UNIVERSITY OF CALIFORNIA
SANTA CRUZ

METRIC LINES IN METABELIAN CARNOT GROUPS

A dissertation submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

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June 2023

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Contents

List of Figures

List of Tables

Abstract

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This work is devoted to metric lines (an isometric embedding of the real line) in metabelian Carnot group $\mathbb{G}$: we say that a group $\mathbb{G}$ is metabelian if $[\mathbb{G}, \mathbb{G}]$ is Abelian. Theorems A and B provide a partial result about the classification of the metric lines in the jet-space of functions from $\mathbb{R}$ to $\mathbb{R}$, denoted by $J^k(\mathbb{R}, \mathbb{R})$. Theorem C is a complete classification of the metric lines in the Engel type Carnot group, denoted by $\text{Eng}(n)$. Both groups, $J^k(\mathbb{R}, \mathbb{R})$ and $\text{Eng}(n)$ are examples of metabelian Carnot groups. The main tools to classify subRiemannian geodesic on $\mathbb{G}$ is a correspondence between the regular subRiemannian geodesics in a metabelian Carnot group $\mathbb{G}$ and the space of solutions to a family of classical electromechanical systems on Euclidean space. The method to prove Theorems A, B and C is to use an intermediate $(n+2)$-dimensional subRiemannian space $\mathbb{R}^{n+2}_F$ lying between the $\mathbb{G}$ and the Euclidean space $\mathbb{R}^{d_1} \simeq \mathbb{G}/[\mathbb{G}, \mathbb{G}]$.

To Heather for be the love of my life,

To my parents for be the support of my life,

To my siblings for be my fellowships of life,

To all my professors for be my compass in life,

Acknowledgments

I want to express my gratitude to my advisor Richard Montgomery for supporting my application to UCSC, for his invaluable help and patience, for sharing his love for math, and for being a good human being and friend. However, the most important lesson that he taught me is that the impossible can be possible since he always believed that complete classification of the metric lines in the jet space was possible and more general in Carnot groups, Theorems A, B and C are the evidence he was correct.

I want to thank Enrico Le Donne for my time at the University of Friuborg, where he introduced the definition of the metabelian Carnot group and the exponential coordinates of the second kind, and also for showing me the proof of Proposition 95. I want to thank Felipe Monroy for being part of my oral exam, his support during my postdoc applications, and his work, which inspired me.

I want to express my gratitude to Anthony Bloch for being part of my thesis committee and supporting my application to the University of Michigan. In addition, I thank Jie Qing and Francois Monard for being part of my oral exam and thesis committee and for their support during my time at UCSC.

I want to thank my bachelor's degree advisor Oscar Palmas-Velazco and master's degree advisor Luis Garcia-Naranjo for sharing their love for math, invaluable help and patience, and for being good human beings and good friends.

I want to thank Nicola Paddeu, Andrei Ardentov, Yuri Sachkov, Felipe Monroy-Perez, Angel Carrillo-Hoyo, and Hector Sanchez-Morgado, for e-mail conversations and talks regarding the course of this work.

I want to thank Oscar Palmas-Velazco, Luis Garcia-Naranjo, Renato Callejas, Gabriel, and Hector Sanchez-Morgado for their letter of recommendation to support my UCSC application.

This research was developed with the support of the scholarship (CVU 619610) from “Consejo de Ciencia y Tecnologia” (CONACYT).

Chapter 1

Introduction

This work is a report of the research done, from July 2019 to April 2023, under the advice of Richard Montgomery. My main goal was to characterize the metric lines on the jet space $J^k(R, R)$, an example of metabelian Carnot group.

1.1 Metric lines in Carnot groups

A Carnot group $\mathbb{G}$ is a simple connected Lie group whose Lie algebra $\mathfrak{g}$ is graded, nilpotent, and its first layer $\mathfrak{g}_1$ generates the Lie algebra $\mathfrak{g}$. Every Carnot group $\mathbb{G}$ has a canonical protection $\pi : \mathbb{G} \to \mathbb{R}^{d_1}$, see (2.1), where $\mathbb{G}/[\mathbb{G}, \mathbb{G}] \simeq \mathfrak{g}_1 \simeq \mathbb{R}^{d_1}$ and d_1 is the dimension of the layer $\mathfrak{g}_1$. To give a subRiemannian structure to $\mathbb{G}$, we define the non-integrable distribution $\mathcal{D}_g := (L_g)_*\mathfrak{g}_1$, and we consider the inner product on $\mathcal{D}$ as the one who makes π a subRiemannian submersion where $\mathbb{R}^{d_1}$ is equipped with the Euclidean product. Let us formalize the subRiemannian submersion.

Definition 1. *Let $(M, \mathcal{D}_M, g_M)$ and $(N, \mathcal{D}_N, g_N)$ be two subRiemannian manifold and* *let $\phi : M \to N$ a submersion, we consider the case $dim(M) \geq dim(N)$. We say that ϕ is a subRiemannian submersion if $\phi_* \mathcal{D}_M = \mathcal{D}_N$ and $\phi^* g_N = g_M$.*

Here we will introduce the definition of a metric line in the context of subRiemannian geometry.

Definition 2. *Let M be a subRiemannian manifold, we denote by $dist_M(\cdot,\cdot)$ the subRiemannian distance on M. Let $|\cdot| : \mathbb{R} \to [0,\infty)$ be the absolute value, we say that a geodesic $\gamma : \mathbb{R} \to M$ is a metric line if $|a-b| = dist_M(\gamma(a),\gamma(b))$ for all compact set $[a,b] \subset \mathbb{R}$.*

A classic result on metric lines is the following.

Proposition 3. *Let $\phi : M \to N$ be a subRiemannian submersion and let $c(t)$ be a metric line in N, then the horizontal lift of $c(t)$ is a metric line in M.*

The proof of Proposition 3 is given in [12, p. 154.].

Definition 4. *Let $\mathbb{G}$ be a Carnot group. We say that a geodesic $\gamma(t)$ is a line if the projected curve $\pi(\gamma(t))$ in $\mathbb{R}^{d_1}$ is a line.*

As an immediate corollary to the Proposition 3, we get:

Corollary 5. *The geodesic lines are metric lines in every Carnot group.*

1.1.0.1 Metric lines in $J^k(\mathbb{R},\mathbb{R})$

In [12], we showed a bijection between the set of pairs (F_μ, I) and the set of geodesics in $J^k(\mathbb{R},\mathbb{R})$, where F_μ is a polynomial defined by (3.1) and I is a hill interval given by Definition 24. In addition, we classified the non-geodesic lines in $J^k(\mathbb{R},\mathbb{R})$ according to their reduced dynamics, that is, the non-geodesic lines in $J^k(\mathbb{R},\mathbb{R})$ are x-periodic, homoclinic, direct-type or turn back, see sub-Section 3.1.1 or Figure 1.1 and 1.2. The Conjecture concerning metric lines in $J^k(\mathbb{R},\mathbb{R})$ is the following.

Conjecture 6. *Besides geodesic lines, the metric lines for $J^k(\mathbb{R},\mathbb{R})$ are homoclinic and direct-type geodesics.*

Theorem A is the first main result and proves Conjecture 6 for the case of direct-type geodesics.

Theorem A. *The direct-type geodesics are metric lines in $J^k(\mathbb{R},\mathbb{R})$.*

The question remains open for homoclinic geodesics. Theorem B is the second principle result and provides a family of homoclinic geodesics that are metric lines.

Theorem B. *The homoclinic-geodesic defined by the polynomial $F(x) = \pm(1 - bx^{2n})$ and hill interval $[0, \sqrt[2n]{\frac{2}{b}}]$ is a metric line in $J^k(\mathbb{R},\mathbb{R})$ for all $k \geq 2n$ and $b > 0$.*

Conjecture 6 was proved by A. Andertov and Y. Sachkov in the case $k = 1$ and $k = 2$, see [7, 6, 5]. The case $k = 1$ corresponds to $\mathbb{G}$ being the Heisenberg group where all the geodesics are x-periodic. The case $k = 2$ corresponds to $\mathbb{G}$ being Engel's group, denoted by Eng; up to a Carnot translation and dilation Eng has a unique metric line such that its projection to the plane $\mathbb{R}^2 \simeq \mathrm{Eng}/[\mathrm{Eng},\mathrm{Eng}]$ is the Euler-soliton. The family of metric lines defined by Theorem B is the generalization of A. Andertov and Y. Sachkov's result from [7, 6, 5]. In [12], we showed that a family of direct-type geodesics with an open condition are metric lines.

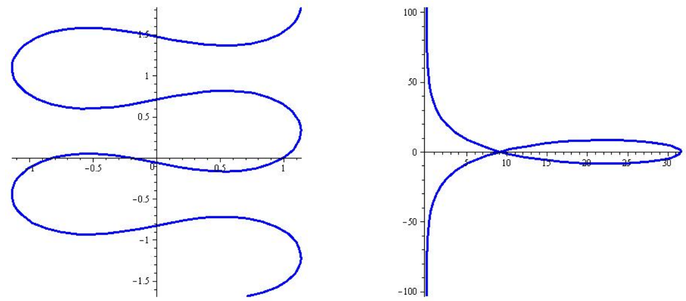

Figure 1.1: Both images show the projection to $\mathbb{R}^2 \simeq J^k(\mathbb{R},\mathbb{R})/[J^k(\mathbb{R},\mathbb{R}), J^k(\mathbb{R},\mathbb{R})]$, with coordinates (x, θ_0), of geodesics in $J^k(\mathbb{R},\mathbb{R})$. The left panel presents a generic x-periodic geodesic, the right panel displays the Euler-soliton solution to the Euler-Elastica problem and whose corresponding geodesic is a metric line, see Theorem B.

1.1.0.2 Metric lines in $\mathrm{Eng}(n)$

Theorem 96 tells that the subRiemannian geodesic flow on $\mathrm{Eng}(n)$ is integrable. We classify the normal geodesics in $\mathrm{Eng}(n)$ according to their reduced dynamics, see 55. Theorem C is the third principle result of this work and makes a complete classification of metric lines in $\mathrm{Eng}(n)$.

Theorem C. *Up to a Carnot translation* $\mathrm{Eng}(n)$ *has one family of metric lines, besides geodesic lines. This family is generated by* $F_\mu(r) = \pm(1 - br^2)$ *with* $0 < b$.

We remark that given a metric line $\gamma(t)$ in the family described by Theorem C, there exists a two-plane in $\mathbb{R}^{n+1} \simeq \mathrm{Eng}(n)/[\mathrm{Eng}(n), \mathrm{Eng}(n)]$ such that the projection of $\gamma(t)$ to this plane is the Euler-Elastica given by case $n = 1$ from Theorem B.

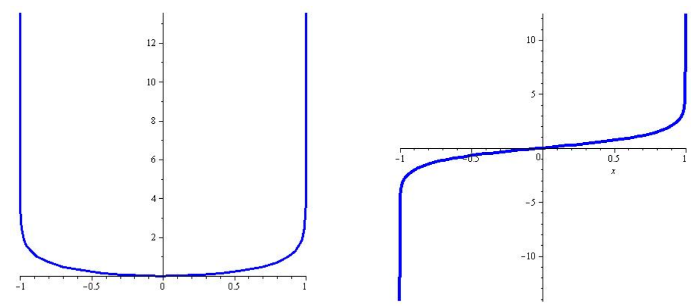

Figure 1.2: Both images show the projection to $\mathbb{R}^2 \simeq J^k(\mathbb{R},\mathbb{R})/[J^k(\mathbb{R},\mathbb{R}), J^k(\mathbb{R},\mathbb{R})]$, with coordinates (x, θ_0), of geodesics in $J^k(\mathbb{R},\mathbb{R})$. The left panel presents the projection of a turn-back geodesic, the right panel displays the projection of a direct-type geodesic and whose corresponding geodesic is a metric line, see Theorem A.

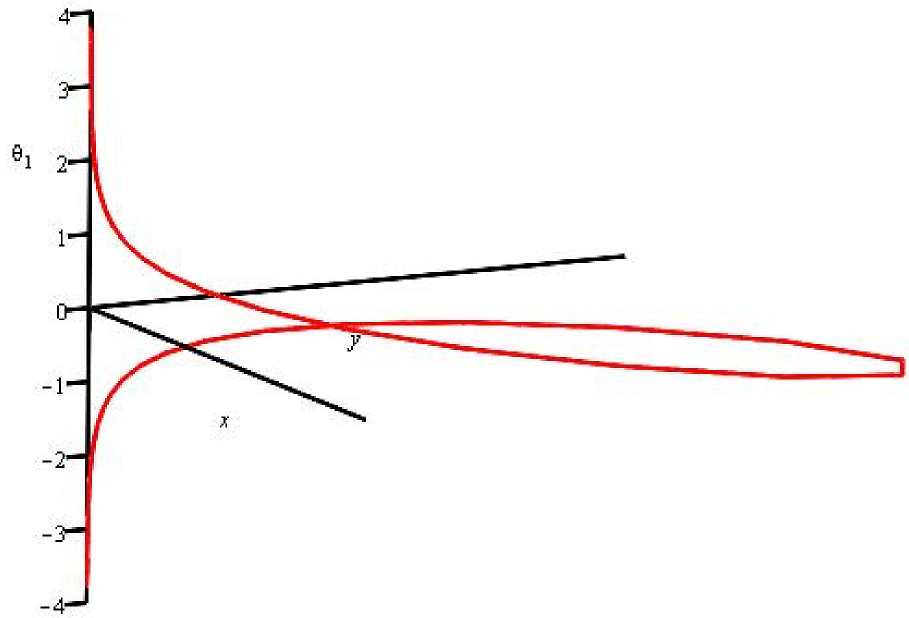

Figure 1.3: The images show the projection to $\mathbb{R}^3 \simeq \mathrm{Eng}(2)/[\mathrm{Eng}(2), \mathrm{Eng}(2)]$, with coordinates (x, y, θ_0), of one metric line in Eng(2) defined by Theorem C

Chapter 2

General theory

2.1 $\mathbb{G}$ as subRiemannian manifold

A Lie algebra $\mathfrak{g}$ is graded stratified if $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and call $\mathfrak{g}_r$ the layers of $\mathfrak{g}$. A graded stratified Lie algebra $\mathfrak{g}$ is nilpotent if $\mathfrak{g}_{s+1} = 0$. We say a $\mathbb{G}$ is a Carnot group if $\mathbb{G}$ is a simply connected Lie group whose Lie algebra $\mathfrak{g}$ is graded stratified, nilpotent and bracket generated by $\mathfrak{g}_1$. We call s the step of $\mathbb{G}$ and denote by $[\mathbb{G}, \mathbb{G}]$ the commutator group of $\mathbb{G}$. Every Carnot group has a canonical projection $\pi : \mathbb{G} \to \mathbb{R}^{d_1}$ where $\mathbb{R}^{d_1} \simeq \mathfrak{g}_1 \simeq \mathbb{G}/[\mathbb{G}, \mathbb{G}]$. If g is in $\mathbb{G}$, then the formal definition is:

$$\pi(g) := g \;\; mod \; [\mathbb{G}, \mathbb{G}]. \tag{2.1}$$

The canonical injection of $[\mathbb{G}, \mathbb{G}]$ into $\mathbb{G}$ and the projection π define a short exact sequence: $0 \to [\mathbb{G}, \mathbb{G}] \hookrightarrow \mathbb{G} \xrightarrow{\pi} \mathbb{R}^{d_1} \to 0$, which tells that $\mathbb{G} \simeq \mathbb{R}^{d_1 + dim\,([\mathbb{G}, \mathbb{G}])}$.

A subRiemannian structure on a smooth manifold M is given by the pair $(\mathcal{D}, (\cdot, \cdot))$, where $\mathcal{D}$ is a non-integrable distribution and $(\cdot, \cdot)$ is an inner product on

$\mathcal{D}$. Every Carnot group is a subRiemannian manifold with the Carnot-Carathéodory distance. Let us define the subRiemannian structure on a Carnot group.

Definition 7. *The subRiemannian structure* $\mathbb{G}$ *is given by* $\mathcal{D}(g) := (L_g)_*\mathfrak{g}_1$ *and inner product on* $\mathcal{D}(g)$ *is such that* π *is a subRiemannian submersion. The Carnot-Carathéodory distance on* $\mathbb{G}$ *is given by*

$$dist_{\mathbb{G}}(g_1, g_2) := \inf \left\{ \int_a^b ||\dot{\gamma}(t)||_{\mathbb{G}} : \gamma(t) : [a,b] \to \mathbb{G} \ \ absolutely\ contiouons \right.$$
$$\left. \gamma(a) = g_1 \ \ \gamma(b) = g_2 \ \ and \ \ \dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)} \ \ for\ a.e.\ t \in [a,b] \right\}$$

The property of $\mathfrak{g}_1$ being bracket generating implies the distribution $\mathcal{D}$ is controllable, see [21].

2.1.1 Metabelian Carnot groups

Let us introduce the formal definition of metabelian group.

Definition 8. *We say* $\mathbb{G}$ *is a metabelian group if* $[\mathbb{G}, \mathbb{G}]$ *is abelian. Every metabelian group has a normal abelian subgroup* $\mathbb{A}$ *containing* $[\mathbb{G}, \mathbb{G}]$.

See [23] for more algebraic details of the definition.

We consider the left action of $\mathbb{A}$ on $\mathbb{G}$, which is proper and free, so the quotient $\mathbb{G}/\mathbb{A}$ is well-defined. Let us denote by $\mathcal{H}$ the quotient $\mathbb{G}/\mathbb{A}$, and let $\pi_{\mathbb{A}} : \mathbb{G} \to \mathcal{H}$ be the canonical projection. Let g be in $\mathbb{G}$, then the canonical projection $\pi_{\mathbb{A}} : \mathbb{G} \to \mathcal{H}$ is given by

$$\pi_{\mathbb{A}}(g) := g \ \ mod \ \ \mathbb{A}. \tag{2.2}$$

Group	Dimension
$\mathbb{G}$	$n+m$
$\mathbb{A} \simeq \mathcal{V} \times [\mathbb{G}, \mathbb{G}]$	$m = n_1 + dim([\mathbb{G}, \mathbb{G}])$
$[\mathbb{G}, \mathbb{G}]$	$m - n_1$
$\mathbb{R}^{d_1} = \mathcal{H} \oplus \mathcal{V} \simeq \mathfrak{g}_1 \simeq \mathbb{G}/[\mathbb{G}, \mathbb{G}]$	$d_1 = n + n_1$
$\mathcal{H} := \mathbb{G}/\mathbb{A}$	n
$\mathcal{V} := \mathcal{H}^{\perp} \subset \mathbb{R}^{d_1}$	n_1

Table 2.1: Dimension of the groups.

The canonical injection of $\mathbb{A}$ into $\mathbb{G}$ and the projection $\pi_{\mathbb{A}}$ define a short sequence; $0 \to \mathbb{A} \hookrightarrow \mathbb{G} \xrightarrow{\pi_{\mathbb{A}}} \mathcal{H} \to 0$, which tells $\mathbb{G} \simeq \mathbb{A} \times \mathcal{H}$, topologically. Thanks to the subRiemannian inner product, given the Lie algebra $\mathfrak{a}$ we can decompose $\mathfrak{g}_1$ as the direct sum of two sub-spaces.

Definition 9. *Let $\mathbb{G}$ be a metabelian Carnot group, and let $\mathfrak{a}$ be the Lie algebra of a maximal abelian subgroup $\mathbb{A}$ containing $[\mathbb{G}, \mathbb{G}]$. Then $\mathfrak{g}_1 = \mathfrak{h} \oplus \mathfrak{v}$, where $\mathfrak{v} := \mathfrak{a} \cap \mathfrak{g}_1$ and $\mathfrak{h}$ is the orthogonal complement of $\mathfrak{v}$ in $\mathfrak{g}_1$. In addition, $\mathcal{D}(g) = \mathcal{D}_{\mathfrak{h}}(g) \oplus \mathcal{D}_{\mathfrak{v}}(g)$ where $\mathcal{D}_{\mathfrak{v}}(g) := (L_g)_* \mathfrak{v}$ and $\mathcal{D}_{\mathfrak{h}}(g) := (L_g)_* \mathfrak{h}$ are left-invariant subspace.*

We want to think in $\mathcal{H}$ inside $\mathbb{R}^{d_1}$. Then $\mathbb{R}^{d_1} = \mathcal{H} \times \mathcal{V}$, where $\mathcal{V}$ is the orthogonal complement of $\mathcal{H}$ with respect of the Euclidean product in $\mathbb{R}^{d_1}$. The map π is compatible with the splitting of $\mathcal{D}(g)$ and $\mathbb{R}^{d_1}$, that is,

$$d\pi_g(\mathcal{D}_{\mathfrak{h}}(g)) = T_{\pi_{\mathbb{A}}(g)}\mathcal{H} \text{ and } d\pi_g(\mathcal{D}_{\mathfrak{v}}(g)) = T_{\pi_{\mathbb{A}(g)}}\mathcal{V}.$$

2.1.2 Geodesic flow and symplectic reduction

Like any subRiemannian structure, the cotangent bundle $T^*\mathbb{G}$ is endowed with a Hamiltonian system whose underlying Hamiltonian H_{sR} has solution curves whose projection to $\mathbb{G}$ are the subRiemannian geodesics. We call this Hamiltonian system the geodesic flow on $\mathbb{G}$. Let $T^*\mathcal{H}$ be the cotangent bundle of $\mathcal{H}$, the Hamiltonian structure for the classical electromechanical system is given by a magnetic potential $\mathcal{A}$ and effective potential ϕ, see [15] or [1] for more details. The mathematical object relating the Hamiltonian structures is a $\mathfrak{a}^*$ valued one-form $\mathcal{A}_{\mathbb{G}} = \mathcal{A}^M_{\mathbb{G}} + \mathcal{A}^E_{\mathbb{G}}$ in $\Omega^1(\mathbb{R}^{d_1}, \mathfrak{a})$, where $\mathcal{A}^M_{\mathbb{G}}$ is in $\Omega^1(\mathcal{H}, \mathfrak{a}^*)$ and $\mathcal{A}^E_{\mathbb{G}}$ is in $\Omega^1(\mathcal{V}, \mathfrak{a}^*)$. Let μ be in $\mathfrak{a}^*$, we define $\mathcal{A}_\mu$ as the paring of $\mathcal{A}_{\mathbb{G}}$ with μ, that is,

$$\mathcal{A}_\mu(x) :=< \mu, \mathcal{A}_{\mathbb{G}} >= \mathcal{A}^M_\mu + \mathcal{A}^E_\mu, \ < \mu, \mathcal{A}^M_{\mathbb{G}} >:= \mathcal{A}^M_\mu \ \text{ and } \ < \mu, \mathcal{A}^E_{\mathbb{G}} >:= \mathcal{A}^E_\mu. \tag{2.3}$$

Then $\mathcal{A}_\mu$ is a one-form on $\mathbb{R}^{d_1}$. The map $\exp : \mathfrak{g} \to \mathbb{G}$ endows $\mathbb{G}$ with the exponential coordinates (x, θ) of the second kind, see sub-section B.2. If we write $\mathcal{A}_\mu$ in terms of these coordinates, then $\mathcal{A}_\mu$ depends only on x in a polynomial way. Moreover, if (p_x, x) are the traditional coordinates for $T^*\mathcal{H} \subset T^*\mathbb{R}^{d_1}$, then $\mathcal{A}_\mu$ defines a Hamiltonian H_μ function in $T^*\mathcal{H}$, given by

$$H_\mu(p_x, x) := \frac{1}{2}||p_x + \mathcal{A}_\mu(x)||^2_{(\mathbb{R}^{d_1})^*} = \frac{1}{2}||p_x + \mathcal{A}^M_\mu(x)||^2_{\mathcal{H}^*} + \frac{1}{2}\phi_\mu(x). \tag{2.4}$$

Where the effective potential $\frac{1}{2}\phi_\mu(x)$ is defined by the function $\phi_\mu(x) = ||\mathcal{A}^E_\mu(x)||^2_{\mathcal{V}^*}$, here $|| \ \ ||_{(\mathbb{R}^{d_1})^*}$, $|| \ \ ||_{\mathcal{H}^*}$, and $|| \ \ ||_{\mathcal{V}^*}$ are the Euclidean norm in $(\mathbb{R}^{d_1})^*$, $\mathcal{H}^*$ and $\mathcal{V}^*$, respectively. Equation (2.4) shows that we can interpret $\mathcal{A}^M_\mu(x)$ and $\mathcal{A}^E_\mu(x)$ as the magnetic potential and effective potential of the reduced Hamiltonian H_μ.

Definition 10. *We call* $(T^*\mathbb{G}, H_{sR} = \frac{1}{2})$ *the subRiemannian geodesic flow in* $\mathbb{G}$. *Let* $J : T^*\mathbb{G} \to \mathfrak{a}^*$ *be the momentum map associated with the action of* $\mathbb{A}$ *and let* μ *in* $\mathfrak{a}^*$. *We say* $\gamma(t)$ *is a geodesic parameterized by arc length and with momentum* μ*, if* $\gamma(t)$ *is the projection of subRiemannian geodesic flow and* $J(p(t), \gamma(t)) = \mu$.

Definition 11. *The reduced Hamiltonian flow is given by* $(T^*\mathcal{H}, H_\mu = \frac{1}{2})$. *We say* $\eta(t)$ *is an* $\mathcal{A}_\mathbb{G}$*-curve for* μ *in* $\mathcal{H}$*, if* $\eta(t)$ *is the projection of the reduced Hamiltonian flow.*

The following result is a consequence of the symplectic reduction made with Nicola Paddeu, see [10].

Background Theorem 1. *Let* $\mathbb{G}$ *be a metabelian Carnot group and* $\mathfrak{a}$ *a choice of maximal abelian ideal* $([\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{a})$. *Then there exists an* $\mathfrak{a}^*$ *valued polynomial one-form* $\mathcal{A}_\mathbb{G}(x)$ *on* $\mathbb{R}^{d_1} = \mathbb{G}/[\mathbb{G}, \mathbb{G}]$ *given by* $\mathcal{A}^M_\mathbb{G}(x) + \mathcal{A}^E_\mathbb{G}(x)$, $x \in \mathcal{H} := \mathbb{G}/\mathbb{A}$ *with the following significance. If* $\gamma(t)$ *is a normal subRiemannian geodesic in* $\mathbb{G}$ *with momentum* μ*, then the curve* $\eta(t) = \pi_\mathbb{A}(\gamma(t))$ *is an* $\mathcal{A}_\mathbb{G}$*-curve for* μ. *Conversely, if* $\eta(t)$ *is an* $\mathcal{A}_\mathbb{G}$*-curve for* μ*, then its horizontal-lift is a normal subRiemannian geodesic in* $\mathbb{G}$ *with momentum* μ.

The first statement of the Background Theorem was proved by showing that the symplectic reduction of the subRiemannian flow on $T^*\mathbb{G}$ yields the reduced Hamiltonian H_μ. In contrast, the converse statement was shown using the symplectic reconstruction. We reduce the study of subRiemannian geodesics in metabelian Carnot groups to the study of the $\mathcal{A}_\mathbb{G}$-curves. The Background Theorem justifies why it is enough to classify the reduced dynamics to classify the subRiemannian geodesic flow.

In [3, 4, 20], A. Anzaldo-Meneses, and F. Monroy-Perez showed the bijection between normal geodesic and the pair (F_μ, I) in the context of the $J^k(\mathbb{R},\mathbb{R})$. In [12], we used their approach to give our partial result of the conjecture 6. Later, thanks to E. Le Donne, we generalize the idea from A. Anzaldo-Meneses, and F. Monroy-Perez to make the syplectic reduction in the metabelian Carnot case.

2.2 Case $[\mathfrak{h},\mathfrak{h}] = 0$

We remark that the condition $[\mathfrak{h},\mathfrak{h}] = 0$ implies that the term $\mathcal{A}_{\mathbb{G}}^M = 0$, then the reduced Hamiltonian is an n-degree of freedom system with polynomial potential given by

$$H_\mu(p_x, x) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\phi_\mu(x).$$

Geodesic lines are the geodesics associated with the constant polynomial $\phi_\mu(x)$. Let us assume $\phi_\mu(x)$ is not constant: there exists a closed set $hill \subset \mathcal{H}$, called the hill region, where the dynamics take place. That is, if x is in $int\ (hill)$, then $0 \leq \phi(x) < 1$ and $p_x \neq 0$, while, if x is in $\partial\ (hill)$, then $\phi_\mu(x) = 1$ and $p_x = 0$. We say that $\eta(t)$ bounces at the boundary of $hill$, the dynamics of the harmonic oscillator is the simplest example of this phenomenon for the Heisenberg group.

2.2.1 Case $dim\ \mathfrak{v} = 1$

The case $[\mathfrak{h},\mathfrak{h}] = 0$ and $dim\ \mathfrak{v} = 1$ will be relevant since, in this general context, we will introduce the subRiemannian submersion π_F to prove that a geodesic in $\mathbb{G}$ is a metric line. We will see that $J^k(\mathbb{R},\mathbb{R})$ and $\mathrm{Eng}(n)$ hold these conditions, see Sections

3.1 and 4.1.

Let $\mathcal{A}_{\mathbb{G}}$ be the $\mathfrak{a}^*$ valued one-form associated to the metabelian Carnot group $\mathbb{G}$. If θ_0 is the exponential coordinate associated to the left invariant vector field Y in $\mathcal{D}_{\mathfrak{v}}$ and μ is in $\mathfrak{a}$. Then, we define the polynomial $F_\mu(x)$ by equation $< \mu, \mathcal{A}_{\mathbb{G}} >:= F_\mu(x) d\theta_0$, so the reduced Hamiltonian is given by

$$H_\mu(p_x, x) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}F_\mu^2(x).$$

In this case, $hill := F_\mu^{-1}[-1,1]$ and the left-invariant vector fields tangent to $\mathcal{D}$ are given by

$$X_i = \frac{\partial}{\partial x_i} \;\; 1 \leq i \leq n, \text{ and } \; Y = \frac{\partial}{\partial \theta_0} + \sum_{\ell=1}^{m-1} \mathcal{A}_\ell^E(x)\frac{\partial}{\partial \theta_\ell}. \tag{2.5}$$

Here, $\mathcal{A}_\ell^E(x)$ are polynomial functions on $\mathcal{H}$, given by Proposition 95. Then the polynomial $F_\mu(x) = a_0 + \sum_{\ell=1}^{m-1} a_\ell \mathcal{A}_\ell^E(x)$, if $\mu = (a_0, \ldots, a_{m-1})$.

2.3 The space $\mathbb{R}_F^{n+2}$

Following the notation from sub-Section 2.2.1. Let $\mathbb{G}$ be a Carnot group such that $[\mathfrak{h}, \mathfrak{h}] = 0$, $dim\ \mathcal{H} = n$, and dim $\mathfrak{v} = 1$, then $\mathcal{D}$ is an $(n+1)$-rank distribution. Let us fix momentum μ in $\mathfrak{a}^*$ and consider the polynomial $F_\mu(x)$ defined in sub-Section 2.2.1. We will introduce an intermediate $(n+2)$-dimensional subRiemannian manifold denoted by $\mathbb{R}_F^{n+2}$ whose geometry depends on $F(x) := F_\mu(x)$.

2.3.1 Factoring a subRiemannian submersion

We denote by $\mathbb{R}_F^{n+2}$, the subRiemannian manifold with the following structure, let $(x_1, \cdots, x_n, y, z)$ be global coordinates, in short way (x, y, z). We define the $(n+1)$-rank non-integrable distribution $\mathcal{D}_F$ by the equation $dz - F(x)dy = 0$. To make $\mathbb{R}_F^{n+2}$ a subRiemannian manifold we define the subRiemannian metric on the distribution $\mathcal{D}_F$ given by $ds^2_{\mathbb{R}_F^{n+2}} = (\sum_{i=1}^n dx_i^2 + dy^2)|_{\mathcal{D}_F}$. We provide a subRiemannian submersion π_F factoring the subRiemannian submersion $\pi : \mathbb{G} \to \mathbb{R}^{n+1}$, that is, $\pi = pr \circ \pi_F$, where the target of π_F is $\mathbb{R}_F^{n+2}$ and the target of pr is $\mathbb{R}^{n+1}$. If $\mu = (a_0, \ldots, a_{m-1})$, then the projections are given in coordinates by

$$\pi_F(x, \theta) = (x, \theta_0, \sum_{\ell=0}^{m-1} a_\ell \theta_\ell) = (x, y, z), \text{ and } pr(x, y, z) := (x, y). \tag{2.6}$$

It follows that π_F maps the frame $\{X^1, \ldots, X^n, Y\}$ defined in 2.5 into the frame $\{\tilde{X}^1, \cdots, \tilde{X}^n, \tilde{Y}\}$, that is,

$$\tilde{X}^i := (\pi_F)_* X^i = \frac{\partial}{\partial x_i};\ 1 \le i \le n, \text{ and } \tilde{Y} := (\pi_F)_* Y = \frac{\partial}{\partial y} + F(x)\frac{\partial}{\partial z},$$

and $\mathcal{D}_F$ is globally framed by the orthonormal vector fields $\{\tilde{X}^1, \cdots, \tilde{X}^n, \tilde{Y}\}$.

2.3.2 $\mathbb{R}_F^{n+2}$-geodesics

The Hamiltonian function governing the subRiemannian geodesic flow in $\mathbb{R}_F^{n+2}$ is

$$H_F(p_x, p_y, p_z, x, y, z) = \frac{1}{2}\sum_{i=1}^n p_{x_i}^2 + \frac{1}{2}(p_y + F(x)p_z)^2. \tag{2.7}$$

Since H_F does not depend on the coordinates y and z, they are cycle coordinates, so the momentum p_y and p_z are constant of motion, see [16] or [8] for the definition of cycle coordinate. This tells us that the translation $\varphi_{(y_0,z_0)}(x,y,z) = (x, y+y_0, z+z_0)$ is an isometry.

Definition 12. *We denote by* $dist_{\mathbb{R}_F^{n+2}}(\ ,\ )$ *and* $Iso(\mathbb{R}_F^{n+2})$*, the subRiemannian distance and the isometry group in* $\mathbb{R}_F^{n+2}$*. In general, we denote by* $Iso(M)$ *the isometry group os the subRiemannian manifold* M

For more details about these definitions see [21] or [2]. Then the translation $\varphi_{(y_0,z_0)}$ is in $Iso(\mathbb{R}_F^{n+2})$.

Definition 13. *We say a curve* $c(t) = (x(t), y(t), z(t))$ *is a* $\mathbb{R}_F^{n+2}$*-geodesic parametrized by arc length in* $\mathbb{R}_F^{n+2}$*, if it is the projection of the subRiemannian geodesic flow with the condition* $H_F = \frac{1}{2}$

Setting $p_y = a$ and $p_z = b$ inspired the following definition:

Definition 14. *We say that the two-dimensional linear space* Pen_F *is the pencil of* $F(x)$*, if* $Pen_F := \{G(x) = a + bF(x) : (a,b) \in \mathbb{R}^2\}$.

We define the lift of a curve in $\mathbb{R}_F^{n+2}$ to a curve in $\mathbb{G}$

Definition 15. *Let* $c(t)$ *be a curve in* $\mathbb{R}_F^{n+2}$*. We say that a curve* $\gamma(t)$ *in* $\mathbb{G}$ *is the lift of* $c(t) = (x(t), y(t), z(t))$ *if* $\gamma(t)$ *solves*

$$\dot{\gamma}(t) = \sum_{i=1}^{n} \dot{x}_i(t) X^i(\gamma(t)) + G(x(t)) Y(\gamma(t)).$$

Now we describe the $\mathbb{R}_F^{n+2}$-geodesics, their lifts, and their relation with the geodesics in $\mathbb{G}$.

Proposition 16. *Let $c(t)$ be a $\mathbb{R}_F^{n+2}$-geodesic for $G(x)$ in Pen_F, then its projection $x(t) := pr(c(t))$ satisfies the n-degree of freedom Hamiltonian equation*

$$H_{(a,b)}(p_x, x) := \frac{1}{2}\sum_{i=1}^{n} p_{x_i}^2 + \frac{1}{2}(a + bF(x))^2 = \frac{1}{2}\sum_{i=1}^{n} p_{x_i}^2 + \frac{1}{2}G^2(x).$$

Having found a solution $(p_{x_1}(t), \ldots, p_{x_n}(t), x_1(t), \ldots, x_n(t))$, the coordinates $y(t)$ and $z(t)$ satisfy

$$\dot{y} = G(x(t)) \ \text{ and } \ \dot{z} = G(x(t))F(x(t)). \tag{2.8}$$

Moreover, every $\mathbb{R}_F^{n+2}$-geodesic is the π_F-projection of a geodesic in $\mathbb{G}$ corresponding to $G(x)$ in Pen_F. Conversely, the lifts of a $\mathbb{R}_F^{n+2}$-geodesic are precisely those geodesics corresponding to polynomials in Pen_F.

The proof is similar to the one exposed in [12, p. 161]

The subRiemannian geometry has two type of geodesics normal geodesics and abnormal geodesics. The following Lemma characterizes the abnormal geodesics in $\mathbb{R}_F^{n+2}$.

Lemma 17. *A curve $c(t)$ in $\mathbb{R}_F^{n+2}$ is an abnormal geodesic if and only if $c(t)$ is tangent to the vector field $\tilde{Y}$ and $pr(c(t)) = x^*$ is a constant point in $\mathcal{H}$ such that $dF|_{x^*} = 0$.*

For more details about abnormal geodesics, see [21], [2] or [13].

Corollary 18. *Let $\gamma(t)$ be a normal geodesic in $\mathbb{G}$ corresponding to the polynomial $F_\mu(x) = F(x)$ and let $c(t)$ be the curve given by $\pi_F(\gamma(t))$, then $c(t)$ is a $\mathbb{R}_F^{n+2}$-geodesic corresponding to the pencil $(a, b) = (0, 1)$.*

Proof. By construction, the pencil $(a, b) = (0, 1)$ correspond to the polynomial $F(x)$. □

2.3.3 Cost map in $\mathbb{R}_F^{n+2}$

Here we will define the *Cost* map, an auxiliary to prove Theorems A, B and C.

Definition 19. *Let $c(t)$ be a $\mathbb{R}_F^{n+2}$-geodesic defined on the interval $[t_0, t_1]$. We define the function $\Delta : (c, [t_0, t_1]) \to [0, \infty] \times \mathbb{R}^2$ is given by*

$$\begin{aligned} \Delta(c, [t_0, t_1]) &:= (\Delta t(c, [t_0, t_1]), \Delta y(c, [t_0, t_1]), \Delta z(c, [t_0, t_1])) \\ &:= (t_1 - t_0, y(t_1) - y(t_0), z(t_1) - z(t_0)). \end{aligned} \tag{2.9}$$

And the function $Cost : (c, [t_0, t_1]) \to [0, \infty] \times \mathbb{R}$ is given by

$$\begin{aligned} Cost(c, [t_0, t_1]) &:= (Cost_t(c, [t_0, t_1]), Cost_y(c, [t_0, t_1]) \\ &:= (\Delta t(c, [t_0, t_1]) - \Delta y(c, [t_0, t_1]), \Delta y(c, [t_0, t_1]) - \Delta z(c, [t_0, t_1])) \end{aligned} \tag{2.10}$$

We call $Cost(c, [t_0, t_1])$ the cost function of $c(t)$.

Let us prove that $Cost(c, [t_0, t_1])$ is well-defined:

Proof. By construction, $|\Delta y(c, [t_0, t_1])| \leq \Delta t(c, [t_0, t_1])$, so $0 \leq Cost_t(c, [t_0, t_1])$. □

The function $Cost_t(c, [t_0, t_1])$ was defined in [12], We interpret $Cost_t(c, [t_0, t_1])$ as the time that takes to the geodesic $c(t)$ travel through the y-component. To give more meaning to this interpretation, we present the following Lemma:

Lemma 20. *Let $c(t)$ and $\tilde{c}(t)$ be two $\mathbb{R}_F^{n+2}$-geodesics. Let us assume that they travel from a point A to a point B in a time interval $[t_0, t_1]$ and $[\tilde{t}_0, \tilde{t}_1]$, respectively. If $Cost_t(c_1, [t_0, t_1]) < tCost_t(c_2, [\tilde{t}_0, \tilde{t}_1])$, then the arc length of $c(t)$ is shorter that the arc length of $\tilde{c}(t)$.*

Proof. We need to show that $\Delta t(c_1,[t_0,t_1]) < \Delta t(c_2,[\tilde{t}_0,\tilde{t}_1])$. Since $A = c(t_0) = \tilde{c}(\tilde{t}_0)$ and $B = c(t_1) = \tilde{c}(\tilde{t}_1)$, it follows that $\Delta y(c_1,[t_0,t_1]) = \Delta y(c_2,[\tilde{t}_0,\tilde{t}_1])$ which implies

$$\Delta t(c_1,[t_0,t_1]) - Cost_t(c_1,[t_0,t_1]) = \Delta t(c_2,[\tilde{t}_0,\tilde{t}_1]) - Cost_t(c_2,[\tilde{t}_0,\tilde{t}_1]),$$

so $0 < Cost_t(c_2,[\tilde{t}_0,\tilde{t}_1]) - Cost_t(c_1,[t_0,t_1]) = \Delta t(c_2,[\tilde{t}_0,\tilde{t}_1]) - \Delta t(c_1,[t_0,t_1])$. □

2.3.4 Sequence of $\mathbb{R}_F^{n+2}$-geodesic

Let us present two classical results on metric spaces.

Lemma 21. *Let $c_n(t)$ be a sequence of minimizing geodesics on the compact interval $\mathcal{T}$ converging uniformly to a geodesic $c(t)$, then $c(t)$ is minimizing in the interval $\mathcal{T}$.*

Proof. Let $[t_0,t_1] \subset \mathcal{T}$, then $dist_{\mathbb{R}_F^{n+2}}(c_n(t_0),c_n(t_1)) = |t_1 - t_0|$ since $c_n(t)$ is sequence of minimizing geodesic. If $n \to \infty$ then $dist_{\mathbb{R}_F^{n+2}}(c(t_0),c(t_1)) = |t_1 - t_0|$, by the uniformly convergence. □

Proposition 22. *Let K be a compact subset of $\mathbb{R}_F^{n+2}$ and let $\mathcal{T}$ be a compact time interval. Let us define the following space of $\mathbb{R}_F^{n+2}$-geodesics*

$$Min(K,\mathcal{T}) := \left\{\mathbb{R}_F^{n+2}\text{-}geodesics\; c(t) : c(\mathcal{T}) \subset K \;\; and\; c(t) \;\; is\; minimizing\; in\; \mathcal{T}\right\}.$$

Then $Min(K,\mathcal{T})$ is a sequentially compact space with respect to the uniform topology.

Proof. We need to prove that every sequence of $\mathbb{R}_F^{n+2}$-geodesics $c_n(t)$ in $Min(K,\mathcal{T})$ has a uniformly convergent subsequence converging to a minimizing $\mathbb{R}_F^{n+2}$-geodesic $c(t)$ in $Min(K,\mathcal{T})$. The space of geodesics $Min(K,\mathcal{T})$ is uniformly bounded and smooth in compact interval $\mathcal{T}$, then $Min(K, \mathcal{T})$ is a equi-continuous family of geodesics. By Arzela-Ascoli theorem, every sequence $c_n(t)$ in $Min(K, \mathcal{T})$ has a convergent subsequence $c_{n_s}(t)$ converging uniformly to a smooth curve $c(t)$. By Lemma 21 $c(t)$ is minimizing in $\mathcal{T}$. $\square$

A useful tool for the proof of Theorem A, B and C is the following

Corollary 23. *Let $c_1(t)$ be a $\mathbb{R}_F^{n+2}$-geodesic in $Min(K, \mathcal{T})$ and let $c_2(t)$ be a $\mathbb{R}_F^{n+2}$-geodesic. If $\varphi(x, y, z)$ is an isometry such that $c_2(\mathcal{T}') \subset \varphi(c(\mathcal{T}))$, then $c_2(t)$ is iminimizing in $\mathcal{T}'$.*

Chapter 3

Metric lines in jet space $J^k(\mathbb{R}, \mathbb{R})$

This Chapter is devoted to proving Theorems A and B.

3.1 $J^k(\mathbb{R}, \mathbb{R})$ as subRiemannian manifold

Let $f(x)$ and $g(x)$ be real-value functions: we say they are related up to order k at x_0 if $f(x) - g(x) = O(|x - x_0|^{k+1})$ holds on a neighborhood of x_0, this relation is an equivalence relation on the space of germs of smooth functions at x_0 and it is called a k-jet at x_0. We identify the k-jet of a function f at x_0 with its k-th order Taylor polynomial of f at x_0, that is, k-jet is determined by the list of its k first derivatives at x_0:

$$u_0 = f(x_0) \text{ and } u_j = \frac{d^\ell f}{dx^\ell}(x_0), \qquad \ell = 1, \cdots, k.$$

When we vary the point and the function, we sweep out the k-jet space $J^k(\mathbb{R}, \mathbb{R})$, a $(k + 2)$-dimensional manifold with global coordinates x and u_ℓ with $0 \leq \ell \leq k$. When fix the function f and let the independent variable x vary, we get a curve $j^k f : \mathbb{R} \to$

$J^k(\mathbb{R},\mathbb{R})$ called the k-jet of f, sending $x \in \mathbb{R}$ to the k-jet of f at x. In coordinates is given by

$$(j^k f)(x) = \{(x, u_k(x), u_{k-1}(x), \cdots, u_1(x), u_0(x)) : \qquad \frac{d^\ell f}{dx^\ell}(x) = u_\ell\}.$$

The k-jet curve itself is tangent to a rank two distribution $\mathcal{D} \subset TJ^k(\mathbb{R},\mathbb{R})$ at every point, and the following two left-invariant vector fields globally frame the distribution $\mathcal{D}$:

$$X = \frac{\partial}{\partial x} + \sum_{\ell=1}^{k} u_\ell \frac{\partial}{\partial u_{\ell-1}} \qquad \text{and} \qquad Y = \frac{\partial}{\partial u_k}.$$

An alternative way to define the subRiemannian structure on $J^k(\mathbb{R},\mathbb{R})$ is to declare these two vector fields orthonormal with the metric in coordinates $ds^2 = dx^2 + du_k^2|_{\mathcal{D}}$. The vector fields X and Y generate the following Lie algebra:

$$Y^1 := [X,Y],\ Y^2 := [X,Y^1],\ \ldots,\ \ Y^k := [X,Y^{k-1}].$$

Otherwise, zero. The Lie algebra $\mathfrak{a}$ is given by the trivialization of Y, $Y^1 \ldots, Y^{k-1}$ and Y^k. In this case $\mathcal{H} = \mathbb{R}$, $\mathcal{V} = \mathbb{R}$ and $[\mathfrak{h},\mathfrak{h}] = 0$, as we required is sub-Section 2.2.1.

Consider the cotangent bundle $T^*J^k(\mathbb{R},\mathbb{R})$ and its traditional coordinates p_x and p_{u_ℓ}. The momentum function associated to the vector fields X and Y are the following: $P_X := p_x + \sum_{\ell=1}^k u_\ell p_{u_{\ell-1}}$ and $P_Y := p_{u_k}$. The Hamiltonian function governing the geodesic flow is given by

$$H_{sR} = \frac{1}{2}P_X^2 + \frac{1}{2}P_Y^2 = \frac{1}{2}(p_x + \sum_{\ell=1}^{k} u_\ell p_{u_\ell})^2 + \frac{1}{2}p_{u_k}^2.$$

The jet space $J^k(\mathbb{R},\mathbb{R})$ has a natural definition using the coordinates x and u_ℓ with $0 \le \ell \le k$. However, these coordinates do not easily show the symmetries of the system, while the exponential coordinates of the second kind do. The left-invariant vector fields X and Y in the exponential coordinates of the second kind $(x, \theta_0, \dots, \theta_n)$ have the following form:

$$X = \frac{\partial}{\partial x} \qquad \text{and} \qquad Y = \sum_{\ell=0}^{k} \frac{x^\ell}{\ell!} \frac{\partial}{\partial \theta_\ell},$$

We rewrite the Hamiltonian function H_{sR} as:

$$H_{sR}(p, g) = \frac{1}{2} p_x^2 + \frac{1}{2} (\sum_{\ell=0}^{k} p_{\theta_\ell} \frac{x^\ell}{\ell!})^2.$$

Since H_{sR} does not depend on the variables θ_ℓ, they are cycle coordinates, and p_{θ_ℓ} are constant of motion. Then the $\mathfrak{a}^*$ valued one-form $\mathcal{A}_{J^k(\mathbb{R},\mathbb{R})}$ is given by

$$\mathcal{A}_{J^k(\mathbb{R},\mathbb{R})} = d\theta_0 \otimes (\sum_{\ell=0}^{k} e^\ell \frac{x^\ell}{\ell!}).$$

If $\mu = (a_0, \dots, a_k)$ is in $\mathfrak{a}^*$, then the reduced Hamiltonian is given by

$$H_\mu(p_x, x) = \frac{1}{2} p_x^2 + \frac{1}{2} F_\mu^2(x) \text{ where } F_\mu(x) = \sum_{\ell=1}^{m} a_\ell \frac{x^\ell}{\ell!}. \tag{3.1}$$

When $a_0 = a_2 = \cdots = a_m = 0$, the reduced system H_μ is the harmonic oscillator, and the corresponding geodesic $\gamma(t)$ in $J^k(\mathbb{R}, \mathbb{R})$ is the lift of a geodesic in the Heisenberg group, see [12]. Let $\eta(t) = x(t)$ be a $\mathcal{A}_{J^k(\mathbb{R},\mathbb{R})}$-curve, then the lift equation is the following:

$$\dot{\gamma} = \dot{x}(t) X(\gamma(t)) + F(x(t)) Y(\gamma(t)). \tag{3.2}$$

As we proved in [12], a geodesic in $J^k(\mathbb{R}, \mathbb{R})$ is determined by a polynomial F_μ and a hill interval I. Let us formalize the hill interval definition:

Definition 24. *We say that a closed interval I is a hill interval associated to $F_\mu(x)$, if $|F_\mu(x)| < 1$ for every x in the interior of I and $|F_\mu(x)| = 1$ for every x in the boundary of I. If I is of the form $[x_0, x_1]$, then we call x_0 and x_1 the endpoints of the hill interval.*

We remark that the reduced dynamics occur in the hill interval. By definition, I is compact if and only if $F(x)$ is not a constant polynomial. In contrast, the constant polynomial $F(x)$ defines a geodesic line.

3.1.1 Classification of geodesic in $J^k(\mathbb{R}, \mathbb{R})$

Let $\gamma(t)$ be a non-geodesic line in $J^k(\mathbb{R}, \mathbb{R})$ corresponding to the pair $(F_\mu(x), I)$, where $I = [x_0, x_1]$, then $\gamma(t)$ is only one of the following options:

- We say $\gamma(t)$ is x-periodic if its reduced dynamics is periodic. The reduced dynamics is periodic if and only if x_0 and x_1 are regular points of $F_\mu(x)$.
- We say $\gamma(t)$ is homoclinic if its reduced dynamics is a homoclinic orbit. The reduced dynamics has a homoclinic orbit if and only if one of the points x_0 and x_1 is regular and the other is a critical point of $F_\mu(x)$.
- We say $\gamma(t)$ is heteroclinic if its reduced dynamics is a heteroclinic orbit. The reduced dynamics has a heteroclinic orbit if and only if both points x_0 and x_1 are critical of $F(x)$.
- We say a heteroclinic geodesic $\gamma(t)$ is turn-back if $F(x_0)F(x_1) = -1$.
- We say a heteroclinic geodesic $\gamma(t)$ is direct-type if $F(x_0)F(x_1) = 1$.

3.1.2 Unitary geodesics

To prove Theorem A and B, we will introduce the concept of a unitary geodesic:

Definition 25. *We say that a geodesic $\gamma(t)$ in $J^k(\mathbb{R},\mathbb{R})$ corresponding to the pair (F_μ, I) is unitary if $I = [0,1]$. We say a direct-type geodesic (or homoclinic) $\gamma(t)$ is unitary, if in addition $F_\mu(x(t)) \to 1$ when $t \to \pm\infty$.*

The reflection $R_{\theta_0}(x, \theta_0, \theta_1, \dots, \theta_k) = (x, -\theta_0, \theta_1, \dots, \theta_k)$ is in the isometry group $Iso(J^k(\mathbb{R},\mathbb{R}))$. If $\gamma(t)$ is a direct type or homoclinic geodesic such that $F_\mu(x(t)) \to -1$ when $t \to \pm\infty$, then $R_{\theta_0}(\gamma(t))$ is such that $F_\mu(x(t)) \to 1$ when $t \to \pm\infty$.

Corollary 26. *Let $\gamma(t)$ be a unitary direct-type geodesic for $F(x)$, then there exists $q(x)$ such that $F(x) = 1 - x^{k_1}(1-x)^{k_2}q(x)$, where $1 < k_1$, $1 < k_2$, and $q(x)$ is polynomial of degree $k - k_1 - k_2$ such that $0 < x^{k_1}(1-x)^{k_2}q(x) < 2$ if x is in $(0,1)$.*

Proof. By construction, $F(x)$ is such that $F(0) = F(1) = 1$, $F'(0) = F'(1) = 0$, and $|F(x)| < 1$ if x is in $(0,1)$, then using the Euclidean algorithm we find the desired result. □

Any geodesic in $J^k(\mathbb{R},\mathbb{R})$ is related to unitary geodesic by a Carnot dilatation and translation.

Proposition 27. *Let $\gamma(t)$ be a geodesic in $J^k(\mathbb{R},\mathbb{R})$ associated to the pair (F_μ, I) and let $h(\tilde{x}) = x_0 + u\tilde{x}$ be the affine map taking $[0,1]$ to $I = [x_0, x_1]$ with $u := x_1 - x_0$. If $\hat{F}_\mu(h(\tilde{x})) = F_\mu(x)$ and $\hat{\gamma}(t)$ is the geodesic in $J^k(\mathbb{R},\mathbb{R})$ corresponding to the pair*

$(\hat{F}_\mu, [0,1])$. *Then* $\gamma(t)$ *is related to* $\hat{\gamma}(t)$ *by Carnot dilatation and translation, that is*

$$\gamma(t) = \delta_u \hat{\gamma}(\frac{t}{u}) * (x_0, 0 \ldots, 0),$$

where δ_u *is the Carnot dilatation.*

Proposition 27 and the reflection R_{θ_0} imply that it is enough to prove Theorem A and B for the unitaryl case.

3.1.3 The space $\mathbb{R}^3_F$

By classical mechanics, we get:

Proposition 28. *Let* $c(t)$ *be a* L*-periodic* $\mathbb{R}^3_F$*-geodesic for the pencil* (a, b) *with a hill interval* I *the period is given by*

$$L(G, I) := 2 \int_I \frac{dx}{\sqrt{1 - G^2(x)}}. \tag{3.3}$$

Moreover, the changes $\Delta y(c, [t, t+L]) = \Delta y(G, I)$ *and* $\Delta z(c, [t, t+L]) = \Delta y(G, I)$ *are given by*

$$\Delta y(G, I) := 2 \int_I \frac{G(x)dx}{\sqrt{1 - G^2(x)}} \quad \text{and} \quad \Delta z(G, I) := 2 \int_I \frac{G(x)F(x)dx}{\sqrt{1 - G^2(x)}}. \tag{3.4}$$

In [12, p. 162], we proved Proposition 28 using classical mechanics. In [11], we showed a similar statement using a generating function of the second type, see [8, Section 15]. $L(G, I)$, $\Delta y(G, I)$ and $\Delta z(G, I)$ are smooth functions with respect to the parameters (a, b) if and only if the corresponding geodesic $c(t)$ for (G, I) is x-periodic. We define an axillary map that will help us to prove Theorems A and B.

Definition 29. *The period map* $\Theta : (G, I) \to [0, \infty] \times \mathbb{R}$ *is given by*

$$\Theta(G, I) := (\Theta_1(G, I), \Theta_2(G, I)) := 2(\int_I \sqrt{\frac{1 - G(x)}{1 + G(x)}} dx, \int_I G(x) \frac{1 - F(x)}{\sqrt{1 - G^2(x)}} dx).$$

Θ_1 and Θ_2 are smooth function with respect the parameters (a, b) not only when the corresponding geodesic $c(t)$ for (G, I) is x-periodic, they are also smooth when $c(t)$ is a direct-type or homoclinic geodesic such that $G(x(t)) \to 1$ when $t \to \pm\infty$.

Corollary 30. *Let* $G(x)$ *be in* Pen_F*. Then:*

(1) $\Theta_1(G, I) = 0$ *if and only if* $G(x) = 1$*.*

(2) If $I = [x_0, x_1]$ *is compact, then* $\Theta_1(G, I)$ *is finite if and only if* x_0 *and* x_1 *are not critical point of* $G(x)$ *with value* -1*.*

We introduce an important concept called the travel interval:

Definition 31. *Let* $c(t)$ *be a* $\mathbb{R}^3_F$*-geodesic traveling during the time interval* $[t_0, t_1]$*. We say that* $\mathcal{I}[t_0, t_1] := x([t_0, t_1])$ *is the travel interval, counting multiplicity, of the* $c(t)$*.*

For instance, if $c(t)$ is a $\mathbb{R}^3_F$-geodesic with hill interval I such that its coordinate $x(t)$ is L-periodic, then $\mathcal{I}[t, t + L] = 2I$.

Corollary 32. *Let* $c(t)$ *be a* $\mathbb{R}^3_F$*-geodesic for* $G(x)$ *in* Pen_F *with travel interval* $\mathcal{I}$*. Then* $\Delta(c, [t_0, t_1])$ *from Definition 19 can be rewritten in terms of polynomial* $G(x)$ *and the travel interval* $\mathcal{I}$ *as follows;*

$$\Delta(c, [t_0, t_1]) = \Delta(G, \mathcal{I}) := (\int_{\mathcal{I}} \frac{dx}{\sqrt{1 - G^2(x)}}, \int_{\mathcal{I}} \frac{G(x)dx}{\sqrt{1 - G^2(x)}}, \int_{\mathcal{I}} \frac{G(x)F(x)dx}{\sqrt{1 - G^2(x)}}).$$

In the same way, the map $Cost(c, [t_0, t_1])$ from Definition 19 can be rewritten as follows:

$$Cost(c, [t_0, t_1]) = Cost(G, \mathcal{I}) := (\int_{\mathcal{I}} \frac{1 - G(x)}{\sqrt{1 - G^2(x)}} dx, \int_{\mathcal{I}} \frac{(1 - F(x))G(x)}{\sqrt{1 - G^2(x)}} dx)$$

Same proof that Proposition 28.

Corollary 33. $\lim_{n \to \infty} Cost_t(c, [-n, n])$ *is finite if and only if* $\lim_{t \to \pm\infty} G(x(t)) = 1$.

3.2 Direct-type geodesic

This section is devoted to proving Theorem A. Let $\gamma_d(t)$ be an arbitrary unitary direct-type geodesic in $J^k(\mathbb{R}, \mathbb{R})$ for a unitary polynomial $F_d(x)$ given by Corollary 26. We will consider the space $\mathbb{R}^3_{F_d}$ and the $\mathbb{R}^3_{F_d}$-geodesic $c_d(t) := \pi_{F_d}(\gamma_d(t))$ and prove the following Theorem:

Theorem 34. *The $\mathbb{R}^3_{F_d}$-geodesic $c_d(t)$ is a metric line $\mathbb{R}^3_{F_d}$.*

The strategy to prove Theorem 34 is the following: We take an arbitrary T and build a $\mathbb{R}^3_{F_d}$-geodesic $c_\infty(t)$ in $Min(K, \mathcal{T})$ and isometry φ in $Iso(\mathbb{R}^3_{F_d})$ such that $c([-T, T)) = \varphi(c_\infty(\mathcal{T}))$, where K is a compact subset of $\mathbb{R}^3_{F_d}$ and $\mathcal{T}$ is a compact interval. By corollary 23, $c(t)$ is minimizing in $[-T, T]$. Since T is arbitrary, $c(t)$ is a metric line.

Let $c_d(t) = (x(t), y(t), z(t))$. Without loss of generality, we can assume that $0 \leq \dot{x}(t)$ and $c_d(0) = (x, 0, 0)$ for every x in $(0, 1)$ since the proof for the case $0 \geq \dot{x}(t)$ is similar and we can use the t, y, and z translations.

3.2.1 The space $\mathbb{R}^3_{F_d}$

Corollary 35. *Let q_{max} be equal to $\max_{x\in[0,1]}\{x^{k_1}(1-x)^{k_2}q(x)\}$, where $q(x)$, k_1 and k_2 are given by Corollary 26. The set of all the direct-type $\mathbb{R}^3_{F_d}$-geodesic with hill interval $[0,1]$ is given by*

$$Pen_d := \{(a,b) = (s, 1-s) : \quad s \in (\frac{2}{q_{max}}, 1)\} \cup \{(a,b) = (-s, s-1) : \quad s \in (\frac{2}{q_{max}}, 1)\}.$$

Moreover, the map $\Theta_2(G,[0,1]) : Pen_d \to \mathbb{R}$ is one to one, and $Cost(c_d,[t_0,t_1])$ is bounded by $\Theta_d := \Theta_1(F,[0,1])$ for all $[t_0,t_1]$.

Proof. Since $F_d(x) \neq -1$ if x is in $[0,1]$, the constant Θ_d is finite. Let us prove that $Cost(c_d,[t_0,t_1])$ is bounded by Θ_d for all $[t_0,t_1]$. Using Corollary (32) and the condition $|F_d(x)| \leq 1$ for x in $[0,1]$, we find that:

$$|Cost_y(c_d,[t_0,t_1])| < Cost_t(c_d,[t_0,t_1]) < 2\int_{[0,1]}\sqrt{\frac{1-F_d(x)}{1+F_d(x)}}dx =: \Theta_1(F,[0,1]).$$

To prove that $\Theta_2(G,[0,1]) : M_d \to \mathbb{R}$ is one to one, we notice that the multiplication by minus sends $(s, 1-s)$ into $(-s, s-1)$ and $\Theta_2(G,I) = -\Theta_2(-G,I)$. Then, it is enough to consider the case $(a,b) = (s,1-s)$. We consider the one-parameter family of unitary polynomials $G_s(x) = s + (1-s)F_d(x)$. Thus, $\Theta_2(G_s,[0,1]) : (0, q_{max}) \to \mathbb{R}$ is one variable function, let us calculate its derivative:

$$\frac{d}{ds}\Theta_2(G_s,[0,1]) = \frac{d}{ds}\int_{[0,1]}\frac{(1-F_d(x))G_s(x)}{\sqrt{1-G_s^2(x)}}dx = \int_{[0,1]}\frac{1-F_d(x)}{(1-G_s^2(x))^{\frac{3}{2}}}dx.$$

Since $0 < 1 - F_d(x)$, then $0 < \frac{d}{ds}\Theta_2(G_s,[0,1])$. □

Lemma 36. *Let $\Omega(F_d) = hill(F_d) \times \mathbb{R}^2$ be the region and let $S_+(x, y, z) : \Omega \to \mathbb{R}$ be the calibration for $c_d(t)$ function given by Proposition 86, then $c_d(t)$ is minimizing between the curves that lay in the region Ω.*

Proof. The proof follows by Proposition 85, since $c_d(t)$ never touches the hill interval boundary in finite time. □

Corollary 37. *There exist $T_d^* > 0$ such that $y_d(t) > 0$ if $T_d^* < t$, and $y_d(t) < 0$ if $-T_d^* > t$.*

Proof. By construction, $\lim_{t\to\infty} y_d(t) = \infty$ and $\lim_{t\to-\infty} \Delta y_d(0) = -\infty$. □

Definition 38. *We define the following set*

$$Com([0,1]) := \big\{(c(t), [t_0, t_1]) : c(t) \text{ is a non-geodesic line, } x(t_0) \in [0,1] \text{ and } x(t_1) \in [0,1]\big\}.$$

Lemma 39. *Let us consider a sequence of pair $(c_n(t), [-n, n])$ in $Com([0, 1])$. If $Cost(c_n, [-n, n])$ is uniformly bounded, then there exists a compact subset $K_{\mathcal{H}}$ of $\mathcal{H}$ such that $\mathcal{I}[-n, n] \subset K_{\mathcal{H}}$ for all n.*

The proof is Appendix A.1.1.

3.2.2 Proof of Theorem 34

3.2.2.1 Set up the Proof of Theorem 34

Let T be arbitrarily large and consider the sequence of points $c_d(-n)$ and $c_d(n)$ where $T < n$ and n is in $\mathbb{N}$. Let $c_n(t) = (x_n(t), y_n(t), z_n(t))$ be a sequence of minimizing

$\mathbb{R}^3_{F_d}$-geodesics, in the interval $[0, T_n]$ such that:

$$c_n(0) = c_d(-n), \;\; c_n(T_n) = c_d(n) \;\text{ and }\; T_n \leq n. \tag{3.5}$$

We call the equations and inequality from 3.5 the endpoint conditions and the shorter condition, respectively. Since the endpoint condition holds for all n, then the sequence $c_n(t)$ holds asymptotic conditions:

$$\lim_{n\to\infty} c_n(0) = (0, -\infty, -\infty), \;\; \lim_{n\to\infty} c_n(T_n) = (1, \infty, \infty), \tag{3.6}$$

and the asymptotic period condition:

$$\lim_{n\to\infty} Cost_y(c_n, [0, T_n]) = \frac{1}{2}\Theta(F, [0,1]) = \frac{1}{2}\Theta_d. \tag{3.7}$$

Corollary 40. *The sequence of $\mathbb{R}^3_{F_d}$-geodesics $c_n(t)$ is not a sequence of geodesic lines and does not converge to a geodesic line. In particular, $c_n(t)$ does not converge to the abnormal geodesic.*

Proof. The Calibration function from Lemma 36 implies that if $c_n(t)$ is shorter than $c_d(t)$, then $c_n(t)$ must leave the region $[0,1] \times \mathbb{R}^2$ and come back, then $c_n(t)$ is a geodesic for non-constant polynomial $G_n(x)$, and $c_n(t)$ is not a geodesic line.

Let $\mathcal{I}_n$ travel interval of $c_n(t)$, then $c_n(t)$ cannot converge to a geodesic line, since $\lim_{n\to\infty} \mathcal{I}_n = [0,1]$ and the only line in the plane (x, θ_0) that travel from $\theta_0 = -\infty$ into $\theta_0 = \infty$ in a fine travel interval is the vertical line, but the vertical line has travel interval $[0,0]$. In particular, Lemma 17 implies $c_n(t)$ cannot converge to an abnormal geodesic. □

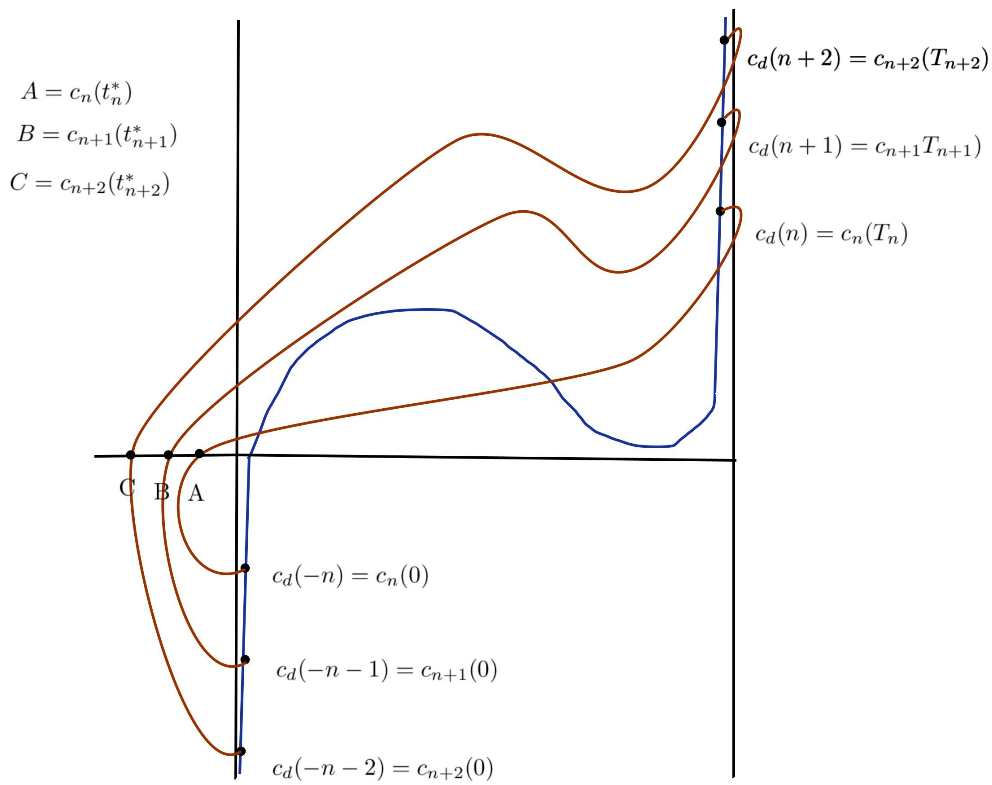

Figure 3.1: The images show the projection to $\mathbb{R}^2$, with coordinates (x, y), of direct type geodesic $c_d(t)$ and the sequence of geodesics $c_n(t)$.

The construction of the $\mathbb{R}^3_{F_d}$-geodesic c_n is such that the initial condition $c_n(0)$ is not bounded. The following Proposition provides a bounded initial condition.

Proposition 41. *Let n be a natural number larger than T_d^*, where T_d^* is given by Corollary 37, and let $K_0 := K_{\mathcal{H}} \times [-1,1] \times K_z$ be the compact set, where $K_{\mathcal{H}}$ is the compact set from Lemma 39 and $K_z := [-\Theta_d, \Theta_d]$. Then there exist a time $t_n^* \in (0, T_n)$ such that $c_n(t_n^*)$ is in K_0 for all $n > T_d^*$.*

Proof. Let n be a natural number larger than T_d^*. By construction, $y_n(0) < 0$ and $y_n(T_n) > 0$, the intermediate value theorem implies that exist a t_n^* in $(0, T_n)$ such that $y_n(t_n^*) = 0$. Since $Cost(c_n, [0, T_n])$ is bounded, by Lemma 39, there exists a compact set $K_{\mathcal{H}}$ such that $x_n(t)$ is in $K_{\mathcal{H}}$ for all t in $[0, T_n]$.

Let us prove that $|z_n(t_n^*)| \leq \Theta_d$: the endpoint conditions imply

$$\Delta y(c_d, [-n, n]) = \Delta y(c_n, [0, T_n]) \text{ and } \Delta z(c_d, [-n, n]) = \Delta z(c_n, [0, T_n]).$$

So $Cost_y(c_d, [-n, n]) = Cost_y(c_n, [0, T_n])$ and Corollary 35 tells us $Cost_y(c_n, [0, T_n])$ is bounded. By definition of $Cost_y$, it follows that:

$$z_n(t_n^*) - z_n(0) = \Delta z(c_n, [0, t_n^*]) = \Delta y(c_n, [0, t_n^*]) - Cost_y(c_n, [0, t_n^*]),$$

$$z_d(0) - z_d(-n) = \Delta z(c_d, [-n, 0]) = \Delta y(c_d, [-n, 0]) - Cost_y(c_d, [-n, 0]).$$

By construction, $\Delta y(c_n, [0, t_n^*]) = \Delta y(c_d, [-n, 0])$, $z_d(0) = 0$ and $z_n(0) = z_d(-n)$, then

$$|z_n(t_n^*)| = |Cost_y(c_n, [0, t_n^*]) - Cost_y(c_d, [-n, 0])| \leq \Theta_d.$$

We just proved $c_n(t_n^*)$ is in K. □

Let us consider the sequence of minimizing $\mathbb{R}^3_{F_d}$-geodesics $\tilde{c}_n(t) := c_n(t + t_n^*)$ in the interval $\mathcal{T}_n := [-t_n^*, T_n - t_n^*]$. $\tilde{c}_n(0)$ is bounded and minimizing $\mathbb{R}^3_{F_d}$-geodesics in the interval $\mathcal{T}_n$.

Corollary 42. *There exists a subsequence $\mathcal{T}_{n_j}$ such that $\mathcal{T}_{n_j} \subset \mathcal{T}_{n_{j+1}}$.*

Proof. Since $\tilde{c}_n(0)$ is bounded and $c(-t_n^*)$ and $c(T_n - t_n^*)$ are unbounded, it follows that $[-t_n^*, T_n - t_n^*] \to [-\infty, \infty]$ when $n \to \infty$. We can take a subsequence of intervals $\mathcal{T}_{n_j}$ such that $\mathcal{T}_{n_j} \subset \mathcal{T}_{n_{j+1}}$. □

For simplicity, we will use the notation $\mathcal{T}_n$ for the subsequence $\mathcal{T}_{n_j}$.

Lemma 43. *Let N be a natural number larger than T_d^*. Then there exist compact set $K_N \subset \mathbb{R}^3_F$ such that $c_n(t)$ is in $Min(K_N, \mathcal{T}_N)$ if $n > N$.*

Proof. Since $\tilde{c}_n(t)$ is minimizing on the interval $\mathcal{T}_n$, it follows that $\tilde{c}_n(t)$ is minimizing on the interval $\mathcal{T}_N \subset \mathcal{T}_n$ if $n > N$. Moreover, there exists a compact set K_N such that $\tilde{c}_n(\mathcal{T}_N) \subset K_N$, since $c_n(0)$ is in K_0 and $c_n(t)$ is a family of smooth functions defined on a compact set $\mathcal{T}_N$. □

Therefore, $\tilde{c}_n(t)$ has a convergent subsequence $\tilde{c}_{n_j}(t)$ converging to a $\mathbb{R}^3_{F_d}$-geodesic $c_\infty(t)$. Corollary 40 implies that $c_\infty(t)$ is a normal $\mathbb{R}^3_{F_d}$-geodesic for a polynomial $G(x)$ in Pen_{F_d}. The following Lemma provides the uniqueness of $G(x) = F_d(x)$:

Lemma 44. *$G(x) = F_d(x)$ is the unique polynomial in the pencil of $F_d(x)$ satisfying the asymptotic conditions given by (3.6) and (3.7).*

Proof. By Proposition 22, $\tilde{c}_n(t)$ has a convergent subsequence $\tilde{c}_{n_s}(t)$ converging to a minimizing geodesic $\tilde{c}(t)$ on the interval $\mathcal{T}_N$. Being a $\mathbb{R}^3_{F_d}$-geodesic, $c(t)$ is associated to a polynomial $G(x) = a+bF_d(x)$. $G(0) = a+b$ must be equal 1, to satisfy the asymptotic conditions given by (3.6). Then (a, b) is in Pen_d, the set defined in Corollary 35. Since the map $\Theta_1(a, b) : Pen_d \to \mathbb{R}$ is one to one, the unique polynomial in Pen_d satisfying the condition (3.6) and (3.7) is $G(x) = F_d(x)$. □

3.2.2.2 Proof of Theorem 34

Proof. Let $\tilde{c}_n(t)$ be the sequence of geodesics defined by the endpoint conditions (3.5). By Lemma 43, for all $N > T^*_d$ there exist a compact set K_N such that $c_n(t)$ is in $Min(K_N, \mathcal{T}_N)$ if $n > N$. By Proposition 22, there exist a subsequence $\tilde{c}_{n_j}(t)$ converging to a $\mathbb{R}^3_{F_d}$-geodesic $c_\infty(t)$ in $Min(K_N, \mathcal{T}_N)$. Corollary 40 implies that $c_\infty(t)$ is a normal geodesic for a polynomial $G(x)$ in Pen_{F_d}. Lemma 43 tells that $G(x) = F_d(x)$.

Since $c_\infty(t)$ and $c_d(t)$ are $\mathbb{R}^3_{F_d}$-geodesics for $F_d(x)$ with the same hill interval, there exists a translation $\varphi_{(y_0,z_0)}$, in $Iso(\mathbb{R}^3_{F_d})$ sending $c_\infty(t)$ to $c_d(t)$. Using N is arbitrary and $c_d([-T, T])$ is bounded, we can find compact set $K := K_N$ and $\mathcal{T} := \mathcal{T}_N$ such that $c_d([-T, T]) \subset \varphi_{(y_0,z_0)}(c_\infty(\mathcal{T}))$ and c_∞ is in $Min(K, \mathcal{T})$. Corollary 23 implies that $c_d(t)$ is minimizing in $[-T, T]$ and T is arbitrarily. Therefore, $c_d(T)$ is a metric line in $\mathbb{R}^3_F$. □

3.2.2.3 Proof of Theorem A

Proof. By Theorem 34, $c_d(t)$ is a metric line. Since π_F is a subRiemannian submersion and $\gamma_d(t)$ is the lift of $c_d(t)$, then Proposition 3 implies that the direct type geodesic $\gamma_d(t)$ is a metric line in $J^k(\mathbb{R}, \mathbb{R})$ $\square$

3.3 Homoclinic geodesics in $J^k(\mathbb{R}, \mathbb{R})$

This chapter is devoted to proving Theorem B. Let $\gamma_h(t)$ be the homoclinic geodesic in $J^k(\mathbb{R}, \mathbb{R})$ for $F_h(x) := 1 - 2x^{2n}$. We will consider the space $\mathbb{R}^3_{F_h}$ and the geodesic $c_h(t) := \pi_{F_h}(\gamma_h(t))$, then we will prove the following Theorem:

Theorem 45. *The geodesic $c_h(t)$ is a metric line $\mathbb{R}^3_{F_h}$.*

The following Theorem shows that the method used to prove Theorem 45 cannot be used to prove the odd case $F(x) := 1 - 2x^{2n+1}$.

Theorem 46. *Let $\gamma(t)$ be the homoclinic geodesic in $J^k(\mathbb{R}, \mathbb{R})$ for $F(x) := 1 - 2x^{2n+1}$ and $c(t) := \pi_{F_h}(\gamma(t))$ be the homoclinic $\mathbb{R}^3_{F_h}$-geodesic. Then $c(t)$ is not a metric line $\mathbb{R}^3_{F_h}$.*

The proof of Theorem 46 is in Section A.2.

3.3.1 The space $\mathbb{R}^3_{F_h}$

Without loss of generality, $c_h(0) = (1, 0, 0)$, by use of the t, y and z translations. By the time reversibility of the reduced Hamiltonian h_μ given by (3.1), it follows that $x(-n) = x(n)$ and $\Delta x(c_h, [-n, n])) := x(n) - x(-n) = 0$ for all n.

Lemma 47. *Let $c_h(t)$ be the homoclinic $\mathbb{R}^3_{F_h}$-geodesic for $F_h(x) := 1 - 2x^{2n}$, then*

$$\Theta_2(F, [0,1]) < 0.$$

Proof. By construction, $-xF_h'(x) = (2n-1)(1 - F_h(x))$. Using integration by parts it follows that

$$\begin{aligned}\Theta_2(F,[0,1]) &= \frac{-2}{2n-1}\int_{[0,1]} \frac{xF_h'(x)F(x)dx}{\sqrt{1-F_h^2(x)}} \\ &= \frac{2}{2n-1} x\sqrt{1-F_h^2(x)}|_0^1 - \frac{2}{2n-1}\int_{[0,1]} \sqrt{1-F_h^2(x)}dx.\end{aligned}$$

$x\sqrt{1-F_h^2(x)}|_0^1 = 0$ implies the desired result. □

Corollary 48. *The set of all the homoclinic $\mathbb{R}^3_{F_h}$-geodesics is given by*

$$Pen_h := \{(a,b) = (s, 1-s) : \ s \in (1,\infty)\} \cup \{(a,b) = (-s, s-1) : \ s \in (1,\infty)\}.$$

Moreover, the map $\Theta_2(G,[0,1]) : Pen_h \to \mathbb{R}$ is one to one and $Cost(c_h,[t_0,t_1])$ is bounded by $\Theta_1(F,[0,1]) := \Theta_h$ for all $[t_0,t_1]$.

Proof. The proof's first part is the same as the one from 35. To prove that $\Theta_1(a,b) : Pen_h \to \mathbb{R}$ is one to one, we notice the multiplication by minus sends $(s, 1-s)$ to $(-s, s-1)$ and $\Theta_2(G,I) = -\Theta_2(-G,I)$. It is enough we consider the one-parameter family of homoclinic polynomial $G_s(x) := s - (1-s)F_h(x)$ with hill interval $[0, \sqrt[2n]{\frac{1}{s}}]$. Thus, $\Theta_1(G_s, [0, \sqrt[2n]{\frac{1}{s}}]) : (0,\infty) \to \mathbb{R}$ is a one variable function and it is enough to show it is a monotone increasing function. Let us set up the change of variable $x = \sqrt[2n]{\frac{1}{s}}\tilde{x}$ so that $F(\tilde{x}) = 1 - 2\tilde{x}^{2n} = F_h(\tilde{x})$ and

$$\Theta_2(G_s, [0, \sqrt[2n]{\frac{1}{s}}]) = \int_{[0,\ \sqrt[2n]{\frac{1}{s}}]} \frac{2x^{2n}G_s(x)}{\sqrt{1-G_s^2(x)}}dx = (\sqrt[2n]{\frac{1}{s}})^{n+1}\Theta_h.$$

Since $(\sqrt[2n]{\frac{1}{s}})^{n+1}$ is monotone decreasing and Θ_h is negative. Then $\Theta_2(G_s,[0, \sqrt[2n]{\frac{1}{s}}])$ is a monotone increasing function with respect to s. □

Corollary 49. *There exist $T_h^* > 0$ such that $y_h(t) > 0$ if $T_h^* < t$ and $y_h(t) < 0$ if $-T_h^* > t$. Moreover, $Cost_y(c_h, [-t,t]) < 0$ if $T_h^* < t$.*

Proof. Since $Cost_y(c_h, [-t,t]) \to \Theta_2(F_h, [0,1])$ when $t \to \infty$ and $\Theta_2(F_h, [0,1]) < 0$, we can find the desired T_h^*. The rest of the proof is equal to Corollary 37. □

3.3.2 Set up the proof of Theorem 45

Let T be arbitrarily large and consider the sequence of points $c_h(-n)$ and $c_h(n)$ where $T < n$ and n is in $\mathbb{N}$. Let $c_n(t) = (x_n(t), y_n(t), z_n(t))$ be a sequence of minimizing $\mathbb{R}^3_{F_h}$-geodesics in the interval $[0, T_n]$ such that:

$$c_n(0) = c_h(-n), \;\; c_n(T_n) = c_h(n) \;\text{ and }\; T_n \leq n. \tag{3.8}$$

We call the equations and inequality from (3.8) the endpoint conditions and the shorter condition, respectively. Since the endpoint condition holds for all n, the sequence $c_n(t)$ has the asymptotic conditions:

$$\lim_{n\to\infty} c_n(0) = (0, -\infty, -\infty), \;\; \lim_{n\to\infty} c_n(T_n) = (0, \infty, \infty), \tag{3.9}$$

and the asymptotic period condition

$$\lim_{n\to\infty} Cost_y(c_n, [0, T_n]) = C_h. \tag{3.10}$$

The following Corollary tells us $c_n(t)$ is not a sequence of line geodesics. We remark that applying the calibration function from proposition 86 is impossible.

Corollary 50. *Let n be larger than T_h^*, where T_h^* is given by Corollary 49, then the sequence of geodesics $c_n(t)$ neither is a sequence of geodesic lines, nor converge to a geodesics line. In particular, $c_n(t)$ does not converge to the abnormal geodesic.*

Proof. Let us assume that $c_n(t)$ is a sequence of geodesic lines. Since $\Delta x(c_h, [-t,t])) = 0$ for all n and $\Delta y(c_h, [-t,t])) > 0$ for all $n > T_h^*$, the unique geodesic line satisfying these conditions is the one generated by the polynomial $G_n(x) = 1$. Since $1 - F_h(x) > 0$ for all x, then $(1 - F_h(x))G_n(x) > 0$ for all x and it follows that:

$$Cost_y(c_n, [0, T_n]) = \int_0^T (1 - F_h(x(t)))G_n(x(t))dt > 0.$$

This contradicts the endpoint conditions given by (3.8) since $Cost_y(c_h, [-t,t]) < 0$ if $T_h^* < t$. The same proof follows if $c_n(t)$ converges to a geodesics line $c(t)$ generated by $G(x) = 1$, since there exists N big enough that $G_n(x) > \frac{1}{2}$ for $n > N$. □

Notice that this proof cannot be done in the case $F_h(x) = 1 - 2x^{2n+1}$. In Section A.2 under the hypothesis $F_h(x) = 1 - 2x^{2n+1}$, we will find a sequence of curves $c_n(t)$ shorter than $c_h(t)$ than $c_h(t)$ that converges to the abnormal geodesic.

The following Proposition provides the bounded initial condition.

Proposition 51. *Let n be a natural number larger than T_h^*, where T_h^* is given by Corollary 49, and let $K_0 = K_{\mathcal{H}} \times [-1,1] \times [-C_h, C_h]$ the compact set, where $K_{\mathcal{H}}$ and C_h is a compact set and the constant defined by Lemma 39 and Corollary 48, respectively. Then there exist a time $t_n^* \in (0, T_n)$ such that $c_n(t_n^*)$ is in K_0 for all $n > T_1^*$.*

Same proof as Proposition 41. Consider the sequence of minimizing $\mathbb{R}^3_{F_h}$-geodesic $\tilde{c}_n(t) := c_n(t + t_n^*)$ in the interval $\mathcal{T}_n := [-t_n^*, T_n - t_n^*]$, so $\tilde{c}_n(0)$ is bounded and minimizing on the interval $\mathcal{T}_n$.

Corollary 52. *There exists a subsequence $\mathcal{T}_{n_j}$ such that $\mathcal{T}_{n_j} \subset \mathcal{T}_{n_{j+1}}$.*

The proof of Corollary 52 is equal as the of Corollary 42. For simplicity, we will use the notation $\mathcal{T}_n$ for the subsequence v_{n_j}.

Lemma 53. *There exist compact set $K_N \subset \mathbb{R}^3_F$ such that $c_n(t)$ is in $Min(K_N, \mathcal{T}_N)$ if $n > N$.*

The proof of Lemma 53 is equal to the of Lemma 43. Therefore, $\tilde{c}_j(t)$ has a convergent subsequence $\tilde{c}_{j_i}(t)$ converging to a $\mathbb{R}^3_{F_h}$-geodesic $c_\infty(t)$. Corollary 50 implies that $c_\infty(t)$ is a normal $\mathbb{R}^3_{F_h}$-geodesic for a polynomial $G(x)$ in Pen_{F_h}. The following Lemma provides the uniqueness of $G(x) = F_h(x)$.

Lemma 54. *$G(x) = F_h(x)$ is the unique polynomial in the pencil of $F_h(x)$ satisfying the asymptotic conditions given by (3.9) and (3.10).*

Proof. By Proposition 22 $\tilde{c}_n(t)$ has a convergent subsequence $\tilde{c}_{n_s}(t)$ converging to a minimizing geodesic $\tilde{c}(t)$ on the interval $\mathcal{T}_N$. Being a geodesic in $\mathbb{R}^3_{F_h}$, $c(t)$ is associated to a polynomial $G(x) = a + bF_h(x)$. $G(0) = a + b$ must be equal 1, to satisfy the asymptotic conditions given by (3.9). Then (a, b) is in Pen_h, the set defined in Corollary 48. Since the map $\Theta_1(G, I) : Pen_h \to \mathbb{R}$ is one to one, the unique polynomial in Pen_h satisfying the condition (3.9) is $G(x) = F_h(x)$. □

The proof of Theorems 45 and B are the same as the proof of Theorems 34 and A, respectively.

Chapter 4

Metric lines in Engel type $\mathrm{Eng}(n)$

This Chapter is devoted to proving Theorem C.

4.1 The Engel type group $\mathrm{Eng}(n)$ as subRiemannian manifold

Let $\mathrm{Eng}(n)$ be the Carnot group with growth vector $(n+1, 2n+1, 2n+2)$ and whose first layer $\mathfrak{g}_1$, framed by $\{E^1, \cdots, E^n, E^0_{\mathfrak{a}}\}$, generates the following Lie algebra:

$$E^i_{\mathfrak{a}} := [E^i, E^0] \;\; i = 1, \cdots, n, \;\; \text{and} \;\; E^{n+1}_{\mathfrak{a}} := [E^i, E^i_{\mathfrak{a}}]. \tag{4.1}$$

Otherwise, zero. The Lie algebra $\mathfrak{a}$ is given by E^0, $E^1_{\mathfrak{a}}, \cdots$, $E^n_{\mathfrak{a}}$ and $E^{n+1}_{\mathfrak{a}}$: In this case $\mathcal{H} \simeq \mathbb{R}^n$, $\mathcal{V} \simeq \mathbb{R}$ and $[\mathfrak{h}, \mathfrak{h}] = 0$. The $\mathfrak{a}^*$ valued one-form $\mathcal{A}_{\mathrm{Eng}(n)}$ is given by

$$\alpha_{\mathrm{Eng}(n)} = d\theta \otimes (e_0 + \sum_{i=1}^{n} x_i e_i + \frac{1}{2}||x||^2_{\mathcal{H}} e_{i+1})$$

If $\mu = \sum_{\ell=0}^{n+1} a_\ell e^\ell$ in $\mathfrak{a}^*$ then the reduced Hamiltonian H_μ is given by

$$H_\mu(p_x, x) = \frac{1}{2}||p_x||^2_{\mathcal{H}} + \frac{1}{2}F^2_\mu(x) \text{ where } F_\mu(x) = a_0 + \sum_{i=1}^{n} a_i x_i + a_{n+1}\frac{1}{2}||x||^2_{\mathcal{H}}. \quad (4.2)$$

Let us consider the case $a_{n+1} \neq 0$, if we set up the change of coordinates

$$(\hat{x}_1, \cdots, \hat{x}_n) = (\frac{a_1}{a_{n+1}} + x_1, \ldots, \frac{a_1}{a_{n+1}} + x_n) \text{ and define } (b_1, b_2) = (a_0 - \frac{1}{2}\sum_{i=1}^{n} a_i^2, \frac{a_{n+1}}{2}).$$

Then

$$H_\mu(p_x, x) = \frac{1}{2}||p_x||^2_{\mathcal{H}} + \frac{1}{2}F^2_\mu(r) \text{ where } F_\mu(x) = b_1 + b_2 r^2. \quad (4.3)$$

Where $r := ||\hat{x}||$. We conclude that after a translation, the reduced Hamiltonian H_μ is the radial an-harmonic oscillator.

4.1.1 History of the notation Eng(n)

In [17], E. Le Donne and F. Tripaldi used the notation $N_{6,3,1a^*}$ to denote the Carnot group Eng(2). After making the symplectic reduction in the general context, we used to find the reduced Hamiltonian H_μ for particular examples from [17], one of these was $N_{6,3,1a^*}$. We consider the subRiemannian geodesic flow on $N_{6,3,1a^*}$ and found that the reduced Hamiltonian H_μ is the plane radial an-harmonic oscillator. We were inspired to define the Carnot group Eng(n) by the work of R. Montgomery, in [22]. Where he considered the subRiemannian geodesic flow in Eng , and he showed that the reduced Hamiltonian H_μ is the an-harmonic oscillator. Latter, we discovered the relation between the homoclinic geodesics in Eng(n) and the Euler-Soliton.

4.2 Geodesics in Eng(n)

We split the dynamics of reduced Hamiltonian H_μ, given by (4.2), into two cases, when $p_{\theta_{n+1}} = a_{n+1} = 0$ and $p_{\theta_{n+1}} = a_{n+1} \neq 0$. In the first case, the Hamiltonian H_μ has a quadratic potential on the x coordinates, so the problem is a small oscillation system, see [8, Chapter 5]. In the second case, the reduced dynamics correspond to the radial an-harmonic oscillator, see (4.3). In sub-Section 4.2.2, we will reduce, again, the radial an-harmonic oscillator into a Hamiltonian $H_{\mu,\ell}(p_r, r)$ with one degree of freedom and effective potential $V_{ef}(r)$, see (4.5). We use the classification of one degree of freedom systems to classify the case $a_{n+1} \neq 0$, as we did in $J^k(\mathbb{R}, \mathbb{R})$.

Definition 55. *Let $\gamma(t)$ be a non-geodesic line in* Eng(n)*, then:*

1. *We say a geodesic $\gamma(t)$ is oscillatory if $a_{n+1} = 0$.*
2. *we say a geodesic $\gamma(t)$ is radial if $a_{n+1} \neq 0$.*
3. *We say a geodesic $\gamma(t)$ is r-periodic if the dynamics of reduced system (4.5) is periodic.*
4. *We say a geodesic $\gamma(t)$ is r-homoclinic if the dynamics of reduced system (4.5) is a homoclinic orbit.*

The following Theorem tells that oscillatory and r-periodic geodesic are not metric lines:

Theorem 56. *The oscillatory and r-periodic geodesics are not metric lines in* Eng(n)*.*

The proof is in Appendix A.5.

4.2.1 Case $a_{n+1} \neq 0$

Proposition 57. *Let $SO(n)$ be the group of rotation of $\mathcal{H}$, then the Lie algebra $\mathfrak{eng}(n)$ is invariant under the action of $SO(n)$ given by*

$$\tilde{E}^j = \sum_{i=1}^{n} Q_{j,i} E^i, \;\; \tilde{E}^0_{\mathfrak{a}} = E^0_{\mathfrak{a}}, \;\; \tilde{E}^j_{\mathfrak{a}} = \sum_{i=1}^{n} Q_{j,i} E^i_{\mathfrak{a}}, \;\; \tilde{E}^{n+1}_{\mathfrak{a}} = E^{n+1}_{\mathfrak{a}},$$

where $Q := (Q_{j,i})$ is in $SO(n)$. Moreover, the action on $\mathfrak{eng}(n)$ induces an isometric action φ_Q on $\mathrm{Eng}(n)$. If (x, θ) are exponential coordinates of the second type defined in 4.1, then $\varphi_Q(x, \theta) = (\tilde{x}, \tilde{\theta})$ is given by

$$\tilde{x}_j = \sum_{i=1}^{n} Q_{j,i} x_i, \;\; \tilde{\theta}_0 = \theta_0, \;\; \tilde{\theta}^2_j = \sum_{i=1}^{n} Q_{j,i} \theta^2_i, \;\; \tilde{\theta}^3_1 = \theta^3_1 \;\; \textit{where} \;\; Q := (Q_{j,i}) \in SO(n).$$

Proof. Let us prove that vectors $\{\tilde{E}^1, \dots, \tilde{E}^n, \tilde{E}^0_{\mathfrak{a}}, \dots, \tilde{E}^{n+1}_{\mathfrak{a}}\}$ satisfy the bracket relations given by (4.1): Let us start with the first layer $\mathfrak{g}_1$,

$$[\tilde{E}^j, \tilde{E}^0_{\mathfrak{a}}] = \sum_{i=1}^{n} Q_{j,i} [E^j, E^0_{\mathfrak{a}}] = \sum_{i=1}^{n} Q_{j,i} E^j_{\mathfrak{a}} = \tilde{E}^j_{\mathfrak{a}}.$$

Let us verify that the bracket relations hold for the second layer $\mathfrak{g}_2$,

$$[\tilde{E}^j, \tilde{E}^k_{\mathfrak{a}}] = \sum_{i=1, i'=1}^{n} Q_{j,i} Q_{j,i'} [E^j, E^k_{\mathfrak{a}}] = \sum_{i=1, i'=1}^{n} Q_{j,i} Q_{j,i'} \delta^k_j E^{n+1}_{\mathfrak{a}} = E^{n+1}_{\mathfrak{a}} = \tilde{E}^{n+1}_{\mathfrak{a}}.$$

□

Definition 58. *Let M_{x_1,x_2} be the 6 dimensional sub-manifold of $\mathrm{Eng}(n)$ given by*

$$M_{x_1,x_2} := \{(x, \theta) \in \mathrm{Eng}(n) : x = (x_1, x_2, 0, \dots, 0) \;\; \textit{and} \;\; \theta = (\theta_0, \theta_1, \theta_2, 0, \dots, 0, \theta_n)\}$$

Lemma 59. *Let $\gamma(t)$ be a geodesic in $\mathrm{Eng}(n)$ such that $\gamma(0)$ is in M_{x_1,x_2} and $\dot{\gamma}(0)$ is in $T_{\gamma(0)} M_{x_1,x_2}$, then $\gamma(t)$ lies in M_{x_1,x_2} for all t.*

Proof. By Hamilton equation we have $\dot{x}_i = p_{x_i}(t)$, $\dot{p}_{x_i} = 2b_2 x_i(t) F_\mu(r(t))$ and $\dot{\theta}_i(t) = x_i(t)F_\mu(r(t))$. The initial condition implies $\dot{x}_i(0) = p_{x_i}(0) = 0$, $\dot{p}_{x_i}(0) = 0$ and $\dot{\theta}_i(0) = 0$ for all $2 < i \leq n$. Therefore, $\dot{x}_i(t) = 0$, $\dot{p}_{x_i}(t) = 0$ and $\theta_i(t) = 0$ for all t and $2 < i \leq n$. □

Corollary 60. *Any geodesic in* $\mathrm{Eng}(n)$ *with* $p_{\theta_{n+1}} \neq 0$ *has the form* $\gamma(t) = \varphi_Q(\gamma_0(t))$*, where* φ_Q *is given by 57 and* $\gamma_0(t)$ *is a geodesic in* M_{x_1,x_2}.

Then, it is enough to understand the dynamics of the plane an-harmonic oscillator to describe the dynamics of the radial an-harmonic oscillator.

4.2.2 The plane radial an-harmonic oscillator

The reduced Hamiltonian H_μ defined by equation (4.3) in polar coordinates is given by

$$H_\mu(p_x, p_\theta, r, \theta) := \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2}) + \frac{1}{2}F_\mu^2(r). \tag{4.4}$$

Since the potential is radial, θ is a cyclic coordinate, and p_θ is constant. If $p_\theta = \ell$, then the effective potential is $\frac{1}{2}V_{ef}(r)$, where $V_{ef}(r) := \frac{\ell^2}{r^2} + F_\mu^2(r)$ and the reduced Hamiltonian H_μ can be reduced, again, to one-degree of freedom Hamiltonian system given by

$$H_{\mu,\ell}(p_x, r) := \frac{1}{2}p_r^2 + \frac{1}{2}V_{ef}(r). \tag{4.5}$$

Fixing the energy level $H_{\mu,\ell} = \frac{1}{2}$ and using Hamilton equation $\dot{r} = p_r$, we reduced the dynamics to a quadrature in the radial coordinate.

Definition 61. *We say an interval $R = [r_{min}, r_{max}]$ is the radial hill interval of the effective potential V_{ef}, if $R = V_{ef}^{-1}[0, 1]$*

Definition 62. *We denote by $hill(\mu, \ell)$ the closed annulus given by*

$$hill(\mu, \ell) := \{(x_1, x_2) \in \mathbb{R}^2 : r_{min}^2 \leq x_1^2 + x_1^2 \leq r_{max}^2\}.$$

We call $hill(\mu, \ell)$ the hill region of the reduced Hamiltonian $H_{\mu,\ell}$ $F_\mu(r)$, where r_{min} and r_{max} are given by Corollary 61.

Corollary 63. *The plane an-harmonic oscillator has an equilibrium at $r = 0$ if and only if $\ell = 0$ and $F_\mu(0) = \pm 1$.*

Proof. Let us assume $(p_r, p_\theta, r, \theta) = (0, 0, 0, \theta_0)$ is an equilibrium point. By, Hamilton's equations for H_μ and $p_\theta = 0$ imply $\ell = 0$. Then, we can read the conservation of the energy $\frac{1}{2} = H_\mu$ as $\frac{1}{2} = \frac{1}{2}(p_r^2 + F_\mu^2(r))$. If we plug $(p_r, p_\theta, r, \theta) = (0, 0, 0, \theta_0)$ into H_μ we have that $F_\mu(0) = 1$.

Conversely, let us assume $\ell = 0$ and $F_\mu(0) = \pm 1$: then the reduced Hamilton equation for H_μ with the conditions $\ell = 0$ imply $\dot{p}_\theta = 0$. The conservation of energy tells $\dot{p}_r = 0$ at $r = 0$. □

4.3 The space $\mathbb{R}_F^{n+2}$

$SO(n) \times \mathbb{R}^2$ acts on $\mathbb{R}_F^{n+2}$ by rotation and translation. If Q is in $SO(\mathcal{H})$ and (y_0, z_0) is in $\mathbb{R}^2$, then $\varphi_{(Q,y_0,z_0)}(x, y, z) = (Qx, y+y_0, z+z_0)$ and $\varphi_{(Q,y_0,z_0)}$ is in $Iso(\mathbb{R}_F^{n+2})$.

Lemma 64. *If* $\mathbb{R}^2_{(x_1,x_2)} := \{(x_1, \ldots, x_n, y, z) \in \mathbb{R}^{n+2}_{F_h} : 0 = x_3 = \cdots = x_n\}$, *then every geodesic* $\mathbb{R}^{n+2}_F$*-geodesic* $c(t)$ *with* $b \neq 0$ *has the form* $\varphi_{(Q,0,0)}(c_0(t))$ *where* $c_0(t)$ *is a* $\mathbb{R}^{n+2}_F$*-geodesic in* $\mathbb{R}^2_{(x_1,x_2)}$ *for all* t*.*

Therefore, it is enough to work on $\mathbb{R}^4_{F_h}$. If $F(r)$ is given by (4.3) and H_F is the Hamiltonian defined by equation (2.7), then $V_{ef}(r) := \frac{\ell^2}{r^2} + G^2(r)$ is the effective potential of the reduced system. This inspires the following definition.

Definition 65. *We say that the three-dimensional space* Pen_V *is the pencil of* $F(r)$*, if* $Pen_V := \{V_{ef}(r) = \frac{\ell^2}{r^2} + G^2_\mu(r) : G(r) \in Pen_F\}$*.*

We define an axillary map that will help us prove Theorems C.

Definition 66. *The period map* $\Theta(G, \ell, R) : (G, \ell, R) \to [0, \infty] \times \mathbb{R}$ *is given by*

$$\Theta(G, \ell, R) := (\Theta_1(G, \ell, R), \Theta_2(G, \ell, R)) := 2\Big(\int_R \frac{1 - G(r)}{\sqrt{1 - V_{ef}(r)}} dr, \int_R \frac{G(r)(1 - F(r))}{\sqrt{1 - V_{ef}(r)}} dr\Big).$$

Corollary 67. *Let* $G(r)$ *be in* Pen_F *and let* ℓ *be the angular momentum. Then:*

(1) $\Theta_1(G, \ell, R) = 0$ *if and only if* $G(r) = 1$ *and* $\ell = 0$*.*

(2) If R *is compact, then* $\Theta_1(G, \ell, R)$ *is finite if and only if* 0 *is in* $\mathcal{R}$ *and* $G(0) = -1$*.*

We introduce an important concept called the radial travel interval:

Definition 68. *Let* $c(t)$ *be a* $\mathbb{R}^{n+2}_F$*-geodesic traveling in the time interval* $[t_0, t_1]$*. We say* $\mathcal{R}[t_0, t_1] := r([t_0, t_1])$ *is the travel interval of the* $c(t)$*, counting multiplicity.*

For instance, if $c(t)$ is an $\mathbb{R}^{n+2}_F$-geodesic with hill interval R such that its coordinate r is L-periodic then $\mathcal{R}[t, t + L] = 2R$.

Corollary 69. *Let $c(t)$ be an $\mathbb{R}_F^{n+2}$-geodesic for $V_{ef}(r)$ in Pen_V and let $\mathcal{R}$ be its radial travel interval. Then $\Delta(c,[t_0,t_1])$ from Definition 19 can be rewritten in terms of the effective potential $V_{ef}(r)$ and the travel radial interval $\mathcal{R}$ as follows;*

$$\Delta(c,[t_0,t_1]) = \Delta(G,\ell,\mathcal{R}) := (\int_{\mathcal{R}} \frac{dr}{\sqrt{1-V_{ef}(r)}}, \int_{\mathcal{R}} \frac{G(r)dr}{\sqrt{1-V_{ef}(r)}}, \int_{\mathcal{R}} \frac{G(r)F(r)dr}{\sqrt{1-V_{ef}(r)}}).$$

In the same way, the map $Cost(c,[t_0,t_1])$ from Definition 19 can be rewritten as follows:

$$Cost(c,[t_0,t_1]) = Cost(G,\ell,\mathcal{R}) := 2(\int_{\mathcal{R}} \frac{1-G(r)}{\sqrt{1-V_{ef}(r)}}dr, \int_{\mathcal{R}} \frac{(1-F(r))G(r)}{\sqrt{1-V_{ef}(r)}}dr)$$

The proof of Corollary 69 is the same proof of Proposition 28.

4.4 Homoclinic geodesics in $\mathrm{Eng}(n)$

This section is devoted to proving C. Without loss of generality, let $\gamma_h(t)$ be the homoclinic geodesic in Eng(2) for $F_h(x) := 1-2r^2$, whose reduced dynamics has initial condition $x = (1,0)$. Using Carnot dilatation and rotation, it is enough to prove this case.

Let $\gamma_h(t)$ be the homoclinic geodesic in $\mathrm{Eng}(n)$ for $F_h(x) := 1-2r^2$. We consider the geodesic $c_h(t) := \pi_{F_h}(\gamma_h(t))$ in the space $\mathbb{R}^4_{F_h}$, and will prove the following Theorem.

Theorem 70. *The direct type geodesic $c_h(t)$ is a metric line $\mathbb{R}^{n+2}_{F_h}$.*

Without loss of generality, we take the initial condition $c_h(0) = (1,0,0,0)$. By construction, $c_h(t) = (x_1(t),0,y(t),z(t))$. Moreover, $x_1(-t) = x_1(t)$ and $\Delta x(c_h,[-n,n])) := x(n) - x(-n) = 0$ for all n.

4.4.1 The space $\mathbb{R}^4_{F_h}$

Lemma 71. *Let $c_h(t)$ be the homoclinic $\mathbb{R}^4_{F_h}$-geodesic for $F_h(r) := 1 - 2r^2$, then*

$$\Theta_2(F, [0,1]) < 0. \tag{4.6}$$

There exist $T^*_h > 0$ such that $y_h(t) > 0$ if $T^*_h < t$ and $y_h(t) < 0$ if $-T^*_h > t$. Moreover, $Cost_y(c_h, [-t,t]) > 0$ if $T^*_h < t$.

Same proof as Lemma 47 and Corollary 49.

Corollary 72. *The set of all the homoclinic $\mathbb{R}^4_F$-geodesic $Pen_h \subset Pen_V$ is given by*

$$Pen_h := \{(a,b,\ell) = (s, 1-s, 0) : \ \ s \in (1,\infty)\} \cup \{(a,b,\ell) = (-s, s-1, 0) : \ \ s \in (1,\infty)\}.$$

Moreover, the map $\Theta_2(G, \ell, \mathcal{R}) : Pen_h \to \mathbb{R}$ is one to one, and $Cost(c_h, [t_0, t_1])$ is bounded by $\Theta_1(F, 0, [0,1]) := \Theta_h$ for all $[t_0, t_1]$.

Same proof as Corollary 48.

Definition 73. *Let $B_{\mathcal{H}}$ be the ball of radius one on $\mathcal{H}$. We define the following set*

$$Com(B_{\mathcal{H}}) := \big\{(c(t), [t_0, t_1]) : c(t) \text{ is a non-geodesic line, } x(t_0) \in B_{\mathcal{H}} \text{ and } x(t_1) \in B_{\mathcal{H}}\big\}.$$

Lemma 74. *Let us consider a sequence of pairs $(c_n(t), [-n, n])$ in $Com(B_{\mathcal{H}})$. If $Cost(c_n, [-n, n])$ is uniformly bounded then there exists a compact subset $K_{\mathcal{H}}$ of $\mathcal{H}$ such that $x([-n, n]) \subset K_{\mathcal{H}}$ for all pair n.*

Same proof as Lemma 39.

4.4.2 Set up for the proof of Theorem 70

Let T be arbitrarily large and consider the sequence of points $c_h(-n)$ and $c_h(n)$ where $T < n$ and n is $\mathbb{N}$. Let $c_n(t) = (x_n(t), y_n(t), z_n(t))$ be a sequence of minimizing geodesics in the interval $[0, T_n]$ such that

$$c_n(0) = c_h(-n), \;\; c_n(T_n) = c_h(n) \;\text{ and }\; T_n \leq n. \tag{4.7}$$

We call the equations and inequality from (4.7) the endpoint conditions and the shorter condition, respectively. Since the endpoint condition holds for all n, the sequence $c_n(t)$ has the asymptotic conditions:

$$\lim_{n\to\infty} c_n(0) = (0, 0, -\infty, -\infty), \;\; \lim_{n\to\infty} c_n(T_n) = (0, 0, \infty, \infty), \tag{4.8}$$

and the asymptotic period condition

$$\lim_{n\to\infty} Cost_y(c_n, [0, T_n]) = C_h. \tag{4.9}$$

The following Proposition provides the bounded initial condition.

Proposition 75. *Let n be a natural number larger than T_h^*, where T_h^* is given by Corollary 71, and let $K = K_{\mathcal{H}} \times [-1, 1] \times [C_h, C_h]$ be the compact set, where $K_{\mathcal{H}}$ and C_h is a compact set and the constant defined by Lemma 74 and Corollary 72, respectively. Then there exists a time $t_n^* \in (0, T_n)$ such that $c_n(t_n^*)$ is in K^* for all $n > T_1^*$.*

Same proof that Proposition 41. Consider the sequence of minimizing geodesics $\tilde{c}_n(t) := c_n(t + t_n^*)$ on the interval $\mathcal{T}_n := [-t_n^*, T_n - t_n^*]$ so that $\tilde{c}_n(0)$ is bounded and minimizing on the interval $\mathcal{T}_n$.

Corollary 76. *There exists a subsequence $\mathcal{T}_{n_j}$ such that $\mathcal{T}_{n_j} \subset \mathcal{T}_{n_{j+1}}$.*

The proof of Corollary 76 is equal as the of Corollary 42. For simplicity we will use the notation $\mathcal{T}_n$ for the subsequence $\mathcal{T}_{n_j}$.

Lemma 77. *There exists compact set $K_N \subset \mathbb{R}^{n+2}_F$ such that $c_n(t)$ is in $Min(K_N, \mathcal{T}_N)$ if $n > N$.*

The proof of Lemma 77 is equal as the of Corollary 43. Therefore $\tilde{c}_n(t)$ has a convergent subsequence $\tilde{c}_{n_j}(t)$ converging to a $\mathbb{R}^{n+2}_{F_h}$-geodesic $c_\infty(t)$, then $c_\infty(t)$ is a $\mathbb{R}^{n+2}_{F_h}$-geodesic for a polynomial $G(x)$ in Pen_{F_h}. The following Lemma provides the uniqueness of $G(x) = F(x)$.

Lemma 78. $(G, \ell) = (F_h, 0)$ *is the unique pair satisfying the asymptotic conditions given by* (4.8) *and* (4.9)

Proof. Corollary 63 tells that the reduced system has an equilibrium point if and only if $G(0) = 1$ and $\ell = 0$. The rest of the proof is the same from Lemma 54. □

4.4.3 Proof of Theorem 70

Proof. Let $\tilde{c}_n(t)$ be the sequence of geodesics defined by the endpoint conditions (4.7). By Lemma 77, for all $N > T^*_d$ there exist a compact set K_N such that $c_n(t)$ is in $Min(K_N, \mathcal{T}_N)$ if $n > N$. By Proposition 22, there exist a subsequence $\tilde{c}_{n_j}(t)$ converging to a $\mathbb{R}^{n2}_{F_h}$-geodesic $c_\infty(t)$ in $Min(K_N, \mathcal{T}_N)$. Corollary 40 implies that $c_\infty(t)$ is a normal geodesic for a polynomial $G(x)$ in Pen_{F_h}. Lemma 43 tells that $G(x) = F_h(x)$.

Since $c_\infty(t)$ and $c_h(t)$ are $\mathbb{R}^{n+2}_{F_h}$-geodesics for $F_h(r)$ with the same radial hill interval, there exists an isometry $\varphi_{(Q,y_0,z_0)}(x, y, z) = (Qx, y+y_0, z+z_0)$ sending $c_\infty(t)$ to $c_h(t)$. Using N is arbitrary and $c_d([-T, T])$ is bounded, we can find compact sets $K := K_N$ and $\mathcal{T} := \mathcal{T}_N$ such that $c_d([-T, T]) \subset \varphi_{(Q,y_0,z_0)}(c_\infty(\mathcal{T}))$ and c_∞ is in $Min(K, \mathcal{T})$. Corollary 23 implies that $c_h(t)$ is minimizing in $[-T, T]$ and T is arbitrarily. Therefore, $c_h(T)$ is a metric line in $\mathbb{R}^{n+2}_{F_h}$. □

The proof of Theorem C is the same as the proof of Theorem A.

Chapter 5

Conclusion

(1) We developed a new method to prove that a geodesic is a metric line. Theorem A proves the Conjecture 6 for the direct-type case, and the problem remains open for the homoclinic case. Theorem 46 says we cannot use the space R^3_F to prove the Conjecture. However, Theorem 46 does not imply that the Conjecture is false. The homoclinic case can be solved by showing the corresponding period map in $J^k(\mathbb{R}, \mathbb{R})$ restricted to the homoclinic geodesics is one-to-one.

(2) The Carnot group $N_{6,3,1}$ has a non-integrable subRiemannian geodesic flow, see Theorem 100. However, $N_{6,3,1}$ has one family of homoclinic geodesics up to a dilatation. This family is related to the Euler-Soliton: there exists a two-plane inside $\mathbb{R}^3$ such that the projection to the homoclinic geodesic is the Euler-Soliton, as we say in $\text{Eng}(n)$. In future work, we will prove that this family's homoclinic geodesics are metric lines in $N_{6,3,1}$.

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Appendix A

Metric lines in$J^k(\mathbb{R},\mathbb{R})$

A.1 Prelude to the proof of Lemma 39

Definition 79. *Let $\mathcal{P}(k)$ be the vector space of polynomial on $\mathcal{H}=\mathbb{R}$ of degree bounded by k, and let $||F||_\infty := \sup_{x\in[0,1]}|F(x)|$ be the uniform norm. We denote by $B(k)$ the closed ball of radius 1.*

Proposition 80. *$B(k)$ is a compact set.*

Proof. Since $B(k)$ is a bounded subset of the finite-dimensional space $\mathcal{P}(k)$, it is enough to prove that $B(k)$ is closed, indeed, by Arzela-Ascoli theorem we just need to prove that $B(k)$ is an equi-continuous set: let $F(x)$ be a polynomial in $C(k)$, then the Markov brothers' inequality implies $|F'(x)|\le k^2$, so $|F(x_1)-F(x_2)|<k^2|x_1-x_2|$. □

Definition 81. *We say a polynomial F is unitary if F has a hill interval $[0,1]$, and let $\mathcal{P}_N(k)$ be the set of unitary polynomials. Let $F_\mu(x)$ be a polynomial with hill interval $[x_0,x_1]$ and let $u:=x_1-x_0$ be the length of the hill interval.*

Corollary 82. *If $G_n(x)$ is a sequence of non-constant polynomials in Pen_F with hill interval $I_n = [x_n, x'_n]$ such that $G_n(x_n) = G_n(x'_n) = 1$, $\lim_{n\to\infty} x_n = -\infty$ and $\lim_{n\to\infty} x'_n = \infty$, then $F(x)$ must be even degree.*

Proof. Let $G_n(x)$ be equal to $a_n + b_n F(x)$. There exists K_x a compact set containing all the roots of $F(x)$, and let n be large enough that $K_x \subset I_n$. Let us assume $F(x)$ is an odd degree. Without loss of generality, let us assume $F(x'_n) > 0$ and $F(x_n) < 0$, then $0 = G(x'_n) - G(x_n) = b_n(F(x'_n) - F(x_n))$, and $b_n = 0$ since $F(x'_n) - F(x_n) > 0$, which is a contradiction to the assumption that $G_n(x)$ is a sequence of non-constant polynomials. □

A.1.1 Proof of Lemma 39

Proof. Let $c_n(t) = (x_n(t), y_n(t), z_n(t))$ be a sequence of $\mathbb{R}^3_F$-geodesics traveling during a time interval $[(t_0)_n, (t_1)_n]$ and with travel interval $\mathcal{I}_n[(t_0)_n, (t_1)_n]$ such that $x_n((t_0)_n)$ and $x_n((t_1)_n)$ are in $[0, 1]$ for all n. Then we will prove that if $\mathcal{I}_n$ is unbounded, then $\Theta(c, [t_0, t_1])$ is unbounded.

The sequence of $c_n(t)$ of $\mathbb{R}^3_F$-geodesics, induces a sequence of $G_n(x)$ polynomials, which induces a sequence of unitary polynomials $\hat{G}_n(\tilde{x}) := G_n(h_n(\tilde{x}))$ where $h_n(\tilde{x})$ is the affine map given by Definition 81, that is, $h_n(\tilde{x}) = (x_0)_n + u_n\tilde{x}$ where $u_n := (x_0)_n - (x_1)_n$. Since $\hat{G}_n(\tilde{x})$ is in $C(k)$. There exists a subsequence $\hat{G}_{n_s}(\tilde{x})$ converging to $\hat{G}(\tilde{x})$. Let us proceed by the following cases: case $\hat{G}(\tilde{x}) \neq 1$ or case $\hat{G}(\tilde{x}) = 1$.

Case $\hat{G}(\tilde{x}) \neq 1$: by Fatou's lemma $0 < Cost(\hat{G}) \leq \liminf_{n\to\infty} Cost(\hat{G}_n)$. Then $u_n \to \infty$ implies $Cost(c, \mathcal{I}_n)$ is unbounded.

Case $\hat{G}(\tilde{x}) = 1$: let $K'_{\mathcal{H}}$ be a compact set such that all the roots of $1 - F(x)$ are in $K'_{\mathcal{H}}$. There exists $n^* > 0$ such that $\hat{G}(\tilde{x}) > \frac{1}{2}$ for all $\tilde{x}$ in $[0,1]$ if $n_s > n^*$. We split the integral for $\Delta z(c, \mathcal{I}_n)$ given by Corollary 32 in the following way

$$\int_{\mathcal{I}_n} \frac{(1-F(x))G_n(x)}{\sqrt{1-G_n^2(x)}}dx = \int_{K'_{\mathcal{H}}\cap\mathcal{I}} \frac{(1-F(x))G_n(x)}{\sqrt{1-G_n^2(x)}}dx + \int_{(K'_{\mathcal{H}})^c\cap\mathcal{I}} \frac{(1-F(x))G_n(x)}{\sqrt{1-G_n^2(x)}}dx.$$

Since the first integral of the right side is finite, it is enough to focus on the second integral.

We proceed by cases: Case $(x_0)_n$ and $(x_1)_n$ are both unbounded and cases (x_0) is bounded and (x_1) is unbounded or (x_0) is unbounded and (x_1) is bounded.

Case $(x_0)_n$ and $(x_1)_n$ unbounded: by Corollary 82 we can assume that $F(x)$ is even, then the condition $\hat{G}(\tilde{x}) > \frac{1}{2}$ implies $|G_n(x)| > \frac{1}{2}$ in the travel interval $\mathcal{I}_n$ and $(1 - F(x))G_n(x)$ does not change sign in the set $\mathcal{I}_n \backslash K'_{\mathcal{H}}$, therefore

$$|\int_{\mathcal{I}_n\backslash K'_{\mathcal{H}}} \frac{(1-F(x))G_n(x)}{\sqrt{1-G_n^2(x)}}dx| > \frac{1}{2}\int_{\mathcal{I}_n\backslash K'_{\mathcal{H}}} |F(x)|dx \to \infty \text{ when } n \to \infty.$$

A similar proof follows if $(x_0)_n$ is bounded and $(x_1)_n$ is unbounded or $(x_0)_n$ is unbounded, and $(x_1)_n$ is bounded. □

A.2 Proof of Theorem 46

For simplicity, we will prove Theorem 46 for the case $F(x) = 1 - 2x^3$. Let $c(t)$ be the $\mathbb{R}^3_F$-geodesic for $F(x) = 1 - 2x^3$ with initial point $c(0) = (1, 0, 0)$ and hill interval $[0, 1]$. Let us consider the travel interval $\mathcal{I}(x) = 2[x, 1]$, then by (**??**), the relation

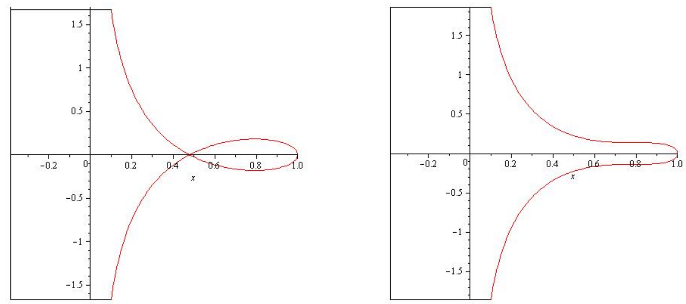

Figure A.1: Both images show the projection of the geodesic $c(t)$ for $F(x) = 1 - 2x^3$ and the curve $\tilde{c}(t)$ to the (x, y) and (x, z) planes, respectively.

between the travel interval and the time is given by

$$2T = 2\int_{[x,1]} \frac{dx}{\sqrt{1 - F^2(x)}}.$$

By equation (**??**), the change in $\Delta y(c, t)$ and $\Delta z(c, t)$ are given by

$$\Delta y(F, x) := 2\int_{[x,1]} \frac{F(x)dx}{\sqrt{1 - F^2(x)}} \quad \text{and} \quad \Delta y(F, x) = 2\int_{[x,1]} \frac{F^2(x)dx}{\sqrt{1 - F^2(x)}}.$$

Therefore

$$c(-T) = (x, -\frac{\Delta y(F, t)}{2}, -\frac{\Delta z(F, t)}{2}) \quad \text{and} \quad c(T) = (x, \frac{\Delta y(F, t)}{2}, \frac{\Delta z(F, t)}{2})$$

Corollary 83. *If* $F(x) = 1 - 2x^3$*, then* $\Delta y(F, x) < \Delta z(F, x)$ *and* $\lim_{x\to 0} \frac{\Delta z(F,x)}{\Delta y(F,x)} = 1$.

Proof. If $F(x) = 1 - 2x^3$, then the same integration by parts, used in the proof of Corollary 47, shows the integral $\Delta y(F, x) - \Delta z(F, x)$ is positive. L'Hopital rules implies $\lim_{x\to 0} \frac{\Delta z(F,x)}{\Delta y(F,x)} = 1$. □

A.2.1 Proof of Theorem 46

Proof. Let us consider $0 < \epsilon < \frac{1}{2}$ and find a x^* such that $\Delta z(F, x) = (1 + \epsilon)\Delta y(F, x)$. The exists $\delta < 0$ such that $F(\delta) = 1 + \epsilon$. If $\delta_1 = x^* + \delta$ and $\delta_2 = \delta_1 + \Delta y(F, t)$, then we define the following curve $\tilde{c}(t)$ in $\mathbb{R}^3_F$.

$$\tilde{c}(t) = \begin{cases} c(-n) + (-t, 0, 0) \text{ where } t \in [0, \delta_1] \\ c(-n) + (-\delta_1, t - \delta_1, 0) \text{ where } t \in [\delta_1, \delta_2] \\ c(-n) + (-\delta_1 + t - \delta_2, \Delta y(F, t), \Delta z(F, t)) \text{ where } t \in [\delta_2, \delta_1 + \delta_2]. \end{cases}$$

See figure A.1. The by construction, $c(-T) = \tilde{c}(0)$ and $c(T) = \tilde{c}(\delta_1 + \delta_2)$, the relation between the T and $\Delta y(F, x^*)$ is given by $2T = \Delta y(F, x^*) + Cost_t(F, x^*)$, while, the relation between $\delta_1 + \delta_2$ and $\Delta y(F, x^*)$ is given by $\delta_1 + \delta_2 = 2(\delta + x^*) + \Delta y(F, x)$. If $x^* \to 0$, then $Cost_t(F, x^*) \to \Theta_1(F, [0, 1]) > 0$, while, $2(\delta + x^*) \to 0$. Thus there exists an x_1 such that $Cost_t(F, x_1) > 2(\delta + x^*)$ and $\tilde{c}(t)$ is shorter that $c(t)$.

□

A.3 The Calibration method

Definition 84. *Let $c(t)$ be an $\mathbb{R}^3_F$-geodesic and let be $\Omega \subset \mathbb{R}^3_F$ a simple connected domain, we say that a functions $S : \Omega \to \mathbb{R}$ is a calibration function for $c(t)$, if $dS(\dot{c}) = 1$ and $dS(v) = (\dot{c}, v)_{\mathbb{R}^3_F}$ for all v tangent to $\mathcal{D}_F$, where $(\dot{;})_{\mathbb{R}^3_F}$ is the subRiemannian inner-product in $\mathbb{R}^3_F$.*

A classical application of a calibration one-form is the following.

Proposition 85. *Let $c(t)$ be a $\mathbb{R}^3_F$-geodesic in $\mathbb{R}^3_F$, if $S : \Omega \to \mathbb{R}$ is a calibration function for $c(t)$, then the $\mathbb{R}^3_F$-geodesic $c(t)$ is a globally minimize within Ω.*

Proof. Let S be calibration function for $c(t)$, let A and B be two points in Ω such that

$c(t)$ travel from A to B with arc length $\ell(c)$. Let us assume $\tilde{c}(t)$ is a curve tangent to $\mathcal{D}_F$ and join the points A and B with arc length $\ell(\tilde{c})$, then by Stoke's theorem, the fact that S is a calibration for $c(t)$ and Cauchy-Schwarz inequality we have

$$\ell(c) = \int_c dS = \int_{\tilde{c}} dS = \int_{\tilde{c}} (\dot{c}, \dot{\tilde{c}})_{\mathbb{R}^3_F} dt \leq \int_{\tilde{c}} ||\dot{\tilde{c}}||_{\mathbb{R}^3_F} dt = \ell(\tilde{c}).$$

By Cauchy-Schwarz inequality we know that $\ell(c) = \ell(\tilde{c})$ if and only if $\dot{\tilde{c}}$ is parallel to $\dot{c}$ a.e.. □

A canonical method to find a calibration function is to solve the Hamilton-Jacobi equation defined for the subRiemannian Hamiltonian function, see [21] or [12, Section 5]. In the context of the space $\mathbb{R}^3_F$ we have the following Proposition

Proposition 86. *A calibration function $S_\pm$ for a $\mathbb{R}^3_F$-geodesic $c(t)$, generated by $G(x) = a + bF(x)$ in the pencil of $F(x)$, is given by*

$$S_\pm(x, u, z) = \pm \int^x \sqrt{1 - G^2(\tilde{x})} d\tilde{x} + ay + bz.$$

$S_\pm$ is smooth inside the region $\Omega(G) := hill(G) \cup \mathbb{R}^2$, where $hill(G) := G^{-1}([-1, 1])$ is the hill region of $G(x)$ and C^1 on the boundary of the hill region, and the abnormal curve does not cross from one connected set to another.

Proof. By Proposition , the $\mathbb{R}^3_F$-geodesic $c(t)$ has derivative

$$\dot{c}(t) = \pm\sqrt{1 - G^2(x(t))} \frac{\partial}{\partial x} + G(x(t))(\frac{\partial}{\partial y} + F(x(t)) \frac{\partial}{\partial z}),$$

we notice $dS_\pm = \pm\sqrt{1 - G^2(x)} dx + ady + bdz$, then

$$dS_\pm(\dot{c}) = 1 - G^2(x) + G(x(t))(a + bF(x(t))) = 1 - G^2(x) + G^2(x(t)) = 1.$$

□

We notice calibration function provided by Proposition 86 is globally defined if and only if $G(x)$ is a constant polynomial, otherwise, it defined in sub-region of $\mathbb{R}^3_F$ and the geodesic is minimizing in the region $\Omega(G)$ until it touches its boundary. It is worth seeing how this argument looks in each of our three cases.

Recall non-line geodesics in J^k come in three "flavors": heteroclinic, homoclinic and x-periodic. It is worth going into details around the time interval $\mathcal{T}$, the domain of the geodesic, for each of the three cases:

(x-Periodic). Choose a time origin so that $x(0) = x_0$ and $x(L/2) = x_1$. Then $\mathcal{T} = (0, L/2)$ or $(L/2, L)$ up to a period shift. The minimizing arcs correspond to half periods of the x-periodic geodesic. The domain Ω projects onto the interior (a, b) of the hill interval.

(Heteroclinic.) If γ is heteroclinic then $\mathcal{T} = \mathbb{R}$ and $c : \mathbb{R} \to \Omega$ is globally minimizing within Ω. If one or both endpoints x_0, x_1 is a local maximum of $F^2(x)$ then Ω projects to an interval (α, β) strictly bigger than (x_0, x_1)

(Homoclinic). In this case, the x curve bounces once off the non-critical endpoint of the hill interval. Say this endpoint is b and that we translate time so that $x(0) = x_1$. Then $\mathcal{T}$ is of the form $(-\infty, 0)$ or $(0, \infty)$. The Hamilton-Jacobi minimality argument does not allow us to include $t = 0$ region $\Omega(G)$ within the domain of γ as $\gamma(0)$ is outside $\Omega(g)$.

A.4 Geodesics in $\mathrm{Eng}(n)$

Here we will introduce the necessary tools to prove Theorems 56. Since we are interested in studying non-line geodesics, we will be restricted to the case $\mu \neq 0$. As we said before, the dynamics are split into two cases; when $p_{\theta_{n+1}} = a_{n+1}$ equals zero or not. Let us start with the case $p_{\theta_{n+1}} = a_{n+1} = 0$

A.4.1 Small oscillations

The condition $p_{\theta_{n+1}} = a_{n+1} = 0$ implies the reduced Hamiltonian H_μ from (4.3) has potential $\frac{1}{2}(a_0 + \sum_{i=1}^{n} a_i x_i)^2$. Using the translation $x_1 \to x_1 - \frac{a_0}{a_1}$, the reduced Hamiltonian is given by

$$H_\mu = \frac{1}{2}(p_x, p_x) + \frac{1}{2}(Bx, x)_{\mathcal{H}}, \tag{A.1}$$

where $(\ ,\ )_{\mathcal{H}}$ is the Euclidean product on $\mathcal{H}$, $Id_{n\times n}$ is the identity matrix and B is the following n by n matrix

$$B = \begin{pmatrix} a_1^2 & a_1a_2 & \dots & a_1a_{n-1} & a_1a_n \\ a_1a_2 & a_2^2 & \dots & a_2a_{n-1} & a_2a_n \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_1a_{n-1} & a_2a_{n-1} & \dots & a_{n-1}^2 & a_{n-1}a_n \\ a_1a_n & a_2a_n & \dots & a_{n-1}a_n & a_n^2 \end{pmatrix} \tag{A.2}$$

Lemma 87. *For any $(a_1, \dots, a_n) \neq 0$, the matrix B has rank one.*

Proof. If $v_i = a_{i+1}e_i, -a_ie_{i+1}$, then $(Bv_i, v_i)_{\mathcal{H}} = 0$. □

We know by linear algebra that the pair of quadratic forms $(Id_{n\times n}p_x, p_x)$ and $(Bx, x)_{\mathcal{H}}$, where the first one is positive-definite, can be reduced to principal axes by a linear change of coordinates $\tilde{x} = Qx$ and the reduced Hamiltonian H_μ, given by (4.5), in the new coordinates $\tilde{x}$ is the following

$$\begin{aligned} H_\mu(p_{\tilde{x}}, \tilde{x}) &= H_{line}(p_{x_1}, \cdots, p_{x_{n_1+1}}) + H_{osc}(p_{x_{n_1+1}}, \cdots, p_{x_n}, x_{n_1+1}, \ldots, x_n) \\ H_{line}\frac{1}{2} &= \sum_{i=1}^{n-1} p^2_{\tilde{x}_i} \quad \text{and} \quad H_{osc} = \frac{1}{2}p^2_{\tilde{x}_n} + \frac{1}{2}\lambda\tilde{x}^2_{n+2}. \end{aligned} \tag{A.3}$$

Then at $H_\mu(p_{\tilde{x}}, \tilde{x})$ has $n-1$ cycle coordinate and $n-1$ constant of motion, namely, x_i and $p_{\tilde{x}_1}$ for $1 \leq i \leq n-1$.

Let us built a solution with initial point the origin. The solution of $\tilde{x}_i$ is $\tilde{x}_i = p_{\tilde{x}_i}t$ for $1 \leq i \leq n-1$, and the energy level $H_\mu(p_{\tilde{x}}, \tilde{x}) = \frac{1}{2}$ implies $H_{line}\frac{1}{2} \leq \frac{1}{2}$. Moreover, $H_{line}\frac{1}{2} = \frac{1}{2}$ implies that the corresponding geodesic is a geodesic line. In contrast, using Hamilton equations for x_n, we find that $\ddot{\tilde{x}}_n = \lambda x_n$, so $\tilde{x}_n(t) = \sqrt{2H_{osc}}\sin(\omega t)$, where $\omega = \frac{1}{\sqrt{\lambda}}$.

$$\tilde{x}(t) = (p_{\tilde{x}_i}t, \ldots, p_{\tilde{x}_i}t, \sqrt{2H_{osc}}\sin(\omega t)). \tag{A.4}$$

Corollary 88. *The solution to the Hamiltonian A.3 is bounded in the coordinates $\tilde{x}_n$ and unbounded in the coordinates $\tilde{x}_i$ such that $1 \leq \lambda_i \leq n-1$ and $p_{\tilde{x}_i}$ is not zero.*

A.5 Proof of Theorem 56

A.5.0.1 Prelude to proof of Theorem 56

The following proofs rely on the method of blowing-down geodesics as explained by E. Hakavuouri and E. Le Donne, in [14]. Suppose that $\gamma : \mathbb{R} \to G$ is a rectifiable curve in a Carnot group G. For $h \in \mathbb{R}^+$ form

$$\gamma_h(t) = \delta_{\frac{1}{h}}\gamma(ht).$$

where $\delta_h : G \to G$ is the Carnot dilation. One easily checks that if γ is a geodesic then so is γ_h for any $h > 0$.

Definition 89 (blow-down)**.** *A blow-down of γ is any limit curve $\tilde{\gamma} = \lim_{k\to\infty} \gamma_{h_k}$ where $h_k \in \mathbb{R}$ is any sequence of scales tending to infinity with k, and the limit being uniform on compact sub-intervals,*

In [14], E. Hakavouri and E. Le Donne proved the following powerful lemma

Lemma 90. *If γ is globally minimizing geodesic parameterized by arc length then every blow-downs $\tilde{\gamma}$ of $\gamma(t)$ is also a globally minimizing geodesic parameterized by arc length.*

A.5.0.2 Proof of Theorem 56

Proof. The strategy of the proof is the same in both cases, we will consider a small oscillation or a r-periodic geodesic $\gamma(t)$, and we will compute one of its blow-down $\tilde{\gamma}(t) = \lim_{k\to\infty} \gamma_{h_k}$ and check that is not parameterized by arc length.

Case r-periodic geodesics: Let L be the period, let us consider the sequence $h_n = nL$ and the compact interval $[0, 1]$. We compute the change undergone by the coordinate θ_0 of $\gamma_{nL}(t)$ after time change by 1:

$$\Delta y(\gamma_{nL}, [0,1]) := \frac{1}{nL}\Delta y(\gamma, [0, nL]) = \frac{1}{L}\Delta y(\gamma, [0, L]) < 1. \tag{A.5}$$

Since $\Delta y(\gamma_{nL}, [0, 1])$ is constant for all n. The change undergone by the coordinate after time change by 1 for the geodesic is equal to $\frac{1}{L}\Delta y(\gamma, [0, 1])$. Being $\gamma_{nL}(t)$ a geodesic in Eng(n), there exists an μ_n in $\mathfrak{a}$ such that $\gamma_{nL}(t)$ has momentum μ_n. The relation between hill regions $h(\mu_n, \ell)$ and $hill(\mu, \ell)$, given by Definition 62, of the geodesics $\gamma_{nL}(t)$ and $\gamma(t)$ is $hill(\mu_n, \ell) = \frac{1}{nL} hill(\mu, \ell)$. Since $hill(\mu, \ell)$ is bounded, $\tilde{\gamma}(t)$ has hill region equal to 0.

Therefore, $\tilde{\gamma}(t)$ is a curve tangent to the vector field Y, $\tilde{\gamma}(t)$ is a line. Instead of being parametrized by arc-length, moving one unit along the line requires a time of $L/\Delta y(\gamma, [0, L]) > 1$ of the blow-down time. We conclude that $\tilde{\gamma}(t)$ is not parameterized by arc length.

Case small oscillations geodesics: The fact that $\gamma(t)$ is not a line implies H_{osc} is a constant different from zero. Let us consider $\gamma(t)$ the geodesic corresponding the solution $\tilde{x}(t)$ given by A.4. We define $\gamma_n(t) = \delta_n \gamma(n\omega t)$, by construction the reduce dynamics of $\tilde{x}_n(t)$ is the following

$$\tilde{x}_n(t) = (p_{\tilde{x}_1} t, \dots, p_{\tilde{x}_{n-1}} t, \frac{\sqrt{2H_{osc}}}{n} \sin(n\omega t)). \tag{A.6}$$

If we take the limit $n \to \infty$, then $\lim_{n\to\infty} \tilde{x}_n(t) = t(p_{\tilde{x}_1}, \dots, p_{\tilde{x}_{n-1}}, 0)$. However, $||\dot{\tilde{x}}_n(0)|| = 2H_{lin} + 2H_{osc} = 1$ for all n. Therefore, $\tilde{\gamma}(t)$ is a curve tangent to $\sum_{i=1}^{n-1} p_{\tilde{x}_i} X_i$, so $\tilde{\gamma}(t)$ is a line. Rather that being parameterized by arc length, moving one unit along the line requires a time $\frac{1}{2H_{lin}}$ of the blow-down time. We conclude that $\tilde{\gamma}(t)$ is not parameterized by arc length. $\square$

Appendix B

$\mathbb{G}$ as $\mathbb{A}$-principle bundle

B.1 The left action of $\mathbb{A}$

Definition 91. *The left-action of* $\mathbb{A}$ *on* $\mathbb{G}$ *is a function* $\varphi : \mathbb{A} \times \mathbb{G} \to \mathbb{G}$ *given by* $\varphi(a, g) := a * g$ *such that* $\varphi(a_1 * a_2, g) := \varphi(a_1, \varphi(a_2, g))$*, where* $*$ *is the Carnot multiplication.*

By construction, $\varphi(a, g)$ is in $Iso(\mathbb{G})$ and since $\mathbb{A}$ is abelian, $\varphi(a_1 * a_2, g) = \varphi(a_2 * a_1, g)$. If ξ is in $\mathfrak{g}$ then the action of $\mathbb{A}$ on $\mathbb{G}$ defines the infinitesimal generator $\sigma : \mathfrak{a} \to \mathfrak{g}$ in the following way

$$\sigma_\xi(g) = \frac{d}{dt}\varphi(\exp(t\xi), g)|_{t=0} = \frac{d}{dt}\exp(t\xi) * g|_{t=0}. \tag{B.1}$$

The map σ sends a vector ξ in $\mathfrak{a}$ to a Killing vector field σ_ξ since φ is in $Iso(\mathbb{G})$. We say the vector field X and the map is σ are $\mathbb{A}$-invariant if $X(a * g) = (L_a)_* X(g)$ and $\sigma_\xi(a * g) = (L_a)_* \sigma(g)$. The infinitesimal generator σ_ξ is equinvariant in general, see

[9, p. 108] or [21, p. 161] for more details. It is a general property of infinitesimal generators that $\mathbb{A}$ abelian implies that $\sigma_\xi(g)$ is $\mathbb{A}$-invariant.

Let $\{E^i\}$ be the base for $\mathfrak{h}$ with $1 \leq i \leq n$, let $\{E^\ell_\mathfrak{a}\}$ be the base for $\mathfrak{a}$ with $1 \leq \ell \leq m$. An alternative notation is; $\{E^k_\mathfrak{a}\}$ be the base for $\mathfrak{v}$ with $1 \leq k \leq n_1$ and let $\{E^j_\mathfrak{a}\}$ be the base for $[\mathfrak{g}, \mathfrak{g}]$ whit $n_1 + 1 \leq j \leq m$. Then, we will use the index i's for vector in $\mathfrak{h}$, ℓ's for vector in $\mathfrak{a}$, and when we want to distinguish between $\mathfrak{v}$ and $[\mathfrak{g}_1, \mathfrak{g}_1]$, we will use k's for vector in $\mathfrak{v}$ and j's for vector in $[\mathfrak{g}_1, \mathfrak{g}_1]$. We denote by X^i and Y^ℓ, the left extension of E^i and E^ℓ, that is, $X^i(g) := (L_g) * E^i$ and $Y^\ell(g) := (L_g) * E^\ell_\mathfrak{a}$, in the same way, we denote by $Y^k := (L_g) * E^k_\mathfrak{a}$ and $Y^j := (L_g) * E^j_\mathfrak{a}$ left extension of E^k and E^j. Then $\{X^i\}$ is a base for $\mathcal{D}_\mathfrak{h}$ with $1 \leq i \leq n$, and $\{Y^k\}$ is the base for $D_\mathfrak{v}$ with $1 \leq k \leq n_1$.

$\sigma(g)$ sends the canonical base E^ℓ for $\mathbb{A}$, with $1 \leq \ell \leq m$, to the frame of Killing vector fields $\sigma^\ell(g)$. Thus, the frame $\sigma^\ell(g)$ defines a canonical co-frame $\omega_\ell(g) \in \mathcal{D}_\mathfrak{h}^\perp$ with $1 \leq \ell \leq m$, such that, $\omega_{\ell_1}(\sigma^{\ell_2})(g) = \delta^{\ell_2}_{\ell_1}$, and $\omega_{\ell_1}(\mathcal{D}_\mathfrak{h})(g) = 0$. It follows that the co-frame σ_ℓ is $\mathbb{A}$-invariant. We also split the base E^ℓ, $\sigma^\ell(g)$ and $\omega_\ell(g)$ into the space corresponding to $\mathfrak{v}$ and $[\mathfrak{g}, \mathfrak{g}]$, as we did with the left-invariant vector field; that is, we use E^k, $\sigma^k(g)$ and $\omega_k(g)$ for $1 \leq k \leq n_1$ and E^j, $\sigma^j(g)$ and $\omega_j(g)$ for $k_1 + 1 \leq j \leq m$ such that, $\omega_{\ell_1}(\sigma^{\ell_2})(g) = \delta^{\ell_2}_{\ell_1}$.

We remark that $(L_g)_*\mathfrak{a}$ and $\sigma(\mathfrak{a})$ are the same as abstract Lie algebras and as sub-vector spaces of $T_g\mathbb{G}$. However, they are different Lie algebras inside $T_g\mathbb{G}$. In general, only the left-invariant vector fields in $\sigma(\mathfrak{a})$ and $(L_g)_*\mathfrak{a}$ are the ones corresponding to the left translation of the last layer $\mathfrak{g}_s$.

B.1.1 $\mathbb{G}$ as $\mathbb{A}$-principle bundle

We can think of $\pi_{\mathbb{A}} : \mathbb{G} \to \mathcal{H}$ as a principle $\mathbb{A}$-bundle. In our case, we have identified $\mathcal{H}$ with a sub-vector space $\mathcal{D}_{\mathfrak{h}} \subset T_g\mathbb{G}$, which is complementary to $(L_g)_*\mathfrak{a} \subset T_g\mathbb{G}$, that is $\mathcal{D}_{\mathfrak{h}} \oplus (L_g)_*\mathfrak{a} = T_g\mathbb{G}$. This way, $\mathcal{H}$ defines a connection on our principal bundle $\pi_{\mathbb{A}}$. Note: $(L_g)_*\mathfrak{a}$ represents the vertical space for $\pi_{\mathbb{A}}$, and $\mathcal{D}_{\mathfrak{h}}$ is an $\mathbb{A}$-invariant choice of horizontal space by left-translation, that is $d\pi_{\mathbb{A}}((L_g)_*\mathfrak{a}) = 0$ and $(L_a)_*\mathcal{D}_{\mathfrak{h}}(g) = \mathcal{D}_{\mathfrak{h}}(ag)$, as a connection on principal $\mathbb{A}$-bundle requires. For more bundles with connections, see [24, Chapter 8], [21, Chapter 12] , or [9, sub-Chapter 2.9].

B.1.2 Connection form

The connection one-form $\omega(g)$ on $\mathbb{G}$ is an $\mathfrak{a}$ valued one-form given by

$$\omega(g) = \sum_{\ell=1}^{m} \omega_\ell \otimes e^\ell(g). \tag{B.2}$$

$\omega(g)$ is $\mathbb{A}$-invariant since $(L_a)_*\omega(g) = \omega(a * g)$. By definition $ker\ \omega(g) = \mathcal{D}_{\mathfrak{h}}(g)$ and $\omega \circ \sigma(g) = Id_{\mathfrak{a}}$.

The canonical projection π is such that $d\pi$ has a canonical inverse map

Definition 92. *If $(v, u) = (v_1, \cdots, v_n, u_1, \cdots, u_{n_1})$ is in $T\mathbb{R}^{d_1}$, then we denote by hor : $T\mathbb{R}^{d_1} \to T\mathbb{G}$ the map given by $hor(v, u) := \sum_{i=1}^{n} v_i X^i + \sum_{k=1}^{n_1} u_k Y^k$;*

$d\pi \circ hor = Id_{\mathfrak{g}_1}$, we say that hor is a horizontal lift with respect to $d\pi$. The horizontal map hor defines a linear projection that formalizes the definition of the $\mathfrak{a}^*$-valued one-form $\mathcal{A}_{\mathbb{G}}$ on $\mathbb{R}^{d_1}$, presented in the introduction.

Definition 93. *We denote by* $\Pi_{\mathbb{R}^{d_1}}$ *the linear projection from* T^*G *to* $T^*\mathcal{H}$ *give by* $\Pi_{\mathbb{R}^{d_1}}(\lambda) := \lambda \circ hor$*. We define the* $\mathfrak{a}^*$*-valued one-form* $\mathcal{A}_{\mathbb{G}}$ *on* $\mathbb{R}^{d_1}$ *by*

$$\mathcal{A}_{\mathbb{G}} := \Pi_{\mathbb{R}^{d_1}}(\omega)(g), \quad \mathcal{A}_{\mathbb{G}}^M := \Pi_{\mathbb{R}^{d_1}}(\omega)(g)|_{\mathcal{H}} \ \text{ and } \ \mathcal{A}_{\mathbb{G}}^E := \Pi_{\mathbb{R}^{d_1}}(\omega)(g)|_{\mathcal{V}}.$$

B.2 Exponential coordinates of the second kind (x, θ)

We use the frame X^i and Y^ℓ to give coordinates to the Carnot group at a point g in the following way: define a map from the coordinates $(x, \theta) \in \mathbb{R}^{n+m}$ to $\mathbb{G}$ by

$$\Phi(x) := \prod_{i=0}^{n-1} \exp(x_{n-i} X^{n-i}) \ \text{ and } \ \Phi(\theta) := \prod_{\ell=1}^{m} \exp(\theta_\ell Y^\ell) = \exp(\sum_{j=1}^{m} \theta_\ell Y^\ell).$$

Definition 94. *The exponential coordinates* (x, θ) *are given by a unique chart* $(\mathbb{R}^{n+m}, \Phi)$ *where a point is given by* $g := \Phi(x, \theta) := \Phi(\theta) * \Phi(x)$.

Proposition 95. *Let* $\mathbb{G}$ *be a metabelian Carnot group and let* $g = (x, \theta)$ *be in* $\mathbb{G}$*. Then the left-invariant vector fields and the left-invariant one-forms on* $\mathbb{G}$ *are given by*

$$X^1(g) = \frac{\partial}{\partial x_1}, \quad X^i = \frac{\partial}{\partial x_i} + \sum_{j=n_1+1}^{m} \mathcal{A}_{ij}^M(x) \frac{\partial}{\partial \theta_j} \qquad 2 \le i \le n,$$

$$Y^k(g) = \frac{\partial}{\partial \theta_k} + \sum_{j=n_1+1}^{m} \mathcal{A}_{kj}^E(x) \frac{\partial}{\partial \theta_j} \qquad 1 \le k \le n_1,$$

$$\Theta_k(g) = d\theta_k \quad \text{and} \quad \Theta_j(g) = d\theta_j - \sum_{i=1}^{n} \mathcal{A}_{ij}^M(x) dx_i - \sum_{k=1}^{n_1} \mathcal{A}_{kj}^E(x) d\theta_k,$$

where $\mathcal{A}_{ij}^M(x)$ *and* $\mathcal{A}_{kj}^E(x)$ *are homogeneous polynomial functions on the horizontal coordinates.*

The proof the Proposition 95 is in [10]

B.3 Examples

This Section will prove that $\mathrm{Eng}(n)$ has integrable subRiemannian geodesic flow and show the Carnot group $N_{6,3,1}$ has non-integrable geodesic flow.

B.3.1 The Engel type group $\mathrm{Eng}(n)$

$\mathrm{Eng}(n)$ is the first example of an arbitrary rank distribution of step 3 whose subRiemannian geodesic flow is integrable, besides metabelian Carnot groups such that $\dim \mathbb{A} = \dim \mathbb{G} - 1$.

Theorem 96. *The subRiemannian geodesic flow on* $\mathrm{Eng}(n)$ *is integrable by meromorphic functions for all n.*

Lemma 97. *Let us consider the following functions*

$$L_{ij} := P_{X_i} P_{Y_j} - P_{X_j} P_{Y_i} \qquad i \neq j \qquad C_N := \frac{1}{2} \sum_{i,j=1}^{N} L_{ij}^2.$$

Then L_{ij} and C_N are constants of motion through the sub-Riemannian geodesic flow in $\mathrm{Eng}(n)$*.*

Proof. Let us use the Poisson bracket to prove that L_{ij} is a constant of motion:

$$\begin{aligned} \{L_{ij}, H\} =& P_{X_i}\{P_{Y_j}, H\} + P_{Y_j}\{P_{X_i}, H\} - P_{X_j}\{P_{Y_i}, H\} - P_{Y_i}\{P_{X_j}, H\} \\ =& - P_{X_i} P_{X_j} P_{Y_{n+1}} + P_{Y_j} P_{Y_0} P_{Y_i} + P_{X_j} P_{X_i} P_{Y_{n+1}} - P_{Y_i} P_{Y_0} P_{Y_j} = 0. \end{aligned}$$

C_N is a constant of motion being the sum of constants of motion. □

Lemma 98. *The functions L_{ij} satisfy the following relationship*

$$\{L_{ij}, L_{kl}\} = P_{Y_{n+1}}(\delta_{ik} L_{jl} + \delta_{jl} L_{ik} - \delta_{il} L_{jk} - \delta_{jk} L_{il}).$$

Proof. Let us compute the following Poisson bracket

$$\{P_{X_i}, L_{kl}\} = P_{X_k}\{P_{X_i}, P_{Y_l}\} + P_{Y_l}\{P_{X_i}, P_{X_k}\} - P_{X_l}\{P_{X_i}, P_{Y_k}\} - P_{Y_k}\{P_{X_i}, P_{X_l}\}$$

$$= P_{Y_{n+1}}(P_{X_k}\delta_{il} - P_{X_l}\delta_{ik}),$$

$$\{P_{Y_j}, L_{kl}\} = P_{X_k}\{P_{Y_j}, P_{Y_l}\} + P_{Y_l}\{P_{Y_j}, P_{X_k}\} - P_{X_l}\{P_{Y_j}, P_{Y_k}\} - P_{Y_k}\{P_{Y_j}, P_{X_l}\}$$

$$= P_{Y_{n+1}}(-P_{Y_l}\delta_{jk} + P_{Y_k}\delta_{jl}).$$

$$\begin{aligned}\{L_{ij}, L_{kl}\} =& P_{X_i}\{P_{Y_j}, L_{kl}\} + P_{Y_j}\{P_{X_i}, L_{kl}\} - P_{X_j}\{P_{Y_i}, L_{kl}\} - P_{Y_i}\{P_{X_j}, L_{kl}\} \\ =& P_{Y_{n+1}}(P_{X_i}(-P_{Y_l}\delta_{jk} + P_{Y_k}\delta_{jl}) + P_{Y_j}(P_{X_k}\delta_{il} - P_{X_l}\delta_{ik}) \\ & - P_{X_j}(-P_{Y_l}\delta_{ik} + P_{Y_k}\delta_{il}) - P_{Y_i}(P_{X_k}\delta_{jl} - P_{X_l}\delta_{jk})) \\ =& P_{Y_{n+1}}(\delta_{ik}L_{jl} + \delta_{jl}L_{ik} - \delta_{il}L_{jk} - \delta_{jk}L_{il}).\end{aligned}$$

□

Lemma 99. *The functions L_{ij} and C_N satisfy the following relationship*

$$\{C_N, L_{kl}\} = 0 \quad \textit{if} \;\; N \le k < l \;\; \textit{or} \;\; k < l \le N.$$

Proof. Let us compute the Poisson bracket

$$\{C_N, L_{kl}\} = \sum_{i,j=1}^{N} L_{ij}\{L_{ij}, L_{kl}\} = P_{Y_{n+1}} \sum_{i,j=1}^{N} L_{ij}(\delta_{ik}L_{jl} + \delta_{jl}L_{ik} - \delta_{il}L_{jk} - \delta_{jk}L_{il})$$

If $k < l < N$, then δ_{ik}, δ_{jl}, $\delta_{il}L_{jk}$ and δ_{jk} are zero. So the non-trivial case is when $k < l \le N$;

$$\begin{aligned}\{C_N, L_{kl}\} = & P_{Y_{n+1}} \sum_{i<j}^{N} L_{ij}(\delta_{ik}L_{jl} + \delta_{jl}L_{ik} - \delta_{il}L_{jk} - \delta_{jk}L_{il}) \\ & - P_{Y_{n+1}} \sum_{j<i}^{N} L_{ij}(\delta_{ik}L_{jl} + \delta_{jl}L_{ik} - \delta_{il}L_{jk} - \delta_{jk}L_{il}) = 0.\end{aligned}$$

Same proof for the case $N \le k < l$. □

Proof. If $n = 2v$, then we consider the following constant of motion

$$H, \underbrace{L_{1,2}, L_{3,4}, \ldots, L_{2v-1,2v}}_{v \text{ constants}}, \underbrace{C_4, C_6, \cdots, C_{2v}}_{v-1 \text{ constants}}.$$

By Lemma 99, the constants of motion are in involution. While, if $n = 2v + 1$, then we consider the following constant of motion

$$H, \underbrace{L_{2,3}, L_{4,5}, \ldots, L_{2v,2v+1}}_{v \text{ constants}}, \underbrace{C_2, C_4, \cdots, C_{2v}}_{v \text{ constants}}$$

By Lemma 99, the constants of motion are in involution. □

B.3.2 $N_{3,6,1}$

Let $N_{3,6,1a}$ be a Carnot group with growth vector $(3, 5, 6)$ and first layer $\mathfrak{g}_1$, framed by $\{E^1, E^2, E_{\mathfrak{a}}\}$, generates the following Lie algebra:

$$E^1_{\mathfrak{a}} := [E^1, E_{\mathfrak{a}}] \;\; E^2_{\mathfrak{a}} := [E^2, E_{\mathfrak{a}}], \;\; E^3_{\mathfrak{a}} := [E^2, E^1_{\mathfrak{a}}] = [E^1, E^2_{\mathfrak{a}}],$$

Otherwise, zero. The Lie algebra $\mathfrak{a}$ is given by $E_{\mathfrak{a}}$, $E^1_{\mathfrak{a}}$, $E^2_{\mathfrak{a}}$ and $E^3_{\mathfrak{a}}$: So in this case $N_{3,6,1a^*} = \mathbb{A} \rtimes \mathbb{R}^2$ and $\mathcal{A}_{N_{3,6,1a^*}} = d\theta_1 \otimes (e_1 + xe_2 + ye_3 + xye_4)$. Then if $\mu = (a_1, a_2, a_3)$ in $\mathfrak{a}^*$, the reduced Hamiltonian H_μ is given by

$$H_\mu(p_x, x) = \frac{1}{2}(p_x^2 + p_y^2 + (a_0 + a_1 x + a_2 y + a_3 xy)^2).$$

Theorem 100. *The subRiemannian geodesic flow on $N_{3,6,1}$ is not integrable by analytic functions.*

The notation $N_{3,6,1}$ was taken from [17].

B.3.2.1 Background Theorem

Here we will use the theory of the Hamiltonian systems with two degrees of freedom two and homogeneous potential of degree $3 \leq k$; that is, we will consider the following Hamiltonian function

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \phi(x, y), \quad \text{where} \quad \phi(\lambda x, \lambda y) = \lambda^k(x, y). \tag{B.3}$$

Let us introduce the **Background Theorem** used to prove the non-integrability of the Hamiltonian from equation (B.3).

Definition 101. *Let ϕ be a homogeneous potential $\phi(x, y)$ of degree k; we say that a point $p = (x, y) \neq 0$ is a Darboux point if $\nabla\phi(p) = p$. Then the homogeneity of the potential implies one eigenvalue of the Hessian $Hess\phi$ is $k-1$, and a second one is given by $\lambda = TrHess(p) - (k-1)$*

Yoshida proved the following Theorem.

Background Theorem 2 (Yoshida)**.** *Let $\mathbb{S}_k$ be the following region*

$$\begin{aligned} \mathbb{S}_k := &\{\lambda < 0, 1 < \lambda < k-1, k+2 < \lambda < 3k-2, \cdots, \\ & j(j-1)\frac{k}{2} + j < \lambda < j(j+1)\frac{k}{2} - j, \cdots\}. \end{aligned}$$

If λ is in $\mathbb{S}_k$, then the Hamiltonian system (B.3) is non-integrable by analytic functions.

B.3.2.2 Proof of Theorem 100

Proof. Let $\mu = (0, 0, 0, a)$ with $a \neq 0$, then $\frac{1}{|a|}(1, 1)$ is a Darboux point and $\lambda = -1$. then λ is in $\mathbb{S}_3$, so H_μ is is not integrable by analytic functions. □

Appendix C

Glossary of mathematical symbols

Symbol	Description	Reference
$J^k(\mathbb{R},\mathbb{R})$	Jet space of function from $\mathbb{R}$ to $\mathbb{R}$	Section 3.1
$\mathrm{Eng}(n)$	Engel type	Section 4.1
$\mathbb{G}$	Carnot group	Definition ??
$\mathfrak{g}$	Lie algebra of $\mathbb{G}$	Definition ??
$[\mathbb{G},\mathbb{G}]$	Commentator group	Definition ??
$\mathfrak{g}_1$	First layer of $\mathfrak{g}$	Definition ??
$\mathbb{R}^{d_1}$	Quotient $\mathbb{G}/[\mathbb{G},\mathbb{G}]$	Definition 7
π	Canonical projection from $\mathbb{G}$ to $\mathbb{R}^{d_1}$	Equation (2.1)
$\mathcal{D}$	Non-integrable distribution	Definition 7
$dist_{\mathbb{R}^{n+2}_F}$	subRiemannian distance on $\mathbb{R}^{n+2}_F$	Definition 2
$F_\mu(x)$	A polynomial on $\mathcal{H}$	sub-Section (2.2.1)
I	Hill interval	Definition 24
$\mathbb{R}^{n+2}_F$	subRiemannian manifold	Section 2.3
π_F	Projection from $\mathbb{G}$ to $\mathbb{R}^{n+2}_F$	Equation (2.6)
$\mathbb{A}$	Maximal normal abelian subgroup of $\mathbb{G}$ containing $[\mathbb{G},\mathbb{G}]$	Equation (8)
$\mathcal{H}$	quotient group $\mathbb{G}/\mathbb{A}$	Definition ??
$\pi_{\mathbb{A}}$	Canonical projection from $\mathbb{G}$ to $\mathcal{H}$	Equation (2.2)
$\mathfrak{a}$	Lie algebra of $\mathbb{A}$	Definition 9
$T^*\mathbb{G}$	Cotangent bundle of $\mathbb{G}$	sub-Section 10
H_{sR}	subRiemannian kinetic energy	Equation (10)
$\gamma(t)$	subRiemannian geodesic on $\mathbb{G}$	Definition 10
$T^*\mathcal{H}$	Cotangent bundle of $\mathbb{G}$	sub-Section 11
$\mathcal{A}_{\mathbb{G}}$	$\mathfrak{a}^*$ value one-form on $\mathbb{R}^{d_1}$	Definition 93
$\mathcal{A}^M_{\mathbb{G}}$	$\mathfrak{a}^*$ value one-form on $\mathcal{H}$	Definition 93
$\mathcal{A}^E_{\mathbb{G}}$	$\mathfrak{a}^*$ value one-form on $\mathcal{V}$	Definition 93
$\eta(t)$	$\alpha_{\mathbb{G}}$-curve on $\mathcal{H}$	Definition 11
(x,θ)	Exponential coordinates of second kind	Definition B.2
H_μ	Reduced Hamiltonian	Equation (2.4)
J	Momentum map induce by $\mathbb{A}$	sub-Section 10
$\mathfrak{v}$	$\mathfrak{g}_1 \cap \mathfrak{v}$	Definition 9
$\mathfrak{h}$	$\mathfrak{a}^\perp$ with respect the subRiemannian inner product	Definition 9
$\mathcal{V}$	$\mathcal{H}^\perp \subset \mathbb{R}^{d_1}$	Definition 9
$\mathcal{D}_F$	The $(n+1)$-rank non-integrable distribution in $\mathbb{R}^{n+2}_F$	sub-Section 2.3.1
π_F	The subRiemannian submersion from $\mathbb{G}$ to $\mathbb{R}^{n+2}_F$	Equation 2.6

Table C.1: Glossary of Mathematical symbols.

Symbol	Description	Reference
pr	The subRiemannian submersion from $\mathbb{R}^{n+2}_F$ to $\mathbb{R}^{n+1}$	Equation 2.6
H_F	Kinetic energy on $T^*\mathbb{R}^{d_1}_F$	Section 2.3
Pen_F	Pencil of F	Definition 14
$c(t)$	subRiemannian geodesic on $\mathbb{R}^{n+2}_F$	Definition 13
$Iso(M)$	Isometry group of the subRiemannian manifold M	
$\Delta t(c,[t_0,t_1])$	Time change in the time interval $[t_0,t_1]$	Definition 19
$\Delta y(c,[t_0,t_1])$	y change in the time interval $[t_0,t_1]$	Definition 19
$\Delta z(c,[t_0,t_1])$	z change in the time interval $[t_0,t_1]$	Definition 19
$Cost(c,[t_0,t_1])$	Cost map in the time interval $[t_0,t_1]$	Definition 19
$Cost_t(c,[t_0,t_1])$	Cost t in the time interval $[t_0,t_1]$ $\mathcal{I}$	Definition 19
$Cost_y(c,[t_0,t_1])$	Cost y in the time interval $[t_0,t_1]$	Definition 19
K	Compact set on $\mathbb{R}^{n+2}_F$	Proposition 22
$Min(K,[t_0,t_1])$	Sequentially compact space of geodesics	Proposition 22
$K_{\mathcal{H}}$	Compact set on $\mathcal{H}$	Lemma 39
$L(G,I)$	The period of (G,I)	Proposition 28
$\Delta y(G,I)$	y change of (G,I)	Proposition 28
$\Delta y(G,I)$	z change of (G,I)	Proposition 28
$\Theta(G,I)$	Period map	Definition 29
$\Theta_t(G,I)$	Period map	Definition 29
$\Theta_y(G,I)$	Period map	Definition 29
$\mathcal{I}$	Travel interval	Definition 31
$\Delta t(G,\mathcal{I})$	Time change during the travel interval $\mathcal{I}$	Proposition 32
$\Delta y(G,\mathcal{I})$	y change on the travel interval $\mathcal{I}$	Corollary 32
$\Delta z(G,\mathcal{I})$	z change on the travel interval $\mathcal{I}$	Corollary 32
$Cost_t(G,\mathcal{I})$	Cost t function on the travel interval $\mathcal{I}$	Corollary 32
$Cost_y(G,\mathcal{I})$	Cost y function on the travel interval $\mathcal{I}$	Corollary 32
Pen_d	The set of all the direct-type $\mathbb{R}^3_{F_{d,t}}$-geodesic	Corollary 35
Pen_h	The set of all the homoclinic $\mathbb{R}^3_{F_h}$-geodesic	Corollary 48
$F_\mu(r)$	A polynomial of a single variable r	Equation (4.3)
$H_\mu(p_r,p_\theta,r,\theta)$	Planar an-harmonic oscillator	Equation (4.4)
$H_{\mu,\ell}(p_r,r)$	Reduced Hamiltonian	Equation (4.5)
$hill(\mu,\ell)$	Plane hill region	Definition 62
Pen_V	Pencil of V_{ef}	Definition 65
$\mathcal{R}$	Radial travel interval	Definition 68
$\Theta(G,\ell,R)$	Radial period map	Definition 66

Table C.2: Glossary of Mathematical symbols.

Symbol	Description	Reference
$\Theta_t(G, \ell, R)$	Radial period map	Definition 66
$\Theta_y(G, \ell, R)$	Radial period map	Definition 66
$Cost_t(G, \ell, \mathcal{I})$	Cost t function on the travel radial interval $\mathcal{R}$	Corollary 32
$Cost_y(G, \ell, \mathcal{I})$	Cost y function on the travel radial interval $\mathcal{R}$	Corollary 32
$\mathcal{P}(k)$	space of polynomial of degree bounded by k	Definition 81
hor	Horizontal lift	Definition 92
φ	Left action of $\mathbb{A}$ on $\mathbb{G}$	Definition 91
σ	Infinitesimal generator	Equation (B.1)
ω	Connection one-form	Equation (B.2)
$\Pi_{\mathbb{R}^{d_1}}$	Linear projection	Definition 93
$So(\mathcal{H})$	Group of rotation on $\mathcal{H}$	sub-Section 57
$N_{6,3,1}$	Carnot group with growth vector $(3, 5, 6)$	sub-section B.3.2

Table C.3: Glossary of Mathematical symbols.