ the formula $\mu - \tau_{\text{gen}} = 8368$. On the other hand, Theorem 19 calculates

$$\begin{aligned}
\mu - \tau_{\text{gen}} &= \left\lfloor \frac{m}{p} \right\rfloor \left\lfloor \frac{(p-1)^2}{4} \right\rfloor - \left\lfloor \frac{p-1}{2} \right\rfloor - \left\lfloor \frac{p_1}{2} \right\rfloor + \sum_{i=1}^{4} \left\lfloor \frac{p_{i-1}}{p_i} \right\rfloor \left\lfloor \frac{p_i^2}{4} \right\rfloor \\
&= \left\lfloor \frac{281}{122} \right\rfloor \left\lfloor \frac{121^2}{4} \right\rfloor - \left\lfloor \frac{121}{2} \right\rfloor - \left\lfloor \frac{37}{2} \right\rfloor + \left\lfloor \frac{122}{37} \right\rfloor \left\lfloor \frac{37^2}{4} \right\rfloor \\
&\quad + \left\lfloor \frac{37}{11} \right\rfloor \left\lfloor \frac{11^2}{4} \right\rfloor + \left\lfloor \frac{11}{4} \right\rfloor \left\lfloor \frac{4^2}{4} \right\rfloor + \left\lfloor \frac{4}{3} \right\rfloor \left\lfloor \frac{3^2}{4} \right\rfloor \\
&= 2 \cdot 3660 - 60 - 18 + 3 \cdot 342 + 3 \cdot 30 + 2 \cdot 4 + 1 \cdot 2 = 8368.
\end{aligned}$$

We have not yet investigated the path that connects these two approaches to calculating the generic Tjurina number.

A Lower Bound for the Number of Value Sets

The minimal Mancala game gives us a lower bound on the number of value sets for coprime Γ in the following way. Since we are looking for all possible increasing Γ-semimodules with the first two generators equal to p and m, we can consider any gap in Γ that is greater than $m+p$ for our Zariski invariant. This gives us our first crude lower bound. Since there are $\mu/2$ gaps in Γ, we can subtract the number of gaps in Γ that are less than $m+p$ which is rather simple to calculate.

If $m = kp + p_1$, then we note that there are only $k+1$ elements in Γ that are less than m, since $0 \in \Gamma$. Thus taking these out we have that there are precisely $m-k-1$ gaps in Γ less than m. Between m and $m+p$ there are precisely $p-2$ gaps. This gives a total of $p+m-k-3$ gaps that are less than $p+m$. We subtract this from $\mu/2$ to get that there are

$$|(\mathbb{N}_0 + m + p) \setminus \Gamma| = \frac{\mu}{2} - m - p + k + 3. \tag{4.46}$$

We can choose any element in the above set difference as our λ_1, or we could choose to throw the bead to infinity for our first move. These would all be increasing Γ-semimodules themselves, and so must be a valid Λ for some plane curve germ.

We can continue with this logic, and see that for each Zariski invariant that we choose we have a resulting u_1. For each u_1 we can play at least as many different games as there are gaps above u_1. Following along with this we can do the same for u_2 so on.

If we choose the minimal λ_1, we now have a recursive formula for the for the minimal generators in the minimal Mancala game, and so we can calculate the number of gaps above u_i remaining after each minimal move. This will give us yet another lower bound on the number of value sets there are for a coprime Γ. At the very least there are $\frac{\mu}{2} - m - p + k + 3 + |G|$ value sets for any coprime Γ where $|G|$ is the number of minimal generators. If G_i is the number of gaps above u_i after each minimal move, then a stronger lower bound on the number of Λ is given by

$$\frac{\mu}{2} - m - p + k + 3 + \sum_{i=1}^{|G|} G_i.$$

These G_i are not simple to calculate, but can be done recursively in a similar way that the $|\Lambda_{\text{gen}} - \Gamma^*|$ was calculated in Example 22. We do not attempt to give a recursive formula for the G_i at this time.

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This concludes the section on the minimal Mancala game and the generic value set of a plane curve germ with coprime semigroup. We now wish to explore the more general case of value sets of plane curve germs where we do not restrict Γ to being a coprime semigroup, equivalently we have a Puiseux characteristic of length greater than 1. We will start by giving some characterizations of the RVT code words that have only one associated value set.

4.2 Λ-Simple Code Words: A Partial Description

Let us begin this section with a definition.

Definition 39. Let W be a critical RVT code word. Then we will call W Λ-*simple* if all of the curve germs with code word W share the same Λ, i.e. the same set of valuations of differential one-forms.

We would like to know which W are Λ-simple. We will make use of the main result in Section 5 of [1] to prove the following useful lemma that will help us determine these words.

Lemma 8. *Let W be an RVT code word that ends in a critical symbol (a V or T) such that the corresponding Puiseux Characteristic is Λ-simple. Then for any $q > 0$ the Puiseux Characteristic corresponding to the word WR^qV is also Λ-simple.*

Proof. This is a straight forward application of the main theorem of Abreu and Hernandes in Section 5 of [1]. Indeed, suppose that φ is any curve germ with Puiseux Characteristic corresponding to the word WR^qV. We assume the Puiseux Characteristic to be of length g. Then any $g-1$ semi-root of φ will have code Puiseux Characteristic corresponding to the RVT code word W, with unique set of valuations of differential one-forms denoted Λ_{g-1}. Since extending the word W by R^qV results in $n_g = 2$ we have by Abreu and Hernandes that

$$\begin{aligned}\Lambda_\varphi \setminus \Gamma_\varphi = &\rho_{g-1}(\Lambda_{g-1} \setminus \Gamma_{g-1}) \dot{\cup} \{v_g - 2\delta : \delta \in \mathbb{N}^* \setminus \Lambda_{g-1}\} \\ &\dot{\cup}\{v_g + 2\delta : \delta \in \mathbb{N} \setminus \Gamma_{g-1}\}.\end{aligned}$$

Now consider any other curve germ $\tilde{\varphi}$ with Puiseux Characteristic corresponding to the word WR^qV. Then likewise for $\tilde{\varphi}$ we have that any $g-1$ semi-root will be a curve germ with Puiseux Characteristic corresponding to the word W. Thus will have the same set of valuations of differential one-forms Λ_{g-1} as any $g-1$ semi-root of φ, as W was assumed to be Λ-simple.

Again by application of Abreu and Hernandes we have that

$$\begin{aligned}\Lambda_{\tilde{\varphi}} \setminus \Gamma_{\tilde{\varphi}} = &\rho_{g-1}(\Lambda_{g-1} \setminus \Gamma_{g-1}) \dot{\cup} \{v_g - 2\delta : \delta \in \mathbb{N}^* \setminus \Lambda_{g-1}\} \\ &\dot{\cup}\{v_g + 2\delta : \delta \in \mathbb{N} \setminus \Gamma_{g-1}\}.\end{aligned}$$

Since φ and $\tilde{\varphi}$ were arbitrary, it follows that any two curve germs with Puiseux Characteristic corresponding to the word WR^qV will share the same set of valuations of differential □

Due to this lemma we can now prove the following theorem in regards to Λ-simple code words.

Theorem 20. *Any set of analytic plane curve germ with Puiseux Characteristic corresponding to a code word with isolated single V's will share the same set of valuations of differential one forms.*

Proof. We go by induction on g (equiv. by induction on the number of V's). Let $g = 1$, then we have the word R^kV, with $k > 0$. The Puiseux Characteristic of this word is $(2; 2k+1)$. Every curve germ with this Puiseux Characteristic is know to be analytically equivalent to the curve germ $(x(t), y(t)) = (t^2, t^{2k+1})$. Therefore there can only be one Λ.

Now suppose $g > 1$ and the claim holds true for $1 \leq i < g$. Note that by our inductive hypothesis any Puiseux Characteristic corresponding to a word with $g-1$ isolated V's is Λ-simple. Thus by Lemma 8 we have that so is any word with g isolated V's, as desired. □

A Special Case: The code word $(\mathrm{RV})^g, g \geq 1$ corresponds to the Puiseux Characteristic with $g+1$ generators given by $(2^g; \beta_{(1,g)}, \ldots, \beta_{(g,g)})$ where

$$\beta_{(i,g)} = 3 \cdot 2^{g-1} + \sum_{j=1}^{i} 2^{g-(j+1)}$$

One can construct this series iteratively. We have $\beta_{(i,g)} = 2 \cdot \beta_{(i,g-1)}$ for $1 \leq i \leq g-1$ and $\beta_{(g,g)} = 2 \cdot \beta_{(g-1,g-1)} + 1$. We have the immediate corollary of Theorem 20:

Corollary 4. *For any $g \geq 0$, the word (RV)^g corresponds to a class of plane curve germs $\varphi\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ which all share the same set of valuations of differential one forms. These plane curves germs are precisely the curves that have the above Puisuex characteristic.*

Further Characterizations: One can check from Example 9 that the code words RVT and RVV are Λ-simple. We to wish give yet another infinite set of code words that are Λ simple.

Proposition 21. *Let W be any critical code word with one critical block. That is, let C be an entirely critical word of only V's and T's and let $W = R^q C$. Then the code word WRV is also Λ-simple.*

Proof. We need to show that all curve germs with word WRV have the same Λ. Let Γ be the semigroup associated to WRV and therefore Γ_1 the word associated to W. Since W has only one critical block there exists $p_1,\ m_1$ coprime such that $\Gamma_1 = \langle p_1, m_1 \rangle$. This implies $\Gamma = \langle 2p_1, 2m_1, 2p_1m_1 + 1 \rangle$. Let γ be any plane curve germ with word WRV. Then we note that there is some parametrization of γ such that

$$\gamma:\ x(t) = t^{2p_1},\ \ y(t) = t^{2m_1} + at^{2m_1+1} + h.o.t.$$

Consider the quasi-homogeneous curve given by $Y^{p_1} - X^{m_1} = 0$. We have

$$y(t)^{p_1} - x(t)^{m_1} = at^{2p_1m_1+1} + h.o.t.,$$

which implies $v_f(y^{p_1} - x^{m_1}) = 2p_1m_1 + 1$. It is therefore clear that the curve (t^p, t^m) will be a 1-semiroot of γ. Since γ was arbitrary, this must be true of any curve with critical word WRV. Since we are extending W by an RV, we get a complete description of our value set of any curve germ with word WRV from [1]. Since all the curve germs with this word share the same 1-semiroot we can use this semiroot to completely characterize our Λ, and hence they must all be equal. This implies the word WRV is indeed Λ-simple. $\square$

For W, and $\tilde{W}$ RVT words, let us from now on use the notation $W \cdot \tilde{W} := W\tilde{W}$. With this notation we have the following corollary.

Corollary 5. *Let W be any critical code word with a single critical block. Then the word*

$$WRV \cdot \Pi_{k=1}^{n} R^{q_k} V$$

with $q_k > 0$ is Λ-simple for all $n \geq 0$.

Proof. This is a direct result of Lemma 8, and the last proposition. □

One can speculate if there are more ways for us to construct Λ-simple code words. The author speculates that there indeed are. One such type of word that is worthy of note is the word given by

$$R^q \cdot (\Pi_{k=1}^{n} RC_k) \cdot RV,$$

where the C_k are entirely critical, $q \geq 0$, and $n \geq 1$. Any curve germ γ with this word will admit in some coordinates a parameterization a nth-semiroot of the form

$$\gamma_n : \ x_n(t) = t^{p_n}, \quad y_n(t) = t^{m_n} + \sum_{k=2}^{n} a_k t^{\beta_k/2} + h.o.t.,$$

where the β_k are the usual PC generators for γ and $p_n = p/2, \ m_n = m/2$. In other words, there is no room for nonessential exponents in between the essential ones for γ_n. To show that the word $WRV \cdot \Pi_{k=1}^{n} R^{q_k} V$ is Λ simple then comes down to showing that for any choice of the a_k the curve $(t^{p_n}, t^{m_n} + \sum_{k=2}^{n} a_k t^{\beta_k/2})$ has the same value set, where we intentionally leave out the higher order terms.

Though the author currently has no formal proof of this, it is highly plausible that this is the case when one considers that a value set is generated using some notion of weighted homogeneous coordinates, and collisions as those presented in the Mancala game. In that light we give the conjecture that in the future we hope to formally prove.

Conjecture 1. *The word*

$$R^q \cdot (\Pi_{k=1}^{n} RC_k) \cdot RV$$

is Λ-simple for any C_k entirely critical, $q \geq 0$, and $n \geq 1$.

To end this section on Λ-simple code words, we wish to give a proof of some sort of a converse to our above theorems. That is, we wish to give a condition for a critical RVT code word so that it is not Λ-simple. We do so with the following conjecture.

Conjecture 2. *Let W be any regular (ending in R) code word of length greater than 2, and C an entirely critical word of length greater than 1. Then WC is not Λ-simple.*

We present some evidence to support our conjecture. If there is only one critical block, namely C, we then refer the reader to Theorem 18 which shows that with our conditions on C and W we must have $p > 2$, and enough R's to ensure that there is a Zariski invariant for the generic value set that is less than ∞. We then note that the quasi-homogeneous value set for WC indeed has that the Zariski invariant is ∞, and so there must be at least 2 value sets. Thus WC is not simple in this case.

Now suppose WC has more than one critical block. Suppose WC has corresponding PC given by $(p; m, \beta_2, \ldots, \beta_g)$. We note that in this case, since $|C| > 1$ we must have $e_{g-1} > 2$. We also note that e_{g-1} is in fact the multiplicity of any curve germ with code word $R^j C$, for any $j > 0$, and that $\beta_i \equiv 0 \pmod{e_{g-1}}$ for any $0 \leq i \leq g-1$, but not for β_g.

Suppose first that $\beta_g \not\equiv -1 \pmod{e_{g-1}}$. Then we claim that there exists some $a \neq 0$ such that the two curves given by

$$\gamma: \ x = t^p, \quad y = t^m + \sum_{i=2}^{g} t^{\beta_i}, \tag{4.47}$$

$$\tilde{\gamma}: \ \tilde{x} = t^p, \quad \tilde{y} = t^m + \sum_{i=2}^{g} t^{\beta_i} + a t^{\beta_g + 1} \tag{4.48}$$

have different value sets. That is, there is some value of a so that either $\nu_g + p + 1$ is in the value set of γ and not in $\tilde{\gamma}$ or vice-versa.

Let us prove this for the case where WC has only two critical blocks. Let us first start with the case where $\beta_2 = m + 1$. In this case we have the two curves

$$\gamma: \ x = t^p, \quad y = t^m + t^{m+1}, \tag{4.49}$$

$$\tilde{\gamma}: \ \tilde{x} = t^p, \quad \tilde{y} = t^m + t^{m+1} + a t^{m+2}. \tag{4.50}$$

We have that our first semiroot for γ is

$$s = y^{p/e_1} - x^{m/e_1} = \frac{p}{e_1} t^{\frac{pm}{e_1}+1} + \binom{\frac{p}{e_1}}{2} t^{\frac{pm}{e_1}+2} + h.o.t.$$

Whereas

$$\tilde{s} = \frac{p}{e_1} t^{\frac{pm}{e_1}+1} + \binom{\frac{p}{e_1}}{2} t^{\frac{pm}{e_1}+2} + \frac{ap}{e_1} t^{\frac{pm}{e_1}+2} + h.o.t.$$

We now start to consider the valuations of one-forms. We have that $v(dx) = v(d\tilde{x}) = p$, $v(dy) = v(d\tilde{y}) = m$ and $v(ds) = v(d\tilde{s}) = pm/e_1 + 1$.

Our next step is to use the Zariski one-form to see that

$$\omega = pxdy - mydx = pt^{m+p}dt$$

and

$$\tilde{\omega} = pt^{m+p} + 2pat^{m+p+1}dt.$$

Thus both $v(\omega) = v(\tilde{\omega}) = m + p + 1$. Let us now consider that all of the valuations in either value set that are between 0 and $pm/e_1 + 1$ can be obtained either from monomials in x and y times dx and dy, or monomials times ω or $\tilde{\omega}$. This is due to the fact that $v(s) = pm/e_1 + 1$

is the first time we can obtain a valuation that is not a result of a valuation of a monomial in x and y, and that therefore every collision of one-forms between 0 and pm/e_1+1 is a result of monomial one-forms with valuations $v(x^iy^jdx)=v(x^ky^ldy)$. Going through what the equality implies we see that if for some $b,c\in\mathbb{C}$ $v(bx^iy^jdx-cx^ky^ldy)>v(x^iy^jdx)$ we must have $bx^iy^jdx-cx^ky^ldy=(x^ry^s)\omega+h.v.t.$, where $h.v.t.$ stands for higher valued terms.

There can be no collisions of one forms that are a result of monomials times z and monomials times dx or dy that are less than pm/e_1+1. This is key to our argument, and comes from the fact that we do not have matching congruency of any of those monomial one forms (mod e_1) until we reach ν_1+p. Therefore all values of Λ and $\tilde{\Lambda}$ that are less than pm/e_1+1 consist precisely of the values in $\langle p,m\rangle^*\cup(\langle p,m\rangle+m+p+1)$ that are less than pm/e_1+1. Here we note that $\Gamma\neq\langle p,m\rangle$.

However, once we have gone past pm/e_1+1 we find that $v(y^{p/e_1-1}z)=\nu_1+p=v(sdx)$. This will be the place where we will find our minimal collision of one-forms that are not exact, and resolve them to get our next generator. We consider the form

$$\omega_2=pxds-(pm/e_1+1)sdx=p\binom{\frac{p}{e_1}}{2}t^{pm/e_1+p+1}dt+h.o.t.dt,$$

and similarly

$$\tilde{\omega}_2=\left(p\binom{\frac{p}{e_1}}{2}+\frac{ap^2}{e_1}\right)t^{pm/e_1+p+1}dt+h.o.t.dt,$$

for our curve $\tilde{\gamma}$. For the value $a=-\frac{e_1}{p}\binom{\frac{p}{e_1}}{2}$ we have $pm/e_1+p+2\notin\tilde{\Lambda}$. This is due to the fact that for this value of a we have $v(\tilde{\omega}_2)>pm/e_1+p+2$, and the fact that there is no other way to get a value of $\tilde{\Lambda}$ that is congruent to 2 (mod e_1). It is immediate from our assessment as well that there is no other way to form one-forms of valuation pm/e_1+p+2 as we would need to find a collision that was less than this, and all of these involve the Zariski one-form or $\tilde{\omega}_2$.

On the other hand it is clear that $pm/e_1+p_2\in\Lambda$ for our curve γ, and from the argument above, we must have that pm/e_1+p+2 is a minimal generator for Λ, since $e_1>2$. This concludes our argument for the case where $\beta_2=m+1$.

The case for which $\beta_2>m+1$ is not so different in terms of computations and arguments about minimal collisions, only this time we argue that any nonzero value of a will result in the opposite effect for our $\tilde{\Lambda}$ versus our Λ. That is, in this case if $a\neq 0$ we have that $pm/e_1+\beta_2-m+p+1\in\tilde{\Lambda}$ and $pm/e_1+\beta_2-m+p+1\notin\Lambda$. This gives us the following proposition

Proposition 22. *If W is a regular code word of length greater than or equal to 2 containing at most one critical block, and C is an entirely critical word of length greater than 1 then WC is not Λ-simple.*

Let us finish this section with an example:

Example 24. The word $RVRVT$ is not Λ-simple. Let us show this using the two forms we used above. The code word corresponds to the semigroup $\Gamma = \langle 6, 9, 19 \rangle$, and $e_1 = 3$. We consider the parameterizations

$$\gamma: \; x = t^6, \quad y = t^9 + t^{10}, \tag{4.51}$$

$$\tilde{\gamma}: \; \tilde{x} = t^6, \quad \tilde{y} = t^9 + t^{10} - \tfrac{1}{2}t^{11}. \tag{4.52}$$

We note that $v(\omega) = v(\tilde{\omega}) = 16$ and so

$$(\Gamma^* \cup (\Gamma + 16))_\mu = \{0, 6, 9, 12, 15, 16, 18, 19, 21, 22, 24, 25, 27, 28, 30, \\ 31, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42\}$$

We have that $\omega_2 = 6t^{25}dt$, whereas $\tilde{\omega}_2 = -12t^{26}dt + \frac{9}{2}t^{27}dt$. Since there is no other way to form a collision of the appropriate valuation we conclude that they must have differing value sets, since $26 \in \Lambda$, and not in $\tilde{\Lambda}$. Therefore the word $RVRVT$ is not Λ-simple.

One can conclude via further analysis that the code word has in fact 4 different value sets. This requires the use of yet another coefficient for the power of t^{13} in the parametrization of $\tilde{y}(t)$. We will not go further into the computations to show these results in this example.

This concludes our example, and section on Λ-simple code words. We look forward to working more on formalizing proofs for the conjectures in this section at a later date. With this section concluded we also have come to the send of our chapter on value sets of singular plane curve germs. We now move on to the invariants of singular curves at the first level of the Monster Tower. We have previously named these contact curve germs or Legendrian curve germs. We devote the next chapter to results concerning the discrete contact invariants of the contact curves.

Chapter 5

Legendrian Semigroups

5.1 Introduction

5.1.1 Goals and Main Questions of the Chapter

It has already been established that for a contact class of Legendrian Curve germs there is a well-defined RVT code word for any contact projection (see directional blowup in Chapter 8 of [19]).

We start with the notion of directional blowup at the first level (see Definition 12), and define what we mean by a contact projection of a Legendrian curve germ. We recall that directional blowup of any plane curve germ at any level n in the Monster Tower gives us a pair of active coordinates $(x^{(i)}, y^{(j)})$. One can ask, what are the invariants attached to directional blowups under symmetries of the Monster tower. By [19] we then really only need to ask what is invariant for the directional blowup(s) of a curve under lifted contact symmetries of the first level. A natural discrete invariant to study is then the semigroups, or valuations of functions, of Legendrian curve germs.

Our first task is to consider Legendrian curve germs in level 1 of the Monster $M(1) = \mathbb{C}^2 \times \mathbb{C}P^1$. We would like to define what a contact projection is, and do so in a coordinate free way. Before we do this, let us first consider a Legendrian curve $\gamma(t) = (x(t), y(t), y'(t))$, where $y'(t) = dy/dx, v(x) < v(y'(t)) < v(y)$. In general, what we are thinking of when we consider a contact projection is the plane curve germ given by $(x(t), y'(t))$, which is what we would call the directional blowup of the plane curve germ $(x(t), y(t))$ at level one.

There are many other ways to project a Legendrian curve onto a plane. We would like to select a set of them that has a special property.

Definition 40. Let $\gamma\colon (\mathbb{C}, 0) \to (M(1), 0)$ be a Legendrian curve germ with respect to the contact distribution defined by contact form α. A *contact projection* of γ is a projection down to a $\mathbb{C}^2$ in any direction that is locally transverse to the contact plane.

In general we will start with coordinates so that $\alpha = dy - y'dx$ and the chosen contact projection is given by $(x(t), y'(t))$. We can also make things more convenient by choosing representatives of contact germs that have a nice form in x and y'. This is really only to help us cleanly work with the curves in our proof.

We have now reached one of the main question of this chapter: what, if any, are the invariants of a contact projection of a Legendrian curve? That is, if we are given a class of Legendrian curve germs under contact transformations and choose two different representatives of this class, what information might a contact projection of one curve share with another? Perhaps more restrictively, if we consider the contact projection (x, y') of our curve germ γ and apply any local contact transformation Φ to γ, what does the new curve $\Phi((x(t), y'(t))$ have in common in terms of the analytic type of the two contact projection plane curves?

We will attempt to answer the last question, at least partially, in this chapter. For now let us state what appears to be the case. Firstly by counter example we can show that the analytic type of the contact projection (x, y') is not a contact invariant. We can even demonstrate that the value set of the contact projection (x, y') is also not an invariant. All would seem lost then. However, there seems to be some discrete analytic information, that is information beyond just the semigroup of the contact projection, that is a contact invariant of the contact projection (x, y'). Namely in any case the Zariski invariant λ of the contact projection is preserved. We also find that the $(\lambda + p - 1)$-jet of the contact projection is preserved up to analytic equivalence. In the case where we find the topological class of contact projections is given by $(p; m)$ with $\gcd(p, m) = 1$ the latter result gives further results in terms of value sets and what information in them is preserved via contact transformation.

The general case still has more room for exploration. Using similar methods γ should have some similar statements available in relation to each critical block defining the Puiseux characteristic of a (and hence any) contact projection.

5.1.2 More Questions

A few questions to keep in mind while we explore the different possibilities for our contact projections: Is there a notion of "generic" value set for any possible choice of Zariski invariant, given a topological class? If so, at least in the case where our PC is of length 1, does this value set form a semigroup if we adjoin 0? If it does or does not, how close is it to the valuations of $\Gamma(\gamma)$, the Legendrian semigroup of γ?

We could take this even further, and ask, what is the generic value set of a given jet of a curve? That is, if we are give an r-jet of a plane curve germ, what are the possible value sets for any curve germ with that r-jet?

For the value sets are semigroup questions, there are cases where the value set of a curve with 0 is not a semigroup. This is because of the fact that thanks to P. Almirón we can take any increasing semimodule in the coprime case, and it will correspond to some curve [3]. In light of this, let $\Gamma = \langle 5, 31 \rangle$ and take the Γ-semimodule $\Lambda = \Gamma^* \cup (\Gamma + 47)$. This

Γ-semimodule does not contain $2 \times 47 = 94$. Hence it cannot become a semigroup simply by adjoining 0 to it. This somewhat suggests that we need to take some sort of "minimal jump" to resolve collisions of valuations of one-forms, once we have obtained our Zariski invariant, in order to form a semigroup.

We now head to our next section to give a counterexample to show that the value set of a contact projection is not necessarily preserved under contact transformation.

5.2 Counterexample to Invariance of Λ

We are looking for a case where two Legendrian curves in the same contact class will yield different value sets for their contact projections. Let us provide an example where this is the case using the topological class $(5;31)$ for our contact projections.

Example 25. Consider the Legendrian curve germ with respect to $\alpha = dy - y'dx$ given by $x(t) = t^5, y(0) = 0$ and

$$\begin{aligned} y'(t) =& t^{31} + t^{32} + \frac{63}{62}t^{33} + \frac{3,008}{2,883}t^{34} + \frac{771,875}{714,984}t^{35} \\ &+ \frac{5,189,184}{4,617,605}t^{36} + \frac{24,232,125,907}{20,612,988,720}t^{37} \\ &+ \frac{49,317,194,752}{39,937,665,645}t^{38} + \frac{22,921,120,093,113}{17,608,073,031,040}t^{39}. \end{aligned}$$

We will explain the complicated coefficients later, but let us consider for now the contact transformation that takes $y' \mapsto y' + y + xy'$, and leaves x alone. By [5] this is a valid contact transformation. Using Delorme's Lemma 14 in [11] and coordinated mancala game (or Delorme's algortihm) we can find the value set of our first contact projection $c_1 := (x, y')$ to be given by $\Gamma^* \cup (\Gamma + 37) \cup (\Gamma + 74)$.

On the other hand the value set of $c_2 := (x, y'+y+xy')$ is given by $\Gamma^* \cup (\Gamma+37) \cup (\Gamma+69)$. One can quickly check that 69 is not in the value set of c_1.

Let us look more closely at this example and try to extract a bit of what is going on here. Clearly we have a very complicated set of coefficients for $y'(t)$, and this would make one think that we would need to search far for any example of this. In reality the goal was clear from the outset: form a curve that has a value set with a small (in this case smallest) Zariski invariant, and from there make it rather sparse. In terms of the mancala game this would be like throwing the beads further at the next collision of valuations of one forms. We again know that we can do this by [3], which also shows that forming a value set for the coprime case is an increasing, step-by-step process.

The next task is to find a curve with this value set, and in particular one that has achieved this value set using coefficients of powers of t in $y'(t)$ that are greater than $31 + 5 = 36$. This is due to the claim that contact transformations locally cannot change the

$(m+p-1)$-jet of a contact projection up to analytic equivalence. This also means that we need to have enough gaps so that we can jump high enough above our next collision. This relates to how the jump sizes are made, as the size of each jump corresponds to the number of successive coefficients that need to be fixed in order to make that jump in valuation past that collision. Recall from Definition 35 (also see Delorme in [11]) the u_i values that we call minimal collisions, and the λ_{i+1} values that we call minimal generators. Here the jumps we mention are given by $j_i = \lambda_{i+1} - u_i$.

What we now have done is formed a rather non-generic value set for the given Zariski invariant, and done so by choosing non-generic coefficients for $y'(t)$. Since we cannot change the $(m+p)$-jet, we try to use y to change the higher order coefficients, all the time knowing that we will move important non-generic coefficients that affect value set of our curve by construction. Though complicated in explanation, one can assume that with rather large multiplicity p one can do this quite regularly. Either way we have shown that the value set is not an invariant under contact transformation.

With our counterexample above we are ready to state our first result of this chapter.

Proposition 23. *Let γ and γ' be two contact equivalent Legendrian curve germs. It is not necessarily the case that the value sets of contact projections of each curve will be equal.*

We now move on to finding what invariants we can given the contact projection (x, y') and any contact transformation of the Legendrian curve germ.

5.3 Contact Invariants of The Contact Projection

This section is dedicated to proving several facts about the invariants of contact projections of a single class of Legendrian curves. We will attempt to start with great generality, then move to the case where the contact projections have coprime PC. Finally we will devote a bit more time to the general case. We also see if that there is more about the generic case that we can say.

We so far have seen that the analytic type of a contact projection is not an invariant under contact transformations. We will now prove a theorem that gives us a partial description of what are the invariants of the set of contact projections.

Theorem 21. *All contact projections of any single class of Legendrian curve germs have the same $\lambda + p - 1$-jet up to analytic equivalence, where λ is the Zariski invariant and p is the multiplicity of any contact projection in that class.*

Proof. Let us start with a Legendrian curve germ γ of the form

$$x(t) = t^p, \quad y'(t) = t^m + t^\lambda + \sum_{\substack{i>\lambda \\ i\notin\Lambda-p}} a_i t^i, \quad y(0) = 0.$$

Here we assume (by abuse of notation) that $y' = dy/dx$ and that Λ is the set of valuations of one-forms on the contact projection (x, y'). Using Neto's description in [6] of contact transformations we see that we need only consider transformations of the form

$$(x, y') \mapsto (x, y' + a(y + xy') + h.v.t.) =: (x, \tilde{y}'),$$

where higher valued terms are to be understood as terms of higher valuation than $v(y) = m+p$. It is important to note that any polynomial in x and y' alone is incapable of changing the $\lambda + p$-jet beyond analytic equivalence. Thus it could only be terms involving nonzero powers of $y(t)$ that could possibly have an effect on the jets of our contact projection.

We have by integration that

$$y(t) = \frac{p}{m+p} t^{m+p} + \frac{p}{\lambda + p} t^{\lambda+p} + p \sum_{\substack{i>\lambda \\ i \notin \Lambda - p}} \frac{a_i}{i+p} t^{i+p}$$

We will first consider the simple case where the h.v.t. are zero in our contact transformation. Since $\tilde{y}' = y' + a(y + xy')$ in this case, we haver that the only power of t added to y' that is less than $\lambda + p$ is t^{m+p}. We also have that $m + p$ is in the semigroup of the contact projection $(x, \tilde{y}')$ and therefore can be removed by the analytic isomorphism $(x, \tilde{y}') \mapsto (x, \tilde{\tilde{y}'} - a(\frac{p}{m+p} + 1)x\tilde{y}')$.

Of course now we have introduced an $m + 2p$ term from our original $y(t)$, but this is also in the semigroup. All other terms that we have added are powers of t greater than $\lambda + p$. We can continue to remove these terms with $x^2\tilde{y}', x^3\tilde{y}'$ etc. until we have reached $\lambda < m + kp < \lambda + p$. In this case we can do our final adjustment to this coefficient again using $x^k\tilde{y}'$ all the while only adding in terms higher than $\lambda + p$ otherwise. In this way we have gotten back identically to our original $\lambda + p - 1$ jet of our original contact projection, all through analytic isomorphism. This proves our theorem in the case where the h.v.t. are all 0.

It is not difficult now to see that a similar argument holds if we were to add higher powers of $y(t)$ to our original contact projection. For instance y^2 has valuation $2m + 2p$ and the next power of t is $\lambda + m + 2p$. Thus a nearly identical argument holds as above for any power of y. Similarly any polynomial given $yP(x, y, y')$ will have the same property, and the argument still holds. This is essentially all possible contact transformations (even really all local analytic transformations of our space $M(1)$) of γ, and so we have proved our theorem. $\square$

Notice that in our proof we have shown that it is impossible to affect the $\lambda + p - 1$-jet by contact transformation, so it is impossible to change analytic equivalence of the λ-jet as well. This implies all contact projections of a single class do have a well-defined Zariski invariant. This gives us the following corollary.

Corollary 6. *The Zariski invariants of any contact projections of any Legendrian curve germs of the same contact class are equal.*

In [10] Delorme gives a criterion for which a plane curve germ with a coprime PC has a generic value set Λ_{gen}, which can be inferred by his lemma 14 in his later work [11]. He states in [10] that for a curve germ with coprime PC to have generic value set, the coefficients of its parameterization between m and $m+p$ must satisfy a certain weighted homogeneous polynomial equation. Namely this given polynomial must be nonzero when evaluated at the coefficients. This is telling us that the generic value set is determined at most by the $m+p$ jet of the plane curve germ. We now can state the following corollary:

Corollary 7. *If a contact projection of Legendrian curve has a two generator semigroup and the generic value set for its topological class, then every contact projection of every curve in the same class has generic value set of the same topological class.*

Proof. This follows from Delorme's generic polynomial in [11], and the theorem above, along with the work by R&Z in [19], which assure us that the topological class of a contact projection is a contact invariant. □

Note that Theorem 18 gives a recursive formula for the generic value set of a contact projection of the above type.

5.3.1 The General Generic Case

We would like to have a similar statement for the general PC generic value set case, but this would require us to know something more about how the generic value set in these cases depends on the coefficients of a parameterization of y'. There is an algorithm to determine the generic value set of any PC given by [21]. In this article the author does give some criterion for the coefficients, and it may be beneficial to dig deeper into this paper.

There is also the consideration of the fact that any parameterization of $y(t)$ has only nonzero coefficients in the same congruency class as $y'(t) \pmod p$ that are of higher order precisely by adding p. It is strongly suspected that the generic coefficients that generate the generic value set have already been determined by the coefficients of y'. It would seemingly be difficult for a higher order term of the same congruency class (mod p) to somehow change the outcome of the generic case. One should think that is the lowest order term of that congruency class that will determine if the plane curve germ is possibly in the generic case.

This leads us to the conjecture that Corollary 7 is in fact true in all cases, but Theorem 21 no longer helps us prove this. We may need instead a stronger version of Theorem 21 that dives more deeply into how the coefficents of $y(t)$ can affect the coefficients of $y'(t)$ close to the essential exponents of $y'(t)$ in terms of Puiseux Characteristic.

Regardless of the speculation for the general case, from this section we now have the ability to discuss what a Zariski invariant of a contact curve might be. The next section is dedicated precisely to this idea.

5.4 The Zariski Invariant and Zariski One-Form of Contact Projections

In general, analytic plane curve germs (x, y) have a Zariski invariant. One definition of the Zariski invariant is the exponent of the smallest nonzero power of t in the Zariski short parameterization of the curve that is greater than m, where m is again the order of vanishing of y. This is very coordinate dependent, so another, perhaps better definition is as follows:

Definition 41. Let γ be a singular plane curve germ at the origin with semigroup Γ and value set Λ. Then the *Zariski invariant* of the curve is given by $\lambda := \min\{\Lambda \setminus \Gamma^*\} - p$, or ∞ if $\Lambda \setminus \Gamma^* = \emptyset$

5.4.1 Directional Blowup and the Zariski Invariant

Let us now continue this section with the assumption that $2v(x) < v(y)$, or in RVT language, a code word with two R's at the beginning. We would now like to state some facts about the Zariski invariant of a contact projection, given a Legendrian curve germ. As we have shown in Theorem 21 the set of contact projections of any class of Legendrian curve all share the same $\lambda + p - 1$-jet up to analytic equivalence, and so they must share the same Zariski invariant. This is the case since any contact projection of the given class must be equivalent to a curve germ of the form $x(t) = t^p, y'(t) = t^m + t^\lambda + h.o.t.$, which clearly has λ as its Zariski invariant (see Cor. 6).

Since part of this chapter is intended to address invariants of directional blowup, let us first ask the question: Given a plane curve germ (x, y) and the Legendrian lift (x, y, y'), does the Zariski invariant of (x, y) determine the Zariski invariant of (x, y'), or vice-versa? Unfortunately the answer to both of these questions is no. Let us now see why this is the case.

First we give a general example to illustrate that we do not necessarily know the Zariski invariant of the plane curve germ we started with, given the Zariski invariant of any contact projection. We do so by showing contact equivalence of two lifted plane curve germs that do not have the same analytic type.

Example 26. Let us start with the topological class $(p; m)$. Within this class, there are always at least two analytic types whenever $p \geq 3$ and we avoid smaller values of m. These two types we will denote γ_f for Frobenious, and γ_{QH} for the quasi-homogeneous.

The defining trait for γ_f is that $\Lambda_f \setminus \Gamma_f = c-1$ the maximum integer Zariski invariant for the topological class. The quasi-homogeneous curve is defined by the property that $\Lambda = \Gamma^*$. According to [13] γ_f is completely determined analytically by its Zariski invariant, hence its value set, and so there are no moduli for γ_f. Let us take the normal form representatives of [13] for our curves:

$$\gamma_f : \quad x = t^p, \quad y = t^m + t^{c-p-1}$$

and

$$\gamma_{QH}: \quad x = t^p, \quad y = t^m$$

We now show that the lifts of these two curves are in fact contactomorphic. We have

$$\gamma_f^1: \quad x = t^p, \quad y = t^m + t^{c-p-1} \quad y' = \frac{m}{p}t^{m-p} + \frac{c-p-1}{p}t^{c-2p-1}$$

$$\gamma_{QH}^1: \quad x = t^p, \quad y = t^m \quad y' = \frac{m}{p}t^{m-p}$$

Let us consider the contact projection of γ_f given by $(x, y') = (t^p, \frac{m}{p}t^{m-p} + \frac{c-p-1}{p}t^{c-2p-1})$. A quick check show that the valuation of $pxdy' - (m-p)y'dx$ is given by $c - p - 1 = mp - m - 2p \geq mp - m - p^2 + 1 = c'$ the conductor of any contact projection. Therefore there is an analytic transformation of the plane taking the curve

$$(t^p, \tfrac{m}{p}t^{m-p} + \tfrac{c-p-1}{p}t^{c-2p-1}) \mapsto (t^p, \tfrac{m}{p}t^{m-p}).$$

We can also use local analytic transformations of $M(1)$ (of 3-space) to take the curve $\gamma_f^1 \mapsto \gamma_{QH}$ since the t^{c-p-1} term can be moved away due to $c-p-1 > c'$. By Zhitomirskii's lemma in [27] we have that the two curves must therefore also be contactomorphic. This concludes our example.

What we have shown in this example is that there are two different analytic types of plane curve germs that both lift to Legendrian curve germs that have ∞ for the Zariski invariant of any contact projection. These two analytic types of plane curve germs have themselves different Zariski invariants. It is therefore impossible to tell which Zariski invariant we started with.

Now we give a short example to show that the both directions fail even if the Zariski invariant of any contact projection is not infinite..

Example 27. Consider the curves

$$\gamma_1: \ x = t^4, \quad y = t^{13} + t^{18} + at^{23}, \ a \notin \left\{0, \frac{18}{13}, \frac{31}{26}\right\}, \tag{5.1}$$

$$\gamma_2: \ x = t^4, \quad y = t^{13} + t^{23}. \tag{5.2}$$

For our first curve $\Lambda_1 \setminus \Gamma_1^* = \{22, 31, 35\}$, and for our second curve $\Lambda_2 \setminus \Gamma_2^* = \{27, 31, 35\}$. Going over to the usual contact projection (x, y') of the lift of each plane curve above we find that they both have Zariski invariant 19. This implies they both have contact projections that are equivalent to the Frobenious curve, which also implies that we cannot determine the Zariski invariant of our original curve even if the Zariski invariant of any contact projection is finite.

Now consider the curve

$$\gamma_3: \quad x = t^4, \quad y = t^{13} + t^{18} + at^{19}, \ a \neq 0.$$

This curve has Zariski invariant 18, as does γ_1. However, this curve does not have the same value set, as it contains 27. Upon lifting the curve and taking the contact projection (x, y') we find that the Zariski invariant is 15, rather than that of γ_1', which is 19 . This shows we cannot even hope to determine the Zarsiki invariant of the contact projection if we know the Zariski invariant of the original plane curve.

The issue here arose when we had that the Zariski invariant of the plane curve was already in the semigroup of the contact projection. One then can say that in this case we generally cannot tell what the Zariski invariant of the contact projections will be given the Zariski invariant of the original plane curve germ.

Let us denote the semigroup of our plane curve germ as Γ and the semigroup of any contact projection Γ'. We would now like to show that if the Zariski invariant λ of our original curve is such that $\lambda \notin \Gamma'$ then we do in fact have that the Zariski invariant of any contact projection is determined by the Zariski invariant of our original plane curve germ. Indeed we can always find coordinates for which our curve γ^1 has the form

$$\gamma^1: \quad x = t^p, \quad y = t^m + t^\lambda + h.o.t., \quad y' = \frac{m}{p}t^{m-p} + \frac{\lambda}{p}t^{\lambda-p} + h.o.t.$$

It follows that the Zariski 1-form for the contact projection (x, y') is given by

$$pxdy' - (m-p)xdy' = \lambda(\lambda - m)t^{\lambda-1} + h.o.t.\ dt,$$

which has valuation $\lambda \notin \Gamma'$. This implies $\lambda - p \notin \Gamma'$. Since the Zariski 1-form has given us a valuation outside of Γ' it must be the minimal one in Λ', the value set of (x, y'). This implies $\lambda - p$ is the Zariski invariant of any contact projection by Corollary 6. This gives us the following result:

Proposition 24. *Let γ be a plane curve germ with semigroup Γ, so that any contact projection has semigroup Γ'. Further suppose the Zariski invariant λ of γ is such that $\lambda \notin \Gamma'$. Then the Zariski invariant of any contact projection is given by $\lambda - p$, where p is the multiplicity of γ.*

In general there seems to be no reverse direction that we can formulate, at least at this time. We would then like to move on to how the Zariski invariant of any contact projection relates to the Legendrian semigroup of the original Legendrian curve germ.

5.4.2 The Zariski Invariant of a Contact Projection and The Semigroup of the Legendrian Curve Germ

In this section we would like to start to address the question of how much of the Legendrian semigroup of a Legendrian curve is determined by value set information of a contact projection. We know here that we have to be careful, as not every contact projection has the same value set. However, given Theorem 21 we certainly retain some analytic information.

One piece of this information is the Zariski invariant of any contact projection. We would now like to state a few basic properties in hopes that we can say something about how the possible value sets of the set of contact projections relate, if at all, to the Legendrian semigroup. Let us begin with a proposition.

Proposition 25. *Let $\gamma = (x, y, y')$ be a Legendrian curve germ with semigroup Γ^1 such that $v(x) = p,\ v(y') = m$. Then the Zariski 1-form of the contact projection $\gamma' = (x, y')$ is an exact form in the set of differential 1-forms on the Legendrian curve. Therefore the value $\lambda' + p$ is in the Legendrian semigroup where λ' is the Zariski invariant of any contact projection of γ.*

Proof. We acknowledge the fact that $y'dx = dy$. Therefore if the Zariski 1-form of γ' is given by $pxdy' - my'dx$ then we find that taking the differential of the regular function $z := pxy' - (m + p)y$ we get

$$dz = d(pxy' - (m + p)y) = pxdy' - my'dx.$$

Thus we have shown that the Zariski 1-form is an exact differential. Should we find a contact projection such that the Zariski 1-form has valuation outside Γ' our contact projection semigroup (which is always possible to find unless $\lambda' = \infty$), then by exactness of our Zariski 1-form, it follows that $\lambda' + p$ must be in the Legendrian semigroup. $\square$

If we restrict our set of Legendrian curves to those that have contact projections with coprime semigroup $\Gamma = \langle p, m\rangle$, then we can use Lemma 5 (12(b) in[11]) to help us determine if we can continue this argument for other minimal generators of the semimodule Λ' and their relation to the Legendrian semigroup of γ.

We see that Lemma 5 provides us with some useful information about our next minimal generator of Λ'. Indeed we must now have that the next generator has a term in it of the form $x^i y'^j dz$. We recall from the previous chapter that $v(dz) = pm - ap - bm$ for some $a, b > 0$. It follows that the next possible minimal collision in Λ' must occur between $x^i y'^j dz$ and something in Γ'. If we suppose that $a < b$ then we must have that $v(x^a dz) = d(x^a z) = pm - bm = m(p - b) = v(d(y'^{p-b}))$. Hence there must be some linear combination of $x^a z$ and y'^{p-b}, such that the valuation of this linear combination is greater than $v(y'^{p-b}) = (p - b)m$. There is also a linear combination of $x^a dz$ and $y'^{p-b-1}dy'$ that has a valuation greater than $(p - b)m$ as well.

The question becomes: If these two linear combinations have valuations that are minimal generators of the Legendrian semigroup and value set Λ', respectively, when are they equal? If the valuations do agree we have found our next minimal generator of Λ' according to Delorme. It is also our next minimal generator of our Legendrian semigroup. This is due to the fact that powers of y cannot generate anything that powers of xy' themselves cannot generate. Thus we are again looking for valuations of functions on γ that agree, but this clearly must involve z. It is clear that we have involved z above in the most minimal way.

We can continue according to Delorme's Lemma 12(b) for each new generator of Λ', as long as we find that the last generator was exact in the differentials of the Legendrian curve γ. The arguments follow along closely to the above, and we note that if the last generator was from an exact form, then our next minimal collision will be obtained using the same monomials, whether in the one-forms of γ' or the regular functions on γ. We note that by Example 25, we certainly cannot have that this process works out in our favor every time. There must be cases where our generators of our Legendrian semigroup do not agree with the generators of Γ'. Thus the word *when* is appropriate in our line of questioning. We will speculate on this in the following chapter.

Chapter 6

Conclusion and Further Questions

6.1 Summary of Results

In summary this work has dedicated itself to several novel results in the field of singular plane curve germs, and Legendrian curve germs, along with new results pertaining to the stratification of points in the Monster Tower.

In Chapter 3 we were able to establish a connection between points $p \in M(n)$ in the Monster Tower at level n, and value sets of plane curve germs. In some cases if the point in the tower was regular, and at a sufficiently high level, we could assign to a single value set. In any case where the point was regular, we associated to it a set of plane curve germs, $\mathrm{Pl}(p)$, that all had the same topological type.

We found that it was possible for $\mathrm{Pl}(p)$ to have curves with differing value sets. We called a point generic if we found that one of the curves in $\mathrm{Pl}(p)$ had the value set Λ_{gen}. We then gave a criterion for which points had the generic value set associated to them.

In Chapter 4 we introduced the coordinated Mancala game for coprime semigroups, and defined the minimal game. This minimal game corresponds to the generic value set Λ_{gen}. We then gave a recursive formula for the minimal generators of the value set Λ_{gen}, which completed the description of the points in the Monster that had generic value set associated to them. We consider this the main result of our work.

From there we began to classify the equi-singularity classes of l=plane curve germs that had only one value set associated to them. In other words the topological classes of curves that had at most only moduli as their analytic invariants. This concluded chapter 4.

Finally Chapter 5, the last of new results, examined Legendrian curve germs in $M(1)$ and gave results on contact invariants of these curve germs. The chapter worked heavily with the notion of the contact projection. We found that some analytic jet information of any contact projection is preserved up to analytic equivalence of the contact projection plane under local contact transformations of $M(1)$. This showed that their is a legitimate notion of the Zariski invariant of a contact curve germ in $M(1)$. We then finally explored to some extent the connection between the value set Λ' of a contact projection, and the Legendrian semigroup of the original curve germ.

We now look forward to future directions of research on the subjects of the Monster Tower, plane curve germs, and Legendrian curve germs.

6.2 Future Directions

6.2.1 The Monster Tower

We have now established that regular points in the Monster tower can be stratified by not only an RVT code word, but also a set of value sets of plane curve germs, which is at times a singleton. These latter results relied on a-equivalence. Under contact equivalence, we still recover the notion of an RVT code word for the point, but in our convention this RVT code word must start with 2 R's.

We could call this the *contact* RVT code word, and stratify the Monster Tower via these code words as well. We have seen already in Chapter 5 that Legendrian curve germs can have the same RVT code word, but not the same Legendrian semigroup. In [19] they assign a set of Legendrian curve germs to a point in $M(n)$, similarly to the way we assign plane curve germs, and call this set $\text{Leg}(p)$. We can then ask a similar question as we did in Chapter 3, but instead about Legendrian curves, and their semigroups. Namely, when is it that $\text{Leg}(p)$ consists of Legendrian curve germs all sharing the same Legendrian semigroup? This is another way that we could consider a finer stratification of points under contact equivalence using a discrete contact invariant.

6.2.2 Plane Curve Germs

There are many future research opportunities for plane curve germs and their analytic invariants. We of course would like to prove both Conjecture 1 and Conjecture 2. We would also like to give a formula for the generic value set of a plane curve germ with 3 generator semigroup, and ultimately n generators. All of this will likely require the notion of semiroots of plane curve germs.

Another question we would like to answer, which is somewhat on the opposite end of the above, is: What is the value set associated to the *prototype* curve, that is, the curve germ in normal form consisting of only essential Puiseux exponents, each with coefficient 1. We would also like to know if the coefficients of the essential exponents can have an effect on the value set of the curve, if all other nonessential exponent coefficients in the parameterization for $y(t)$ are set to 0. The answer to these questions will greatly facilitate completing the classification of Λ-simple code words.

6.2.3 Legendrian Curve Germs

As mentioned at the end of the previous chapter, we would like to know how closely related the minimal generators are to each other between the minimal generators of the Legendrian semigroup, and the minimal generators of a contact projection. It is clear that the Zariski one-form is exact, and so in the case of our curve having contact projections with coprime semigroups, we can try to use Delorme's Lemma 12(b) to explore this further.

We note that there is some notion of a *generic* contact projection. Indeed when we look at what contact transformations do to a given contact projection, we see there must be notion of generic one in the sense of obtaining a generic curve germ with fixed $\lambda' + p$-jet, in terms of the notation of Chapter 5. Since we cannot guarantee that any higher jet is preserved under contact transformation, it makes sense to speculate that the generic $\lambda' + p$-jet will have a value set Λ' that most closely coincides with our Legendrian semigroup. We then have the conjecture:

Conjecture 3. *Suppose we have a Legendrian curve germ γ in $M(1)$ such that the contact projection γ' of the curve has generic contact projection (that is, γ' is a generic curve with a fixed $\lambda' + p$-jet). Then the minimal generators of the value set Λ' of γ' are the same as the nonzero minimal generators of the Legendrian semigroup $L(\gamma)$.*

It is not clear if the result is true in the greatest of generality, that is, in the case of an arbitrary number of generators for the semigroup of the contact projection. However there is much more reason to believe that it is true in the case where the contact projection has coprime semigroup. This reasoning follows from the discussion at the end of Chapter 5 in Section 5.4.2. In the future we hope to dedicate more time to proving this conjecture in both the coprime and general case. A good place to start is looking at Legendrian curve germs with multiplicity 4. Some of this work has already been started recently by Montgomery in [18], concerning the embedding dimension of semigroups with multiplicity 4.

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