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# CHAINS AND MONSTERS

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**Abstract**

Chains and Monsters

by

Alex L. Castro

This dissertation is an extended version of earlier works published by the author and my supervisor Prof. Richard W. Montgomery dealing with the solution of two different problems in differential geometry. Although these problems are different in nature, the techniques involved in their solution share a good deal of commonality and coincidently both had their origins in the works of Élie Cartan and his school.

Part one concerns the chains of left-invariant structures Cauchy-Riemann structures on the three-sphere viewed as a Lie group. Cauchy-Riemann manifolds borrow their name from the problems in several complex variables they first arouse, and É. Cartan was the first person to systematically study them. The moduli space of these left-invariant structures form a half-line, and using Fefferman's characterization of chains as null-geodesics of an associated conformal structure we compute these chains by applying tools from geometric mechanics to the geodesic flow. We show that for almost all values of the modulus parameter, either two or three types of chains occur. To the authors knowledge this has been the first time the computations of the chains for these structures have appeared in the literature.

In the second part we change gears and construct a tower of fibrations associated to space curves generalizing earlier work by Montgomery-Zhitomirskii for planar

curves, and having its origins in the classification problem of Goursat distribution flags up to local diffeomorphism.

Cartan introduced the method of prolongation which can be applied either to manifolds with distributions (Pfaffian systems) or integral curves to these distributions. Repeated application of prolongation to $n$-space endowed with its tangent bundle yields the Monster tower, a sequence of manifolds, each a projective $\mathbb{P}^{n-1}$-bundle over the previous one, each endowed with a rank $n$ distribution.

The pseudo-group of diffeomorphisms of $n$-space acts on each level of the extended tower. We take initial steps toward classifying points of this extended Monster tower under this pseudogroup action. Arnol'd's list of stable simple curve singularities plays a central role in these initial steps.

To my parents, José Lúcio de Castro and Sueli Maria Ribeiro,

for their unconditional love, because none of them had a clue of why their first born spent so many hours incarcerated in a cubicle reading about abstract nonsense and scribbling frenetically on endless piles of scratch paper.

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Many thanks to my research adviser, Prof. Richard Montgomery, for taking me along on this beautiful but demanding journey that mathematical research consists of. He has chosen some endurable trails wherein I learned about the nitty-gritty of survival in the mathematical profession and where, I believe, I was also taught the gut-feelings for what is relevant and important in professional research. Due to him I was spared getting lost in the allegorical forest of mathematical ideas, which if truth be told verged on happening in many occasions.

A cheers to many friends and colleagues who have also contributed directly and indirectly along the painstaking saga my research projects led to, with all their long and unpredictable detours and sometimes dead ends. They all dealt with my blues playing the right song.

Special thanks to Jie Qing, Gil Bor and Misha Zhitomirskii for their selfless tutelage, Robin Graham, and Tony Tromba for all their invaluable advice and mentoring being extended way beyond a mere 'epsilon'-contribution. And last but not least, Viktor Ginzburg from whom I learned a plethora of mathematical ideas, and whose crystalline exposés of mathematics were exemplary. Much of my research interests were molded

one way or the other by direct interaction with these men.

# Part I

# Cauchy-Riemann geometry

# Chapter 1

# Introduction and Historical Digression

"Eu não sei quase nada, mas desconfio de muita coisa".
Grande sertão: veredas - Página 16 de João Guimarães Rosa - Publicado por J. Olympio, 1958 - 571 páginas.

## 1.1 Poincaré and Several complex variables

Higher-dimensional complex analysis, or several complex variables as its nowadays commonly known, is rather young compared to its one-dimensional cousin. There are various names that could be mentioned here in relation to its early development, including of course K. Weierstrass work on multi-variable analytic function theory. But instead we shall fast forward our tale a tad bit, and start with H. Poincaré's works on the subject. And though Poincaré started pioneering the field in the late nineteenth century, his piece of work which have certainly caused the most bewilderment in the community was a note dating from 1907 where he investigates the question to whether or not Riemann's mapping theorem admits a multidimensional generalization.

Our reader may recall that Riemann mapping theorem occupies a central place in analytic function theory. For the purpose of illustration we shall state here a weaker version of the full-blown theorem found in most standard treatises in complex analysis, though we extracted our present version from M. Levi's delightful book [Lev09]:

**Theorem 1.1.1** *Let $D$ be the open region bounded by a simple closed curve $C$ in the complex plane and let $z_0$ be a point in $D$. There exists an analytic function $f$ that maps $D$ onto the unit disk $\Delta = \{z : |z| < 1\}$ in a one-to-one fashion, with $f(z_0)$ and $f'(z_0) > 0$.*

How should one generalize holomorphic maps to higher dimensions? A fundamental property of holomorphic functions, rather useful in practice, is that $f$ preserves angles at all points where $f' \neq 0$. But requiring a mapping in dimension $n > 2$ to be conformal reduces drastically the class of maps one can study. According to J. Liouville's theorem, a smooth conformal map in dimensions in dimension three of higher must be linear (cf. [DFN92]).

It is more fruitful to look at maps $F : D_1 \to D_2$ between domains of $\mathbb{C}^2$, or more generally $\mathbb{C}^n$, and requiring each component $f_i$ of $F = (f_1, \ldots, f_n)$ to be holomorphic in the classic sense. We shall call these maps *holomorhic* too, and if they admit a holomorphic inverse they are said to be *biholomorphic.*

Poincaré (cf. [Jac85], mainly introduction) tackled the question of whether or not Riemann's mapping theorem extends to domains in $\mathbb{C}^2$. He derived a negative answer to the problem by computing the group of biholomorphic self-mappings of the unit ball in $\mathbb{C}^2$

$$\{(z,w) : |z|^2 + |w|^2 < 1\}$$

and of the polydisk

$$\{(z,w) : |z|^2 < 1, |w|^2 < 1\}$$

and verifying these two groups are not isormorphic. But Poincaré did not stop there. In the same paper he showed that two real hypersurfaces in $\mathbb{C}^2$ are in general biholomorphically inequivalent. Roughly, his argument was to show that the smooth boundary of a domain in $\mathbb{C}^2$ contains intrinsic holomorphic invariant information about its 'shape'.

The gist of Poincaré's argument, can be summarized by the following amusing 'physicists's argument' provided by R. Penrose [Pen83]. Consider the freedom involved in specifying the real hypersurface boundary of a region in $\mathbb{C}^2$. This is provided by one real function of 3 variables (e.g. $\Im w$ in terms of $\Re z, \Im z, \Re w$).

For the *intrinsic* structure of this boundary, we must factor it out by the freedom provided by the local holomorphic maps of $\mathbb{C}^2$ to itself. This is provided locally by two holomorphic functions of two variables. But a holomorphic function is determined by its analytic values on any real environment, e.g. on its values where its arguments, say $z$ and $w$ are real. The real and imaginary parts are each, in effect, independent real analytic functions, so the freedom to be factored out by is that of four real functions of two real variables. The amount of intrinsic freedom in the structure of the boundary is therefore

$$\frac{\text{1 real function of 3 real variables}}{\text{4 real functions of 2 real variables}}.$$

However, any finite number of functions of two variables must be regarded as 'peanuts' in the context of free functions of three variables. This is a heuristic indication that there must be "intrinsic invariants" of the boundary shape.

Monsieur Poincaré concludes his ingenious note by posing the question of finding differential invariants to distinguish one real hypersurface from another. Smooth hypersurfaces $M^3$ in $\mathbb{C}^2$, or in $\mathbb{C}^n$ were eventually christened *Cauchy-Riemann manifolds*, or just CR-manifolds in the several complex variables trade. By restricting the Cauchy-Riemann operator to a smooth hypersurface $M^3$, we are led to a complex valued vector section $L : M \to TM \otimes C$ which describes the interaction of the hypersurface with the ambient complex structure. We shall have more to say about Cauchy-Riemman manifolds in the next section.

## 1.2 S.S.-Chern, J. Moser's and differential invariants of CR-manifolds

Poincaré's question for hypersurfaces in $\mathbb{C}^2$ was first completely answered by É. Cartan, using his powerful equivalence method. We refer the reader to the beautiful monograph by Howard Jacobowitz [Jac90] where Cartan's paper is distilled and rewritten in modern language for the benefit of our entire community! Cartan's equivalence method, used to construct differential invariants in the solution of Poncarés question, is explained in a very didactic way, and the reader is put in Cartan's shoes and led to trace back all the insights involved in his solution.

More recent work of S. S. Chern and J. K. Moser [CM83] on the classification of hypersurfaces $M^{2n-1}$ in $\mathbb{C}^n$ with nondegenerate Levi form, a family of curves in $M$, called "chains", has been singled out. Simply stated, a chain is a curve in $M$, such that, after a local holomorphic change of coordinates, M can be approximated to high order by a quadratic surface $Q$, e.g. $\Im w = |z|^2$ that makes contact with $M$ along . One of the main results in the paper of Chern and Moser (op. cit.) states that, for every $p \in M$ and every tangent vector $v \in T_pM$ transversal to the holomorphic tangent space at $p$, there is a chain through p in the direction of $v$, and that is unique, subject to some further conditions.

Permit us to be more explicit here. In modern language, a CR structure on a three-dimensional manifold $M^3$ is a 2-plane distribution $H \subset TM^3$ together with a fibre preserving $J : H \to H$ with $J^2 = -I$, an *almost complex structure.* Given such a structure, we may choose a real 1-form $\omega$ which annihilates $H$ and a complex 1-form $\omega_1$ which annihilates all vectors of the form $X + iJX$, $X \in H$. These choices can be done in such a way that $\omega \wedge \omega_1 \wedge \overline{\omega_1}$ is different from zero in a neighborhood of a given point. Since Cartan was originally interested in results of a local nature on $M^3$, so we may shrink $M$ and assume $\omega \wedge \omega_1 \wedge \overline{\omega_1}$ is everywhere different from zero.[1] Any three-dimensional submanifold $M$ of $\mathbb{C}^2$ has an induced CR structure. Let $\tilde{J} : \mathbb{C}^2 \to \mathbb{C}^2$ be the standard complex structure of $\mathbb{C}^2$. Then $H = TM \cap \tilde{J}TM$ and $J = \tilde{J}|_H$. Note that if $F : D_1 \to D_2$ is a biholomorphism of open sets in $\mathbb{C}^2$ , then $M \cap D_1$ and $F(M) \cap V$ have the same CR structure. The forms $\omega$ and $\omega_1$ can also be directly determined for

[1] Conversely, given $\omega$ and $\omega_1$ with $\omega \wedge \omega_1 \wedge \overline{\omega_1}$ we may easily construct H and $J$.

$M \subset C^2$ . To do this let $M$ be given by $r(Z, \bar{Z}) = 0$ with $Z = (z, w)$ and $dr \neq 0$ at points of $M$. For definiteness assume that at a given point $p \in M$ we have $dz \wedge dr \neq 0$. Now let $\omega = i\partial r$ and $\omega_1 = dz$. The operator $\partial$ and $\bar{\partial}$ are the coboundary operators associated to the Dobeault complex in $\mathbb{C}^2$. Loosely speaking, $\partial = \frac{\partial}{\partial z}dz + \frac{\partial}{\partial w}dw$ acting on forms. Since $dr = \partial r + \bar{\partial} r = 0 \Rightarrow \partial r$ is imaginary.

We are now guaranteed to have $\omega \wedge \omega_1 \wedge \overline{\omega_1} \neq 0$.[2]

**Definition 1.2.1** *A CR structure on $M^3$ is strictly pseudo-convex if*

$$d\omega = ig\omega_1 \wedge \overline{\omega_1}(\ mod\ \omega) \neq 0. \tag{1.2.1}$$

On the other hand, in several complex variables there is also a notion *strictly-pseudoconvexity* of a hypersurface $M^3 = \{r = 0\}$.

**Definition 1.2.2** *A hypersurface $M^3 = \{r = 0\} \subset \mathbb{C}^2$ is said to be strictly-pseudoconvex at p if the quadratic form, also known as* Levi form*,*

$$H(p) = \frac{\partial^2 r}{\partial \bar{z} \partial z}(p) z\bar{z} + \frac{\partial^2 r}{\partial \bar{z} \partial w}(p) \bar{z} w + \frac{\partial^2 r}{\partial z \partial \bar{w}}(p) z\bar{w} + \frac{\partial^2 r}{\partial \bar{w} \partial z w}(p) w\bar{w},$$

*is positive definite. The vector $v = (z, w)$ sits in $T_p M^3 \cap J T_p M^3$.*

A dilation $\tilde{r} = \lambda r$, $\lambda \neq 0$ changes $H(p)$ to a conformally equivalent quadratic form.

Later on, when we are presenting the bulk of our work we shall explain how these two apparent different notions of pseudoconvexity in $M^3$ are actually equivalent.

As shown by Jacobowitz [Jac85], following Chern and Moser, there is always a local biholomorphism which brings strictly-pseudoconvex hypersurface to the normal form

[2] It is easy to show that this gives the same CR structure as the previously defined $H$ and $J$.

$$\{(z_1, z_2) : \Im w = |z|^2 + G(z, \bar{z}, \Re w), G \in O(4)\}.$$

Let us set, $w = u + iv$. Then $(z, u)$ provides local coordinates on any hypersurface in the normal form above.

To find $\omega$ for the induced CR structure on a submanifold of the form given previously, we take the defining function $r$ to be

$$r = \frac{1}{2i}(w - \bar{w}) - z\bar{z} - G,$$

where $G$ is extended off the $(z, u)$-plane. Thus

$$\omega = i\partial r = i(\frac{\partial}{\partial z}dz + \frac{\partial}{\partial w}dw)(\frac{1}{2i}(w - \bar{w}) - z\bar{z} - G) =$$

$$= i\bar{z}dz + \frac{1}{2}dw - iG_z dz - i \underbrace{G_w}_{=\frac{1}{2}G_u} dw.$$

But

$$dw = du + idv = du + i(zd\bar{z} + \bar{z}dz + \underbrace{dG}_{G_z dz + G_{\bar{z}} d\bar{z} + G_u du}).$$

So after collecting all similar terms as well as it is possible in the expression for $\omega$ above we end up with:

$$\omega = \frac{1}{2}((1 + A)du + (iz + B)d\bar{z} - (i\bar{z} - \bar{B})dz),$$

where $A = O(6)$ and $B = O(3)$, recalling that $G = O(4)$. By replacing $\omega$ by a $(1+A)^{-1}\omega$, which still annihilates the subbundle $H$, we can write

$$\tilde{\omega} = \frac{1}{2}(du + (iz + C)d\bar{z} - (i\bar{z} - \bar{C})dz).$$

Differentiating,

$$d\tilde{\omega} = \frac{1}{2}(dC \wedge d\bar{z} + d\bar{C} \wedge dz + 2dz \wedge d\bar{z}).$$

Next if we expand $dC$ in the basis $\omega, dz, d\bar{z}$, we can write $dC = C_0\omega + C_1 dz + \overline{C_1} d\bar{z}$ and reexpress $d\omega$ as

$$d\omega = (\underbrace{1 - i\frac{1}{2}C_1 + i\frac{1}{2}\overline{C_1}}_{=D})idz \wedge d\bar{z}(\mod \omega).$$

Thus $d(D^{-1}\omega) = D^{-1}d\omega(\mod \omega) = idz \wedge d\bar{z}(\mod \omega)$ which allows us to take $\tilde{\tilde{\omega}} = D^{-1}\omega$ and $w_1 = dz$.

In conclusion, we have just seen that on a strictly pseudo-convex hypersurface in $C^2$ it is always possible to choose $\omega$ and $\omega_1$ such that $d\omega = i\omega_1 \wedge \overline{\omega_1}(\mod \omega)$. This choice is not unique. Cartan [Car33] has shown how to use any one choice to calculate quantities which are in fact independent of the choice.

Consider for instance, a set of equations

$$\omega_1 = -\mu\omega, \tag{1.2.2}$$

$$d\mu = F(\mu, x)\omega,$$

defined on $M^3 \times \mathbb{C}$ where $F$ is assumed to be smooth. When written in local coordinates, the corresponding direction field gives rise to a second order differential equation. Through each point $p$ and in each direction $v$ transverse to H there is a unique (**unparameterized**) curve which satisfies (1.2.2). From the work of Cartan [Car33], given a CR structure one can always write one such system with $F$ depending only on the differential invariants associated to the CR structure under study, defining in this form a CR invariant system of curves. The resulting curves are called *chains*. For example, for the oscullating hyperquadric $v = |z|^2$ reduce to:

$$\omega_1 = -\mu\omega,$$

$$d\mu = i\mu|\mu|^2\omega.$$

This particular system can be integrated by hand.

Chains have certain similarities to geodesics in Riemannian manifolds. H. Jacobowitz [Jac85], and later Lisa Koch [Koc88], showed that any two sufficiently close points in a CR-manifolds can be joined by a chain. But chains also present a some 'pathological' behavior not akin to geodesics. Fefferman to our knowledge was the first to notice it. Next we digress on how the ideas of Fefferman permitted the realization of the present work.

## 1.3 Fefferman shedding light on the subject

In the work of Cartan, Chern and Moser chains appear as solutions of a complicated second order system of differential equations, and it seems very hard to actually compute them. But it is clear that detailed information about chains is very important for a deeper understanding of the complex geometry of hyper-surfaces.

One such example is J.H. Cheng's theorem [Chê88] stating that chain preserving diffeomorphisms are either CR isomorphisms or anti-isomorphisms. G. Sparling and P. Nurowski [NS03] have investigated, much more in the spirit of Cartans original treatment, also the relation between the 'path-geometry' of second-order ODE's and three-dimensional CR structures and it seems that it is possible to almost entirely reconstruct the underlying CR structure from the path geometry associated to the corresponding system of chains. This seems to be a *raison d'étre* for Cheng's theorem.

Back Charles Fefferman in [Fef76] introduced some brilliant new ideas that, ultimately, lead to a computable characterization of chains on the boundary $\partial D$ of a strictly pseudoconvex domain $D$. Below is an account originally told by Michael Range in the "Mathematical Reviews on the Web."

The starting point was an idea conceived by the Japanese mathematician I. Naruki, who introduced an extra variable to $\mathbb{C}^n$ and used the Bergman kernel $K_D$ of $D$ to define a metric $ds^2$ on $\overline{D} \times \mathbb{C}^*$, here $D$ denotes a strictly pseudoconvex domain in $\mathbb{C}^n$. However, since $K_D$ is really unknown in the interior of $D$, $ds^2$ cannot be calculated. Fefferman solved this problem as follows. First, by restricting the above metric $ds^2$ to $\partial D \times S^1$ one obtains a nondegenerate Lorentz metric on $\partial D \times S^1$, denoted again by $ds^2$, whose conformal class is a biholomorphic invariant. The central role of this Lorentz metric is revealed by the next step: it turns out that the chains on $\partial D$ are obtained by projecting the light rays (geodesics with tangent of length 0) defined by $ds^2$ from $\partial D \times S^1$ to $\partial D$. By invoking some machinery Fefferman himself developed earlier, more specifically an asymptotic expansion of the Bergman kernel near the boundary one could now, in principle, compute the Lorentz metric $ds^2$. But Fefferman takes one further step: he notes that to calculate $ds^2$ one does not really need the Bergman kernel; instead, one can use the solution $u$ of a suitable boundary value problem associated with a second order nonlinear partial differential equation of the Monge-Ampére type.

However all that is *really* needed for the problem at hand is the formal second order approximation to the solution $u$, and this can be computed! Beginning with a function that defines the boundary $\partial D$, the author calculates explicitly the Lorentz metric $ds^2$ and the Hamiltonian system that defines the light rays. As a nifty application of this approach, Fefferman goes on to show that the hypersurface $v = |z|^2 + 2u|z|^8$ has chains which spiral in to the origin! This is one example of the unusual behavior of chains when compared to geodesics.

And it is here that *our story* begins.

## 1.4 Burns-Rossi's example and pseudo-Hermitean geometry of Farris and Lee

Fefferman's construction has only one immediate shortfall : it does not automatically generalize to abstract CR structures. And non-realizable CR structures, i.e. CR structures which cannot be viewed as hypersurfaces in some complex ambient space, even locally, abound.[3]

The first such example was described by H. Lewy in [Lew57]. He introduced a complex valued smooth linear differential operator defined on smooth functions, nowadays known as Lewy operator, and showed that in three dimensions the resulting PDE admits no solution in the smooth category. Lewy operators, typically denoted by $L$, are commonly used to describe abstract CR structures on three manifold and are rep-

[3] Real analytic CR structures are always locally embedabble. Cf. [Jac90].

resented by a complex-valued vector field. One can comfortably say it “describes the interaction of $M$ with the ambient complex case”, supposing $M$ is embedded.

And even if we restrict ourselves to the subcategory of analytic CR manifolds there are still examples of locally non-embeddable three-manifolds that are not globally embeddable! There is a one-dimensional deformation of the standard CR structure in $S^3$, all of which are left-invariant, and that does not admit a global realization in $\mathbb{C}^2$. To our knowledge, this is example is due to Dan Burns [Bur79], who adapted it from an older paper by H. Rossi [Ros65]. In Appendix A we re-derive Burns' proof using purely representation theoretic arguments, first suggested to us by Gil Bor. We include it here for sake of illustration and for its aesthetic properties.

Inspired by Bill Goldman's book [Gol99], and the curious behavior of chains first pointed out by Fefferman, we decided to investigate the behavior of chains for the Burns-Rossi structures described above.

To implement our program, we had to surmount Fefferman's ambient construction and use a more ‘canonical’ procedure using only differential geometric data intrinsic to $M^3$, removing thus the need for an ambient complex structure. Luckily, for that, Frank Farris and John Lee had already thought up a plan.

Farris [Far86] and Lee's [Lee86] working definition of CR manifold is similar to the one we gave earlier. Lee's characterization, which we have ultimately opted for, rely on a choice of a one-form, the so-called pseudo-Hermitian structure on $M$, which annihilates the maximum complex subbundle of the tangent bundle $TM$. Lee, and Farris, defines Lorentz metrics on an intrinsically defined circle bundle $Z \xrightarrow{S^1} M$. The characterization is in terms of differential forms which are intrinsic to the bundle $Z$ and which are normalized using the chosen pseudo-Hermitian structure. This construction renders a metric that transforms conformally under a change of pseudo-Hermitian structure; and that they agree with the Fefferman metric when $M$ is embedded.

Much to our surprise, we found Fefferman's spiralling behavior in the chains from the Burns-Rossi example. In our work $Z \cong SU(2) \times S^1$. The Fefferman dynamics is governed by a Hamiltonian system in $T^*Z$, a Hamiltonian system in the contangent bundle of Lie group. By applying Lie-Poisson reduction we faced now with a non-linear system of ordinary differential equations resembling Euler's equation for a free rigid body, and that is also integrable. Spiralling behavior here is due to the existence of homoclinic orbits in the reduced dynamics. It remains to investigate whether qualitative dynamical behavior of chains can be read off the conformal invariants of Fefferman's metric, the first obvious example being its Weyl curvature tensor.

Now that we have described the background of our play, let the first act begin!

# Chapter 2

# Our results : behavior of Chains

This is a formatted reproduction of our paper which has been published
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The left-invariant CR structures on the three-sphere $S^3 = SU(2)$ form a family of CR structures containing the standard structure. After the standard structure, these form the most symmetric CR structures possible in dimension 3. See Cartan [Car33]. The purpose of this note is to compute the chains for these structures. (Computations of Cartan curvature type invariants for the left-invariant CR structures can be found in [Čap06]. )

The chains on a strictly pseudoconvex CR manifold are a family of curves on the manifold invariantly associated to its CR structure. Chains were defined by Cartan [Car33]and further elucidated by Chern-Moser [CM83], and Fefferman [Fef76]. Chains play a role in CR geometry somewhat similar to that of geodesics in Riemannian geometry. The left-invariant CR structures on $S^3$ are strictly pseudoconvex. Our computation of the chains for these structures appears here, apparently for the first time.

The space of left-invariant structures on $S^3 = SU(2)$ modulo conjugation is a half-line parameterized by a single real variable $a$. Any left-invariant CR structure is conjugate to one of those presented in the normal form below (section 2.1, equations ( 2.1.2), (2.1.3). The standard structure corresponds to $a = 1$. Its chains are obtained by intersecting $S^3 \subset \mathbb{C}^2$ with complex affine lines in $\mathbb{C}^2$. (See [Gol99] for especially good visual descriptions.) In particular all chains for the standard structure are closed curves. Here is our main result:

**Theorem 2.0.1** *Consider the left-invariant CR structures on the three-sphere. They form a one-parameter space, with parameter $a$ and $a = 1$ corresponding to the standard structure, as given by the normal form of section 2.1, equations ( 2.1.2), (2.1.3). Then, for all but a discrete set of values of $a$ two types of chains are present: closed chains and quasi-periodic chains dense on two-torii. The curves of each type are dense in $S^3$. A bifurcation occurs at $a = \sqrt{3}$ so that for $a > \sqrt{3}$ a third type of chain occurs, corresponding to a homoclinic orbit and which accumulates onto a periodic chain ( a geometric circle). For all $a > \sqrt{3}$ all three types of chains: periodic, quasi-periodic, and homoclinic are present and every chain is one of these three types. For $a < \sqrt{3}$ only the closed chains and quasi-periodic chains are present.*

**Remark.** We have left open the possibility that for a finite set of $a \in [1, \sqrt{3}]$ all chains are closed.)

The computations leading to the theorem are based on a construction of Fefferman [Fef76], refined and generalized by Lee [Lee86] and Farris [Far86]. Starting with a strictly pseudoconvex CR manifold $M$ the Fefferman construction yields a circle bundle $S^1 \to X \to M$ together with a conformal class of Lorentzian metrics on $X$. The chains are then the projections to $M$ of the light-like geodesics on $X$. It follows that we can look for chains by solving Hamiltonian differential equations.

Once we have the Hamiltonian system for Fefferman's metric, a simple picture from geometric mechanics underlies this theorem. For our left-invariant structures this Hamiltonian system is very similar to that of a free rigid body, but with configuration space being $SU(2) = S^3$ instead of the rotation group $SO(3)$. Like the rigid body, this Hamiltonian system is integrable. Its solutions – the chains – lie on torii, the Arnol'd-Liouville torii. As in the case of the rigid body, the non-Abelian symmetry group forces resonances between the a priori three frequencies on the torii: so that the torii are in fact two-dimensional, not the expected three dimensions, of $3 = dim(S^3)$.. When the frequencies are rationally related we get closed chains. Otherwise we get the quasi-periodic chains. The phase portrait (figure 2.2 below) changes with $a$ and the bifurcation at $a = \sqrt{3}$ corresponds to the origin turning from an elliptic to a hyperbolic fixed point in a bifurcation sometimes known as the Hamiltonian figure eight bifurcation.

### 2.0.1 Outline

There are five steps to the proof of the theorem. The paper is organized along these steps.

0. Find the normal form for the left-invariant structures on $SU(2)$.

1. Compute the Fefferman metric on $SU(2) \times S^1$ for the left-invariant CR structures.

2. Reduce the Hamiltonian system for the Fefferman geodesics by the symmetry group $SU(2) \times S^1$.

3. Integrate the reduced system.

4. Compute the geometric phases ( holonomies) relating the full motion to the reduced motion.

We briefly describe the methods and ideas involved in each one of the steps above, and in so doing link that step to the section in which it is completed.

Step 0. **Finding a normal form. Section2.1 )** In section 2.1 we derive the normal form (2.1.2), (2.1.3) for the left-invariant CR structures with single real parameter $a$. This normal form is well-known and standard. Its derivation is routine. The normal form can be found for example in Hitchin [Hit95] p. 34, and especially the first sentence of the proof of Theorem 10 on p. 99 there. Hitchin provided no derivation of the normal form. For completeness we present the derivation on the normal form in section 2.1

Step 1. **Finding the Fefferman metric. Section 2.2.**

In section 2.2 we compute the Fefferman metric associated to our normal forms. We follow primarily [Lee86]. Inverting this metric yields the Hamiltonian $H = H_a$ whose solution curves correspond to chains.

Step 2. **Constructing the reduced dynamics. Sections 2.4 and 2.3.**The chains for the left-invariant CR structures are the projections to $S^3$ of the light-like geodesics for the metrics computed in step 1. These geodesics are solutions to Hamiltonian systems on $T^*(S^3 \times S^1)$ whose Hamiltonians we write $H = H_a : T^*(S^3 \times S^1) \to \mathbb{R}$. As with all "kinetic energy" Hamiltonians, $H$ is a fiber-quadratic functions on the cotangent bundle. To specify that the geodesics are light like, we only look at those solutions with $H = 0$. The Fefferman metrics are always invariant under the circle action. In our case of left-invariant CR structures the metrics are also invariant under the left action of $S^3 = SU(2)$ (extended in the standard way to the cotangent bundle). Consequently we can reduce the Fefferman dynamics by the groups $S^1$ and $SU(2)$. This reduction is performed in sections section 2.4 and 2.3. Section 2.4 provides generalities concerning reducing left-invariant flows on Lie groups, and as such helps to orient the overall discussion. In section 2.3 we compute the reduced flow. In order to perform the reduction fix the standard basis $e_1, e_2, e_3$ for the $su(2)$. Write its dual basis, viewed as left-invariant one-forms, as $\omega^1, \omega^2, \omega^3$. Write $(g, \gamma)$ for a point of $S^3 \times S^1$ and $d\gamma$ for the one-form associated to the angular coordinate $\gamma$. Any covector $\beta \in T^*_{g,\gamma}(S^3 \times S^1)$ can be expanded as $\beta = M_1\omega^1(g) + M_2\omega^2(g) + M_3\omega^3(g) + Pd\gamma$ so we can write have $H = H(g, \gamma; M_1, M_2, M_3, P)$. Left-invariance implies that $H$ does not depend on $g$ or $\gamma$ so we can think of the Hamiltonian as a function $H = H(M_1, M_2, M_3, P)$ on $\mathbb{R}^3 \times \mathbb{R}$. The Euclidean space $\mathbb{R}^3 \times \mathbb{R}$ represents $su(2)^* \times \mathbb{R}^*$, the dual of the Lie algebra of our Lie group, $SU(2) \times S^1$. Equivalently, $\mathbb{R}^3 \times \mathbb{R}$ is the quotient space $T^*(S^3 \times S^1)/(S^3 \times S^1)$. The reduced dynamics is a flow on this space. The coordinate function $P$ is the momentum map for the action of the circle factor and as such is constant along solutions for the reduced dynamics. The function $H$ generates the reduced dynamics: $\dot{M}_i = \{M_i, H\}$ and $\dot{P} = \{P, H\}(= 0)$ where $\{\cdot, \cdot\}$ is the 'Lie-Poisson bracket'. See section 2.4.

Step 3. **Solving the reduced dynamics. Section 2.5.** The phase portrait found in figures 2.1, and 2.2 summarizes the reduced dynamics. . The computations proceed as follows. The functions $P$ and $K = M_1^2 + M_2^2 + M_3^2$ are Casimirs for the Lie-Poisson structure, meaning that $\{K, h\} = \{P, h\} = 0$ for any Hamiltonian $h$ used to generate the reduced dynamics. The solutions to the reduced dynamical equations thus lie on the curves formed by the intersections of the three surfaces $P = const.$, $K = const.$ and $H = 0$ in $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$. For typical values of these constants , these curves are closed curves. At special values the curves may be isolated points, or may be singular, like in the case of the homoclinic eight (figure 2.2).

When $P = 0$ we can solve for the dynamics explicitly. The corresponding chains are the left translates of a particular one-parameter subgroup in $G = S^3$. The case $P \neq 0$ can be reduced to $P = 1$ by the following scaling argument. We have $H(\lambda M_1, \lambda M_2, \lambda M_3, \lambda P) = \lambda^2 H(M_1, M_2, M_3, P)$. Up on $S^3 \times S^1$ this scaling represents leaving positions alone and scaling momenta, and hence velocities. Thus the reduced solution curves with initial conditions $(\lambda M_1, \lambda M_2, \lambda M_3, \lambda P)$ and those with initial conditions $(M_1, M_2, M_3, P)$ represent the same geodesics, and so the same chains, just parameterized differently. Choosing $\lambda = 1/P$ we can always scale the case $P \neq 0$ to the case $P = 1$. Now we have a single Hamiltonian $h = H(M_1, M_2, M_3, 1)$ on the standard rigid body phase space $\mathbb{R}^3$. We represent the surface $h = 0$ as a graph $M_3 = q(M_1, M_2; a)$

over the $M_1 - M_2$ plane, where $q$ is an even quartic function of $M_1, M_2$. We form the solution curves by intersecting this graph with the level sets of $K$. To simplify the analysis we project the resulting curves onto the $M_1 M_2$ plane. A critical point analysis of $K$ restricted to the graph locates the bifurcation value $a = \sqrt{3}$ for the reduced phase portrait as described in theorem 1.

Step 4. **Geometric phases. Section 2.6.** We follow the idea presented in the paper [Mon91] in order to reconstruct the chains in $S^3$ from the reduced solution curves. Some mild modifications are needed to that idea, since our initial group is $SU(2) \times S^1$ rather than the group $SO(3)$ of that paper. Fix $P = 1$ and a value of $K$ so that the reduced curve $C$ of step 1 is closed. The left action of $SU(2) \times S^1$ on $T^*(S^3 \times S^1)$ has a momentum map with values in $su(2) \times \mathbb{R} = \mathbb{R}^4$ and solutions (chains) must lie on constant level sets of this momentum map. One factor of this momentum map is $P$ from steps 2 and 3 which we have set to 1. Upon projecting the level set onto $T^*S^3$ via the projection $T^*S^3 \times T^*S^1 \to T^*S^3$ we obtain an embedded $S^3 \subset T^*S^3$ (the graph of a right-invaraint one-form) together with a projection onto the reduced phase space $\mathbb{R}^3 \times \{1\}$ of step 3. The inverse image of $C$ under this projection is a two-torus, and all the chains whose reduced dynamics is represented by $C$ and whose momentum map has the given fixed value lie on this two-torus within $T^*(S^3)$. One angle of this torus represents the reduced curve. The relevant question is: as we go once around the reduced curve, how much does the other angle change? Call this amount $\Delta\theta$. If the value of $\Delta\theta$ is an irrational multiple of $2\pi$ then the chain is not closed and forms one of the quasi-periodic chains of theorem 1, dense on its two-torus. If its value of $\Delta\theta$ is a rational multiple $\frac{p}{q}$ of $2\pi$ then the chain is closed, corresponding to some $p, q$ winding on its torus. With certain modifications, the basic integral formula for $\Delta\theta$ from [Mon91] is valid. One term in this formula corresponds to a holonomy of a connection, and is termed the "geometric phase", explaining the subtitle we have given to this step 4. The values of $\Delta\theta$ depends only on the values of $a$ and $K$ and its dependence is analytic in these variables. Thus the proof of the theorem will be complete once we have shown there is a value of $a$ for which $K \to \Delta\theta(K, a)$ is not constant.

In order to prove non-constancy of $\Delta\theta(K, a)$ , take $a > \sqrt{3}$ so that the reduced dynamics has a homoclinic eight. Denote the value of $K$ on the eight by $k(a)$. We show that as $K \to k(a)$ we have that $\Delta\theta(a, K) \to \infty$.

Steps 0 – 4 now completed, theorem 1 is proved.

**Appendices** We finish the paper with two appendices. In appendix A.1 we verify that when $a = 1$ the Fefferman geodesics for the Hamiltonian computed here (eq 2.3.3) correspond to the well-known chains for the standard three-sphere. In appendix 2 we show that the left-invariant CR structures for $a \neq 1$ correspond to the family of non-embeddable CR structures on $S^3$ discovered by Rossi, and frequently found in the CR literature.

**An Open problem.** We end appendix A.2 with an open problem inspired by the Rossi embedding of $S^3$/(antipodal map) and a conversation with Dan Burns.

## 2.1 A normal form for the left-invariant CR structures (step 0).

### 2.1.1 Preliminaries. Basic Definitions.

A contact structure in dimension 3 is defined by the vanishing of a one-form $\theta$ having the property that $\theta \wedge d\theta \neq 0$. Let $M$ be the underlying 3-manifold and $TM$ its tangent bundle. The contact structure is the field of 2-planes $\xi = \{(m, v) \in TM : \theta(m)(v) = 0\} \subset TM$. It is a rank 2 sub-bundle of the tangent bundle. The one-form $\theta$ and $f\theta$, for $f \neq 0$ a function, define the same contact structure.

**Definition 2.1.1** *A strictly pseudoconvex CR structure on a 3-manifold $M$ consists of contact structure $\xi$ on $M$ together with an almost complex structure $J$ defined on the contact planes $\xi$.*

We will primarily be using the following alternative, equivalent definition

**Definition 2.1.2** *A strictly pseudoconvex CR structure on a 3-manifold $M$ consists of an oriented contact structure $\xi$ on $M$ together with a conformal equivalence class of metrics defined on contact planes $\xi$.*

To pass from the first definition to the second, we construct the conformal structure from the almost complex structure $J$ in the standard way. Namely, the conformal structure is determined by knowing what an orthogonal frame is, and we declare $e, J(e)$ to be such a frame, for any nonzero vector $e \in \xi$. An alternative to this construction is to choose a contact form $\theta$ for the contact structure and then construct its associated *Levi form*

$$L_\theta(v, w) = d\theta(v, Jw) \tag{2.1.1}$$

which is a quadratic symmetric form on the contact planes. The contact condition implies that the Levi form is either negative definite or positive definite. If it is negative definite, replace $\theta$ with $-\theta$ to make it positive definite. We henceforth insist that $\theta, J$ are taken so the Levi form is positive definite. This assumption on $(\theta, J)$ is equivalent to assuming that the orientation on the contact planes induced by $\theta$ and induced by $J$ agree. (Note that a choice of contact one-form orients the contact planes. ) The conformal structure associated to $(\theta, J)$ from definition 2.1.1 is generated by the Levi form. If we change $\theta \to f\theta$ with $f > 0$ then the Levi form changes by $L_\theta \to fL_\theta$, showing that this definition of conformal structure is independent of (oriented) contact form $\theta$.

To go from definition 2.1.2 to definition 2.1.1, take any oriented orthogonal basis vectors $E_1, E_2$ having the same length relative to some metric in the conformal class. Define $J$ by $J(E_1) = E_2, J(E_2) = -E_1$. Thus in dimension 3 we can define a CR structure by a contact form $\theta$, defined up to positive scale factor, together with an inner product on the contact planes $\omega = 0$ to represent the conformal structure, also only defined up to a positive scaling. Choosing the scale factor of either the contact form or the quadratic form fixes the scalar factor of the other one through the Levi-form relation, eq. (2.1.1).

### 2.1.2 The left-invariant case.

We take $M = S^3$ which we identify with the Lie group $SU(2)$ in the standard way, via the action of $SU(2)$ on $S^3 \subset \mathbb{C}^2$. A left-invariant CR structure on $S^3$ is then given by the Lie algebraic data on $su(2)$. This data consists of a ray in $su(2)^*$ representing the left-invariant contact form $\theta$ up to positive scale and a quadratic form on $su(2)$ defined modulo $\theta$, and positive definite when restricted to $ker(\theta)$. Conjugation on $SU(2)$ maps left invariant CR structures to left-invariant CR structures, and induces the co-adjoint action on $su(2)^*$. This action is equivalent, as a representation, to the standard action of the rotation group $SO(3)$ on $\mathbb{R}^3$ via the 2:1 homomorphism $SU(2) \to SO(3)$. Consequently, we can rotate the contact form $\theta$ to anti-align with the basis element $\omega_3$. Thus we take $\theta = -\omega^3$. The contact planes are then framed by the left-invariant vector fields $e_1, e_2 \in su(2)$. The choice of $-\omega^3$ is made so that $e_1, e_2$ is the correct orientation of the plane, as follows from the structure equation

$$d\omega^3 = -\omega^1 \wedge \omega^2.$$

This structure equation also proves that the plane field $-\omega^3 = 0$ is indeed contact, so that the corresponding CR structure (no matter the choice of $J$) will be strictly pseudoconvex. A quadratic form on the contact plane is given by a positive definite quadratic expression in $\omega^1, \omega^2$ , that is: $A(\omega^1)^2 + 2B\omega^1\omega^2 + C(\omega^2)^2$, viewed mod $\omega^3$. The isotropy group of $\omega^3$ acts by rotations of the contact plane (the $e_1, e_2$ plane). A quadratic form can be diagonalized by rotations, so upon conjugation by some element of the isotropy subgroup of $\omega^3$ we can put the quadratic form in the diagonal form

$A(\omega^1)^2 + B(\omega^2)^2$ with $A, B > 0$. The form is only well-defined up to scale, and we can scale it so that $A = 1/B$, i.e the conformal structure is that of $(1/a)(\omega^1)^2 + a(\omega^2)^2$, $a > 0$. We have proved the bulk of :

**Proposition 2.1.1 (Normal form)** *Every left-invariant CR structure on $S^3$ is conjugate to one whose contact form is given by*

$$\theta = -\omega^3 \tag{2.1.2}$$

*and whose associated conformal structure is*

$$L_\theta = \frac{1}{a}(\omega^1)^2 + a(\omega^2)^2 \tag{2.1.3}$$

*The associated almost complex structure $J = J_a$ is defined by $J(e_1) = \frac{1}{a}e_2$, $J(e_2) = -ae_2$. The structure defined by $a$ is isomorphic to the structure defined by $1/a$. As the notation indicates, the quadratic form $L_\theta$ is indeed the Levi-form associated to $\theta, J$ as per eq. (2.1.1).*

To see that $J$ in the proposition is correct, note that the choice $\theta = -\omega^3$ as contact form induces the orientation $\{e_1, e_2\}$ to the contact planes, and that $(e_1, \frac{1}{a}e_2)$ are orthogonal vectors having the same squared length $(1/a)$relative to the given metric $L_\theta$. To see that the structure defined by $a$ is isomorphic to the structure defined by $\frac{1}{a}$ observe that rotation by 90 degrees converts $(1/a)(\omega^1)^2 + a(\omega^2)^2$ to $a(\omega^1)^2 + (1/a)(\omega^2)^2$. Finally, compute from $d\theta = \omega^1 \wedge \omega^2$ and the form of $J$ that indeed, the Levi form is the given quadratic form $L_\theta$.

## 2.2 Fefferman's metric (step 1).

When the strictly convex CR structure on $M$ is induced by an embedding $M \subset \mathbb{C}^2$, Fefferman [Fef76] constructed a circle bundle $Z \to M$ together with a conformal Lorentzian metric on $Z$ invariantly associated to the CR structure. Farris [Far86] and then Lee [Lee86] generalized Fefferman's construction to the case of an abstract strictly pseudoconvex CR structure, i.e. one not necessarily induced by an embedding into $\mathbb{C}^2$. In this section we construct the Fefferman metric for the family of left-invariant CR structures from step 1 (proposition 2.1.1 there). We most closely follow Lee's presentation.

We begin with a general construction. Let $\pi : Z \to M$ be any circle bundle over $M$. Fix a contact form $\theta$. Recall that the Reeb vector field associated to $\theta$ is the vector field on $M$ uniquely defined by the two conditions

$$\theta(R) = 1$$

$$i_R d\theta = 0.$$

Changing $\theta$ to $g\theta$, $g$ a function, changes $R$ to $\frac{1}{g}R + X_g$ where $X_g$ lies in the contact plane field and is determined pointwise by a linear equation involving $dg$ and $d\theta$ which is reminescent of the equation relating a Hamiltonian to its Hamiltonian vector field. We extend the Levi form (2.1.1) to all of $TM$ by insisting that $L_\theta(R, v) = 0$ for all $v \in TM$ and continue to write $L_\theta$ for this extended form. Let $\sigma$ be any one-form on $Z$ with the property that $\sigma$ is nonzero on the vertical vectors (the kernel of $d\pi$), Then

$$g_\theta = \pi^* L_\theta + 4(\pi^*\theta) \odot \sigma \tag{2.2.1}$$

is a Lorentzian metric on $Z$. Here $\odot$ denotes the symmetric product of one-forms: $\theta \odot \sigma = \frac{1}{2}(\theta \otimes \sigma + \sigma \otimes \theta)$.

> *The trick needed is a way of defining $\sigma$ in terms of the contact form, and $J$, in such a way that a "conformal change" $\theta \mapsto g\theta$ of the contact structure induces a conformal change of the metric $g_\theta$.*

**Warning.** Farris and Lee, use a different definition of the symmetric product $\odot$: their $\theta \odot \sigma$ is twice ours, so that in their formula for the metric our 4 is replaced by a 2. We have chosen our definition so that, using it, $(dx + dy)^2 = dx^2 + 2(dx \odot dy) + dy^2$, where $\theta^2 = \theta \otimes \theta$.

### 2.2.1 Forming the circle bundle from the canonical bundle. (2,0) forms.

The circle bundle $Z \to M$ will be a bundle of complex-valued 2-forms, defined up to real scale factor. A choice of contact form $\theta$ on $M$ induces various one-forms on $Z$ in a canonical way. One of these one-forms will be the form $\sigma$ needed for the Fefferman metric, eq (2.2.1). Here are the main steps leading to the construction of $Z$ and its one-form $\sigma$.

The complexified contact plane $\xi_{\mathbb{C}} = \xi \otimes \mathbb{C}$ splits under $J$ into the holomorphic and anti-holomorphic directions, these being the $+i$ and $-i$ eigenspaces of $J$, where $J$ is extended from $\xi$ to $\xi_{\mathbb{C}}$ by complex linearity. In the case of 3-dimensional CR manifold, if we start with any non-zero vector field $E$ tangent to $\xi$, then $Z = E - iJE$ spans the holomorphic direction, while $\bar{Z} = E + iJE$ spans the anti-holomorphic direction. In our case

$$Z = e_1 - \frac{i}{a} e_2 \tag{2.2.2}$$

is holomorphic, while

$$\bar{Z}_a = e_1 + \frac{i}{a} e_2 \tag{2.2.3}$$

is the anti-holomorphic vector field.

**Remark. Third definition of a 3-dimensional CR manifold.** Eq (2.2.2) corresponds to yet a 3rd definition of a CR manifold.

**Definition 2.2.1** *(CR structure, 3rd time 'round). A CR structure on $M^3$ is a complex line field, i.e. a rank 1 subbundle of the complexified tangent bundle $TM \otimes \mathbb{C}$ which is nowhere real.*

Such a complex line field is locally spanned by a "holomorphic" vector field $Z$ as in eq 2.2.2. Writing $Z = E_1 - iE_2$ with $E_1, E_2$ real vector fields, we define the 2-plane field $\xi$ to be the real span of $E_1, E_2$, and we set $J(E_1) = E_2$, $J(E_2) = -E_1$. The "strictly pseudoconvex" condition, which is the condition that $\xi$ be contact, is that $E_1, E_2$ together with the Lie bracket $[E_1, E_2]$ span the real tangent bundle $TM$.

The almost complex structure $J$ on the contact planes of a CR manifold induces a splitting of the space of complex-valued differential forms into types $\Omega^{p,q}$ similar to the splitting of forms on a complex manifolds. We declare that a complex valued k-form $\beta$ is or of type $(k, 0)$ (that is to say "holomorphic") if $i_{\bar{Z}}\beta = 0$ for all anti-holomorphic vector fields $\bar{Z}$. In dimension 3, one only needs to check this equality for a single nonzero such vector field, such as $\bar{Z}$ of eq ( 2.2.3) as above.

**Our case.** The (1,0) forms for the left-invariant structure for the parameter value $a$ of Proposition 2.1.1 is spanned by

$$\theta = -\omega^3\,; \omega_a = \omega^1 + ia\omega^2 \qquad\qquad : (1,0) \text{ forms} \tag{2.2.4}$$

The (2,0) forms are spanned (over $\mathbb{C}$) by

$$\theta \wedge \omega_a \qquad\qquad : (2,0) \text{ forms} \tag{2.2.5}$$

In dimension 3 the space of all $(2,0)$ forms, considered pointwise, forms a complex line bundle, denoted by $K$ and called the canonical bundle as in complex differential geometry. $Z$ is defined to be the "ray projectivization" of $K$:

$$Z = K \setminus \{ \text{ zero section } \}/\mathbb{R}^+.$$

We next recall from Lee [Lee86] how a choice of contact form $\theta$ determines the one-form $\sigma$.

1. Volume normalization equation. Fix the contact form $\theta$ on $M$. The volume normalization equation is

$$\sqrt{-1}\;\theta \wedge i_R\zeta \wedge i_R\bar{\zeta} = \theta \wedge d\theta. \tag{2.2.6}$$

The right hand side is the standard volume form defined by a choice of contact structure. On the left-hand side, $R = R_\theta$ is the Reeb vector field for $\theta$. The 2-form $\zeta \in \Gamma(K)$, a section of the canonical bundle is to viewed as the unknown. The equation is quadratic in the unknown since multiplying $\zeta$ by the complex function $f$ multiplies the left hand side of the volume normalization equation by $|f|^2$. It follows by this scaling that there is a solution, $\zeta_0$ to the volume normalization which is unique up to unit complex multiple $\zeta \mapsto e^{i\gamma}\zeta$.

Said slightly differently, eq. (2.2.6) defines a section of

$$s = s_\theta : Z \to K$$

of the ray bundle $K \to Z$, since once we fix the complex phase of $\zeta$, the equation uniquely determines the real phase factor. Fix a solution, which is to say, a smoothly varying pointwise choice of solutions

$$\zeta_0 : M \to K$$

to eq (2.2.6). Such a solution choice defines a global trivialization of $Z$, since we can express any point $z$ of $Z$ can be uniquely expressed via this section as

$$s_\theta(z) = e^{i\gamma}\zeta_0(\pi(z))$$

where $m = \pi(z) \in M$. Thus the choice $\zeta_0$ induces a global trivialization:

$$Z \cong M \times S^1.$$

(A more pictorial, equivalent description of this trivialization of $Z$ is as follows. Form the ray generated by $\zeta_0(m)$, which is a point in the circle fiber $Z_m$, over $m$. Rotate this ray by the angle $\gamma$ until you hit the ray $z \in Z_m$, thus associating to $z$ a point $(m, \gamma) \in M \times S^1$) We henceforth use this identification $Z = M \times S^1$ and define a global one-form on $Z$ by

$$\zeta(m, \gamma) = e^{i\gamma}\zeta_0(m). \tag{2.2.7}$$

We check now that the two-form $\zeta$ depends only on the choice of contact form $\theta$, and so, up to this choice, is intrinsic to $Z$. The total space $K$ of the canonical bundle , like any total space constructed as a bundles of $k$-forms, has *on it* a canonical form $k$-form $\Xi$. To describe $\Xi$ write a typical point of $K$ as $(m, \beta) \in K$, $m \in M$, $\beta \in \Lambda^{(2,0)} T_x M$. Then we can set $\Xi(x, \beta) = \pi_x^* \beta$ where $\beta \in \Lambda^{(2,0)} T_x M$ and $\pi : K \to M$ denotes the projection. This canonical form, like all such canonical forms, enjoys the *reproducing property* that if $\beta : M \to K$ is any section, then $\beta^* \Xi = \zeta$. We use the section $s = s_\theta : Z \to K$ to pull back $\Xi$:

$$\zeta := s_\theta^* \Xi.$$

The reproducing property shows that, under the global trivialization of $Z$ induced by $\zeta_0$, we have that $\zeta$ is given by formula (2.2.10) above.

**Our case.** Return to the left-invariant situation: Choosing $\theta = -\omega^3$ we get $\theta \wedge d\theta = -\omega^1 \omega^2 \omega^3$. The Reeb field

$$R = -e_3. \tag{2.2.8}$$

Writing $\zeta_0 = g\theta \wedge \omega_a$ we compute that $i_R \zeta_0 = g\omega_a$. Using $\omega_a \wedge \bar{\omega}_a = -2ia\omega^1 \wedge \omega^2$ we compute that the left-hand side of the volume normalization equation (2.2.6) expands out to $-2a|g|^2 \omega^1 \omega^2 \omega^3$. Th volume normalization equation (2.2.6) then implies that $|g|^2 = 1/2a$. Thus

$$\zeta_0 = \frac{1}{\sqrt{2a}} \theta \wedge \omega_a \tag{2.2.9}$$

is a global normalized section of $K$. It induces a global trivialization of $Z$, as just described, so that we can think of $Z$ as $S^3 \times S^1$. with $(m, e^{i\gamma})$ being the ray through the (2,0) form $e^{i\gamma}\zeta_a(m)$. The two-form $\zeta$ on $Z$ is given, under this identification, by this same algebraic relation:

$$\zeta = e^{i\gamma}\frac{1}{\sqrt{2a}}\theta \wedge \omega_a \tag{2.2.10}$$

where we are not using different symbols to differentiate between a form $\beta$ on $M$ and its pull-backs $\pi^*\beta$ to $Z$.

**Proposition 2.2.1 (Lee: [Lee86], p. 417)** *Fix the contact form $\theta$ for the CR manifold $M$. Let $\zeta$ be the induced one-forms on $Z$ as just described. Let $R$ be the Reeb vector field for $\theta$.*

*A. There is a complex valued one-form $\eta$ on $Z$, uniquely determined by the conditions: .*

$$\zeta = \theta \wedge \eta \tag{2.2.11}$$

$$i_v\eta = 0 \textit{ whenever } \pi_* v = R \tag{2.2.12}$$

*B. With $\eta$ as in A, there is a unique real-valued one form $\sigma$ on $Z$ determined by the equations*

$$d\zeta = 3i\sigma \wedge \zeta \tag{2.2.13}$$

$$\sigma \wedge d\eta \wedge \bar{\eta} = Tr(d\sigma)i\sigma \wedge \theta \wedge \eta \wedge \bar{\eta}. \tag{2.2.14}$$

*The meaning of Trace in this last equation is as follows. Any solution $\sigma$ to (2.2.13) has the property that $d\sigma$ is basic, i.e. is the pull-back of a two-form on $M$, which by abuse* *of notation we also denote by $d\sigma$. Any two-form on $M$ can be expressed as $fd\theta + \theta \wedge \beta$. Set $Trace(fd\theta + \theta \wedge \beta) = f$.*

*C. The form $\sigma = \sigma(\theta)$ determined by the equations (2.2.11, 2.2.12, 2.2.13, 2.2.14) is the form $\sigma$ appearing in the Fefferman metric $g_\theta$ of eq (2.2.1). If $\theta \mapsto f\theta$, $f > 0$ then the Reeb extended Levi form $L_\theta$ and $\sigma$ transform in such a way that $g_{f\theta} = fg_\theta$, i.e. the conformal class of the Fefferman metric is indeed invariantly attached to the CR structure.*

Remark. An equivalent definition of the trace used in eq (2.2.14) is as follows. Take a two-form such as $d\sigma$ on $M$, restrict it to the contact plane and then use the Levi form $L_\theta$ to raise its indices and thus define its trace, $Tr(d\sigma)$.

**The forms on $Z$ in the left-invariant case.** In our left-invariant situation the forms $\theta, \zeta$ of the theorem have been described above in equations (2.1.2), (2.2.10). They are $\theta = -\omega^3$, $\zeta = \theta \wedge \eta$ with

$$\eta = \frac{1}{\sqrt{1/2a}}(e^{i\gamma}\omega_a) \tag{2.2.15}$$

and

$$\omega_a = (\omega^1 + ia\omega^2)$$

This $\eta$ is indeed the $\eta$ of part A of the theorem, since if $V$ is any vector field on $Z$ satisfying $\pi_* V = R$ then $i_V \pi^* \eta = i_R \eta = 0$, down in $M = S^3$. (Recall we use $\eta$ for $\pi^*\eta$ as forms on $Z$.)

Now we move to the computations o part B of the Proposition for the one-form

$\sigma$. We compute:

$$\sigma = \frac{d\gamma}{3} + f\theta \quad , \quad f = \frac{1}{8}(a + 1/a) \tag{2.2.16}$$

Here are key steps along the way of the computation.

$$d\eta = id\gamma \wedge \eta + \frac{1}{\sqrt{2a}} e^{i\gamma} d\omega_a \tag{2.2.17}$$

$$= id\gamma \wedge \eta + \frac{1}{\sqrt{2a}} e^{i\gamma} \theta \wedge (-\omega^2 + ia\omega^1) \tag{2.2.18}$$

Then

$$d\zeta = id\gamma \wedge \zeta$$

It then follows from the first equation in part B of the theorem, and the reality of $\sigma$ that

$$\sigma = \frac{d\gamma}{3} + f\theta$$

for some real function $f$. We have $Tr(d\sigma) = f$. Setting $dvol = d\gamma \wedge \theta \wedge \omega^1 \wedge \omega^2$ we compute the right hand side of eq (2.2.14) to be $(f/3)dvol$, while its left hand side is equal to $[(1/3)(1 + a^2)/2a - f]dvol$. Setting the two 4-forms equal and solving for $f$ yields $f = (1 + a^2)/8a$ as claimed.

Returning now to the form of the Fefferman metric, eq. (2.2.1), and using $\theta = -\omega^3$ we see that the metric is given (up to conformality) by

$$ds^2 = \{\frac{1}{a}(\omega^1)^2 + a(\omega^2)^2\} + 4\omega^3 \odot (\frac{1}{8}(a + \frac{1}{a})\omega^3 - \frac{d\gamma}{3}) \tag{2.2.19}$$

Written in terms of the basis $\{e_1, e_2, e_3, \frac{\partial}{\partial\gamma}\}$ this metric is

$$g(a) = \begin{pmatrix} \frac{1}{a} & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & \frac{1}{2}(a+\frac{1}{a}) & -\frac{2}{3} \\ 0 & 0 & -\frac{2}{3} & 0 \end{pmatrix}. \tag{2.2.20}$$

## 2.3 Reduced light ray equations (step 2.)

The geodesics for any metric $ds^2 = \Sigma g_{ij} dx^i dx^j$, Riemannian or Lorentzian, can be characterized as the solutions to Hamilton's equations for the Hamiltonian defined by inverting the metric, and viewing the result as a fiber quadratic function on the cotangent bundle:

$$H(x,p) = \frac{1}{2}\Sigma g^{ij}(x) p_i p_j. \tag{2.3.1}$$

(See for example, [AM78], [Arn99], or [Mon02].) Here $g^{ij}(x)$ is the matrix pointwise inverse to the matrix with entries $g_{ij}(x)$.

If we are only interested in light-like geodesics, then we restrict to solutions for which $H = 0$. It is important that these geodesics are conformally invariant. If $\tilde{ds}^2 = f ds^2$ is a metric conformal to the original, then the corresponding Hamiltonians are related by $\tilde{H} = H/f$ and the two Hamiltionian vector fields, are related *on their common zero level set* $\{H = 0\}$ by $X_{\tilde{H}} = (1/f)X_H$ *provided* $H = 0$. This proportionality of vector fields says that the set of light rays for any two conformally related metrics $ds^2, \tilde{ds}^2$ are the same as sets o unparameterized curves.

The Hamiltonian for the Fefferman metric lives on $T^*Z$. Any covector $p \in T^*_z Z$

can be expanded in the basis $\omega_1, \omega_2, \omega_3, d\gamma$ dual to the basis in which the matrix (2.2.20) was computed:

$$p = M_1\omega_1 + M_2\omega_2 + M_3\omega_3 + Pd\gamma$$

The inverse matrix to (2.2.20) is

$$g(a) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \frac{1}{a} & 0 & 0 \\ 0 & 0 & 0 & -3/2 \\ 0 & 0 & -3/2 & -\frac{9}{8}(a+\frac{1}{a}) \end{pmatrix}, \tag{2.3.2}$$

It follows that the Fefferman Hamiltonian for our left-invariant CR structure with parameter $a$ is given by

$$H_a(g, \gamma; M_1, M_2, M_3, P) = \frac{1}{2}\{aM_1^2 + \frac{1}{a}M_2^2 - 3M_3P - \frac{9}{8}(a + \frac{1}{a})P^2\}. \tag{2.3.3}$$

## 2.4 Left-invariant geodesic flows.

Our Hamiltonian (2.3.3, 2.2.20) generates the geodesic flow for a left-invariant (Lorentzian) metric on the Lie group $G = SU(2) \times S^1$. In this section we review some general facts regarding left-invariant geodesic flows, and specify to our situation. We refer the reader to [AM78], especially chapter 4, or [Arn99], especially Appendix 2, for background and more details regarding the material of this section and the next.

### 2.4.1 Generalities

Let $Q$ be a manifold. Let $ds^2$ be a metric on $Q$ as above. The geodesic flow for $ds^2$ is encoded by a Hamitlonian vector field $X$ on $T^*Q$ which is defined in terms of the Hamiltonian above in eq (2.3.1). The vector field $X$ can be defined by the canonical Poisson brackets $\{,\}$ on $T^*Q$ according to $X[f] = \{f, H\}$, for $f$ any smooth function on $T^*Q$. It is worth noting that the momentum scaling property $H(q, \lambda p) = \lambda^2 H(q,p)$, for $p \in T^*_q Q$ corresponds to the fact that the geodesic $\tilde{\gamma}(t)$ with initial conditions $(q, \lambda p)$ is simply the same geodesic $\gamma(t)$ as represented by the initial conditions $(q,p)$ but just parameterized at a different speed: $\tilde{\gamma}(t) = \gamma(\lambda t)$

Now suppose that $Q = G$ is a finite dimensional Lie group and the metric is left-invariant, i.e. left translation by any element of $G$ acts by isometries relative $ds^2$. The left action of $G$ on itself canonically lifts to $T^*G$, and left-invariance of the metric implies that the Hamiltonian $H$ is left-invariant under this lifted action. Write $\mathfrak{g}$ for the Lie algebra of $G$, and $\mathfrak{g}^*$ for the dual vector space to $\mathfrak{g}$, which we identify with $T^*_e G$, where $e \in G$ is the identity. Using the codifferential of left-translation, we left-trivialize $T^*G = G \times \mathfrak{g}^*$, and use corresponding notation $(g, M) \in G \times \mathfrak{g}^*$ for points in the trivialized cotangent bundle. Then the left-invariance of $H$ means that, relative to this trivialization we have

$$H(g, M) = H(M)$$

depending on $M$ alone.

Let $e_a$ be a basis for $\mathfrak{g}$, the Lie algebra of $G$, and $\omega^a$ the corresponding dual basis for $\mathfrak{g}^*$. Then we can expand

$$M = \Sigma M_a \omega^a$$

and

$$H = \frac{1}{2} \Sigma g^{ab} M_a M_b$$

where $g^{ab}$ is the matrix inverse to the inner product matrix $g_{ab} = ds^2(e_a, e_b)$. We find that

$$\{M_a, M_b\} = -\Sigma c^d_{ab} M_d$$

where $c^d_{ab}$ are the structure constants of $\mathfrak{g}$ relative to the basis $e_a$.

It follows that the geodesic flow can be pushed down to the quotient space $(T^*G)/G = \mathfrak{g}^*$, and as such it is represented in coordinates by

$$\dot{M}_a = -\Sigma_{k,b,r} g^{rb} c^k_{ab} M_r M_k$$

We will call these the "reduced equations", or "Lie-Poisson equations". They are a system of ODE's on $\mathfrak{g}^*$. We will call the quotient map $T^*G \to (T^*G)/G = \mathfrak{g}^*$ the reduction map. (Warning: This map is not the reduction map of symplectic reduction.)

#### 2.4.1.1 Momentum Map

The left-action of $G$ on itself, lifted to $T^*G$ has for its MOMENTUM MAP the map $J : T^*G \to \mathfrak{g}^*$ of *right trivialization.* In terms of our left-trivialized identification $J(g, M) = Ad^*_{g^{-1}} M$ where $Ad^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$ denotes the dual of the adjoint representation $Ad_g$ of $G$ on $\mathfrak{g}$. The left-invariance of $H$ implies that each integral curve for the

Hamiltonian vector field $X$, - i.e. the geodesics when viewed as curves in the cotangent bundle, lies within a constant level set of $J$.

Each individual constant level-set $J^{-1}(\mu)$ is the image of a right-invariant one-form $G \to T^*G$, and as such is a copy of $G$ in $T^*G$. The projection of such a level set onto $\mathfrak{g}^*$ by the reduction map yields as image the co-adjoint orbit through $\mu$ , thus: $\pi(J^{-1}(\mu)) = G \cdot \mu$ where $G \cdot \mu = \{M : M = Ad_g^*\mu, g \in G\} \subset \mathfrak{g}^*$. Since the integral curves in $T^*G$ lie on level sets of $J$, the integral curves of the reduced dynamics lie on such co-adjoint orbits.

#### 2.4.1.2 Unreducing

Let $G_\mu$ denote the isotropy group of $\mu \in \mathfrak{g}^*$ under the co-adjoint action. As smooth $G$-spaces we have $\pi(J^{-1}(\mu)) = G \cdot \mu = G/G_\mu$ , and the projection of $J^{-1}(\mu) \to \pi(J^{-1}(\mu)$ is isomorphic to the canonical bundle projection $G \to G/G_\mu$ with fiber $G_\mu$. When $G$ is compact then for generic $\mu$ we have that $G_\mu \cong T$, where $T$ is the maximal torus $T$ of $G$. If the typical integral curves $C$ for the reduced dynamics are closed curves $C \subset G \cdot \mu \subset \mathfrak{g}^*$, then the integral curves for the original dynamics sit on manifolds $\pi^{-1}(C) \cap J^{-1}(\mu)$ which is a $T$-bundle over the circle $T$. In our particular situation this bundle will be trivial, so that it is itself a torus of one more dimension than $T$.

#### 2.4.1.3 Casimirs

A Casimir on $\mathfrak{g}^*$ is a smooth function such that for all smooth functions $h$ on $\mathfrak{g}^*$ we have that $\{C, h\} = 0$. The values of a Casimir stay constant on the solutions to the reduced equation. For $G$ compact with maximal torus $T$ the algebra of Casimirs is functionally generated by $r = dim(T)$ polynomial generators, these generators being polynomials invariant under the co-adjoint action. The common level set $C_1 = c_1, \ldots, C_r = c_r$ of these $r$ Casimirs is, for generic values of the constants $c_i$, a co-adjoint orbit $G \cdot \mu$ for which $G_\mu = T$.

### 2.4.2 The case of Lorentzian metrics on $SU(2) \times S^1$

The Hamiltonian for the Fefferman metric (eq 2.3.3)) computed from step 1 is that of a left-invariant Lorentzian metric on $G = SU(2) \times S^1$. We specialize the discussion of the last few paragraphs to this situation. Then the dual of the Lie algebra of $G$ splits as $\mathfrak{g}^* = \mathbb{R}^3 \times \mathbb{R}$. The $\mathbb{R}^3$ factor acts like the well-known angular momentum from physics. The coordinates as $M_1, M_2, M_3, P$ appearing in eq 2.3.3) are linear coordinates on $\mathfrak{g}^* = \mathbb{R}^3 \times \mathbb{R}$. . Their Lie-Poisson brackets are

$$\{M_1, M_2\} = -M_3, \ \ \{M_3, M_1\} = -M_2, \ \ \{M_2, M_3\} = -M_1$$

together with

$$\{M_i, P\} = 0.$$

The rank of $G$ is 2. The algebra of Casimirs is generated by two Casimirs

$$P \text{ and } K = M_1^2 + M_2^2 + M_3^2 \quad \text{(Casimirs)}$$

Using momentum scaling, we can split the analysis of the reduced geodesic flow into two cases, $P = 0$, and $P = 1$.

#### 2.4.2.1 Case 1: $P = 0$

We will see that for our Hamiltonian the first case is easily solved. The reduced dynamics will be trivial: $M_1 = M_2 = 0$, $M_3 = const$. Up on $G$, the corresponding geodesics are left translates of the one-parameter subgroup corresponding to the 3 direction.

#### 2.4.2.2 Case 2: $P = 1$

When $P = 1$ we have for our Hamiltonian the function $H(M, 1)$ on $\mathbb{R}^3 = \mathbb{R}^3 \times \{1\} \subset \mathfrak{g}^*$. We are only interested in the light-like geodesics, which means we will set $H(M, 1) = 0$. This defines a paraboloid in $\mathbb{R}^3$. The integral curves for the reduced dynamics lie on the intersections of this paraboloid with the spheres $K = r_0^2$. These intersections typically consist of one or two closed curves, which are the closed integral curves of the reduced dynamics.

#### 2.4.2.3 Co-adjoint action and identifications

The co-adjoint action of $G$ on $\mathfrak{g}^* = \mathbb{R}^3 \times \mathbb{R}$ acts trivially on the $\mathbb{R}$ factor, since that corresponds to the Abelian factor $S^1$. The $\mathbb{R}^3$ factor of $\mathfrak{g}^*$ is identified with both $su(2)$ and $su(2)^*$ and the identification is such that the co-adjoint (or adjoint) action corresponds to the standard action of $SO(3)$ on $\mathbb{R}^3$ by way of composition with the 2:1 cover $SU(2) \to SO(3)$. (The $S^1$ factor of $G$ acts trivially on $\mathbb{R}^3$.) Under this identification, the co-isotropy subgroup $SU(2)_L \subset SU(2)$ of a non-zero vector $L \in \mathbb{R}^3$ consists of the one-parameter subgroup generated by $L$, and in $SO(3)$ to rotations about the axis $L$.

#### 2.4.2.4 Unreducing

The momentum map $J : T^*G \to \mathbb{R}^3 \times \mathbb{R}$ splits into

$$J = (L, J_0) = ((L_1, L_2, L_3), J_0) \quad \text{with } J_0 = P.$$

The fact that $J_0 = P$ is the $\mathbb{R}$ component of $J$ is a reflection of the triviality of the co-adjoint action on the $\mathbb{R}$ factor of $\mathfrak{g}^* = \mathbb{R}^3 \times \mathbb{R}$.

The solution curves back up on $T^*G$ corresponding to a given reduced solution curve $C$ lie on submanifolds $J^{-1}(\mu) \cap \pi^{-1}(C)$. The value of $\mu = (L, P)$ is constrained by the co-adjoint orbit on which $C$ lives. This constraint is simply $K = \Sigma L_i^2$. Only the case $K \neq 0$ is interesting. Then the isotropy $G_\mu$ is one of the maximal torii $G_\mu = SU(2)_L \times S^1 = S^1 \times S^1 \subset SU(2) \times S^1$. The first $S^1$ factor is the circle $SU(2)_L$ as in the paragraph 2.4.2.3. It follows from the dicussion of (2.4.1.2) that $J^{-1}(\mu) \cap \pi^{-1}(C)$ is a $G_\mu = S^1 \times S^1$ bundle over $C$. We also saw in (2.4.1.1) that $J^{-1}(\mu) \cong G = S^3 \times S^1$. The projection $\pi$ restricted to $J^{-1}(\mu)$ is the composition $S^3 \times S^1 \to S^3 \to S^2 \subset \mathbb{R}^3 \times \{P = 1\}$ where the last map is the Hopf fibration. The Hopf fibration is trivial over $S^2 \setminus \{P\}$ for any point $P \in S^2$. It follows that $J^{-1}(\mu) \cap \pi^{-1}(C)$ in isomorphic to a three torus, $T^3$. One factor of this three-torus is the $S^1$ factor of $SU(2) \times S^1$, and corresponds to the extra angle $\gamma$ we add when constructing the circle bundle on which Fefferman's metric lives. We project out this angle when forming the chains. Thus the chains lie on two-torii $T^2 \subset SU(2)$. One angle of the two-torus corresponds to a coordinate around a curve $C$ in the reduced dynamics. The other angle is generated by the circle $SU(2)_L \subset SU(2)$.

## 2.5 The reduced Fefferman dynamics.

### 2.5.1 The case $P = 0$

When $P = 0$ we see that $H = \frac{1}{2}(aM_1^2 + \frac{1}{a}M_2^2)$. Since $H = 0$ we have that $M_1 = M_2 = 0$ along light-like solutions with $P = 0$. From the constancy of the Casimir $K$ it follows that $M_3 = const.$ also, so that the reduced solution is a constant curve. Generally speaking, for a left-invariant metric on a Lie group $G$, the geodesics in $G$ which correspond to a constant solution $M(t) = const. = M_*$ of the reduced equations consist of the one-parameter subgroup $exp(t\xi)$ and its left translates $gexp(t\xi)$, where $\mathbb{I}\xi = M_*$ and $\mathbb{I}$ is the "inertial tensor", i.e. the index lowering operator corresponding to the metric at the identity. In our case $\mathbb{I}$ maps the $e_3$ axis to the $M_3$ axis, so that the corresponding geodesic is the 1-parameter subgroup $exp(te_3$ and its translations $gexp(te_3)$. (More accurately, $\mathbb{I}^{-1}(0, 0, M_3, 0)$ is a linear combination of $e_3$ and the basis vector $\frac{\partial}{\partial\gamma}$. We project out the angle $\gamma$ to form the chain corresponding to a light-like geodesic, so these chains are indeed generated by $e_3$.) These $P = 0$ chains are precisely circles of the Hopf fibration $S^3 = SU(2) \to S^2 = SU(2)/S^1$, where the $S^1$ is generated by $e_3$ and acts by right multiplication.

### 2.5.2 The case $P = 1$.

Set $P = 1$ in $H$ to get

$$H_a(M_1, M_2, M_3; 1) = \frac{1}{2}(aM_1^2 + \frac{1}{a}M_2^2 - 3M_3 - c(a))$$

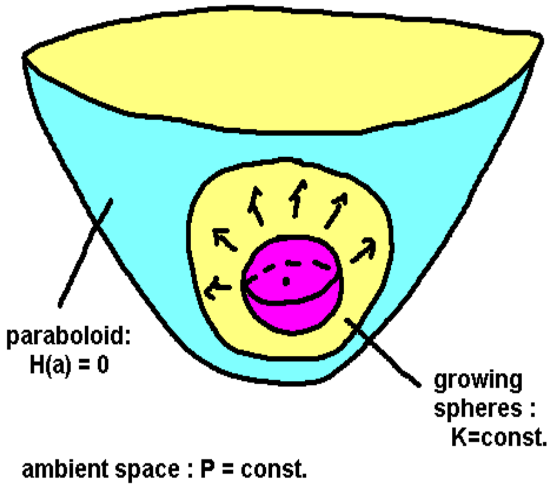


Figure 2.1: A schematic diagram of a $P$-slice. Reduced solutions lie in the intersection of the paraboloid $H = 0$ and the growing level spheres $K$ =const.

where we have set

$$c(a) = -\frac{9}{8}(a + \frac{1}{a}).$$

Recall that we are only interested in the solutions for which $H = 0$. The surface $H = 0$ is a paraboloid which we can express as the graph of a function of $M_1, M_2$:

$$\{H = 0\} = \{(M_1, M_2, M_3) : M_3 = \frac{1}{3}(aM_1^2 + \frac{1}{a}M_2^2 - c(a))\} \tag{2.5.1}$$

The solution curves must also lie on level sets of $K = M_1^2 + M_2^2 + M_3^2$. In other words, the solution curves are formed by the intersection of the paraboloid $H = 0$ with the spheres $K = r_0^2$. See figure 2.1. These intersection curves are easily understood by using $M_1, M_2$ as coordinates on the paraboloids, i.e. by projecting the paraboloid onto the $M_1 - M_2$ plane. They are depicted in figure 2.2.

Eq 2.5.1 yields $M_3$ in terms of $M_1$ and $M_2$ on the paraboloid. Plug this

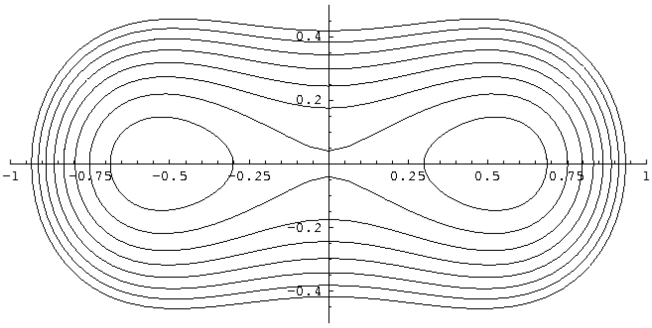


Figure 2.2: Phase portrait for the reduced equations obtained via the projection in figure 1, for $a = 2$ , $P = 1$, and $K$ is small.

expression for $M_3$ into $K$ to find that on the paraboloid

$$K = (1 - \frac{2}{9}c(a)a)M_1^2 + (1 - \frac{2}{9}\frac{c(a)}{a})M_2^2 + \frac{1}{9}(aM_1^2 + \frac{1}{a}M_2^2)^2 + c(a)^2.$$

For $a$ close to 1 the coefficients of the quadratic terms, $M_1^2$ and $M_2^2$ are positive, and close to $1/2$. The only critical point for $K$ is the origin and is a nondegenerate minimum. It follows that all the intersection curves are closed curves, circling the origin. As $a$ increases the sign of the coefficient in front of the $M_1^2$ term eventually crosses 0 and becomes negative. This happens when $1 - \frac{2}{9}c(a)a = 0$ which works out to $a = \sqrt{3}$. After that the origin becomes a saddle point for $K$, and the level set of $K$ passing through the origin has the shape of a figure 8, with the cross at the origin. Inside each lobe of the eight is a new critical point. See figure 2.2 below. This change as $a$ crosses past $\sqrt{3}$ is an instance of what is known as a "Hamiltonian pitchfork bifurcation" or "Hamiltonian figure eight" bifurcation among specialists in Hamiltonian bifurcation theory.

To re-iterate: for $1 < a < \sqrt{3}$ all reduced solution curves are closed and surround the origin. For $a > \sqrt{3}$ the origin becomes a saddle point, and the level set of

$K$ passing through the origin consists of three solution curves: the origin itself which is now an unstable equilibrium, and two homoclinic orbits corresponding to the two lobes of the eight. Being homoclinic to the unstable equilibrium, it takes an infinite time to traverse either one of these homoclinic lobes.

The situation is symmetric as $a$ decreases, with the bifurcation occurring at $a = 1/\sqrt{3}$. This is as it must be, from the discrete symmetry alluded to in Proposition 2.1.1, namely that $a \mapsto 1/a$ while $M_1 \mapsto M_2, M_2 \mapsto M_1$ is a symmetry of the system.

## 2.6 Step 4: Berry phase and unreducing.

As per the discussion in (2.4.2.4), associated to each choice of closed solution curve $C \subset \mathbb{R}^3 \times \{1\}$ and each choice $\mu \neq 0$ of momentum, we have a family of chains which lie on a fixed two torus $T^2 = T^2(C; \mu) \subset T^*S^3$. Our question is : are the chains on this $T^2$ closed? The Fefferman dynamics restricted to $T^2$ is that of linear flow on a torus. Let $\phi$ be a choice of angular variable around $C$, which we call the base angle. Let $\theta$ be the other angle of the torus, which we call the 'vertical angle' chosen so that the projection $T^2 \to C$ is $(\phi, \theta) \mapsto \phi$. We take both angles defined mod $2\pi$. As we traverse the chain, every time that the base angle $\phi$ varies from 0 to $2\pi$, (which is to say we travel once around $C$) the vertical angle $\theta$ will have varied by some amount $\Delta\theta$. The amount $\Delta\theta$ does not depend on the choice of chain within $T^2$. If $\Delta\theta$ is a rational multiple of $2\pi$ then the chains in $T^2$ are all closed. If $\Delta\theta$ is an irrational multiple of $2\pi$, then none of the chains in $T^2$ close up, and we have the case of quasi-periodic chains corresponding to irrational flow on $T^2$.

Without loss of generality we can suppose that $\mu = r_0 e_3$ where $e_3$ denotes the final element of the standard basis of $su(2)^* = \mathbb{R}^3$. For why we can assume this without loss of generality refer to subsection 2.4.2.3 above. In this case $K = r_0^2$ and this fixing of $K$almost fixes the reduced curve $C$. (See the second paragraph in the proof of the proposition immediately below for details.) Remembering the modulus parameter $a$, we see that

$$\Delta\theta = \Delta\theta(K, a).$$

Since the dynamical system defined by the Fefferman metric depends analytically on initial conditions and on the parameter $a$, we see that $\Delta\theta(K, a)$ is an analytic function of $a$ and $K$. *It follows that in order to prove theorem 1, all we need to do is show that for a single value of $a$, the function $K \mapsto \Delta\theta(K, a)$ is non-constant.* We see that in order to prove Theorem 1 it only remains to prove:

**Proposition 2.6.1** *For $a > \sqrt{3}$ the function $K \mapsto \Delta\theta(K, a)$ is non-constant.*

**Proof of Proposition.**

Fix $a > \sqrt{3}$. Consider the value $K = c(a)^2$ corresponding to the homoclinic figure eight through the origin in the $M_1 M_2$ plane. We will show that

$$\lim_{K \to c(a)^2_-} \Delta\theta(K, a) = +\infty. \tag{2.6.1}$$

and that for $K$ slightly less than $c(a)^2$ the value of $\Delta\theta(K, a)$ is finite. It follows that the function $K \mapsto \Delta\theta(K, a)$ varies, as required.

Let $m(a)$ denote the absolute minimum of $K$ on the paraboloid. The minimum is achieved at two points, the elliptic fixed points inside each lobe of the homoclinic eight. For values of $r_0^2$ between $m(a)$ and $c(a)^2$ the level set $K = r_0^2$ consists of two disjoint closed curves $C_1, C_2$, one inside each lobe of the eight. These two curves are related by the reflection $(M_1, M_2) \mapsto (-M_1, M_2)$. The entire dynamics is invariant under this reflection, so that the value of $\Delta\theta$ on $C_1$ equals its value on $C_2$. (The two components are traversed in the same sense.) Consequently $\Delta\theta(K, a)$ is well-defined and finite for $m(a) < K < c(a)^2$, being equal to the common value of $\Delta\theta(C_i)$.

In what follows we arbitrarily fix one of the two components of $K = r_0^2$ and call it $C$.

The key to establishing the limit (2.6.1) is a Berry phase formula for $\Delta\theta$ which mimics earlier work of one of us ([Mon91]). The formula expresses $\Delta\theta$ as the sum of two integrals:

$$\Delta\theta(K, a) = \text{dynamic} + \text{ geometric} \tag{2.6.2}$$

where

$$\text{dynamic } = \frac{1}{\sqrt{K}} \int_0^T f dt$$

and

$$\text{geometric} = -(\text{oriented solid angle}).$$

Both the dynamic and the geometric terms can be expressed as line integrals around $C$. In the dynamic term, $T = T(K)$ is the period of the curve $C$, and where

$$f = \frac{1}{2}[aM_1(t)^2 + \frac{1}{a}M_2(t)^2 + c(a)]. \tag{2.6.3}$$

The integral is done around the projection of the curve $C$ to the $M_1 M_2$. The time $t$ is the time parameter occuring in the reduced equations, which is the same as the geodesic time. In the second formula, the oriented solid angle is the standard oriented solid angle enclosed by a closed curve such as $C$ in space. The absolute value of an oriented solid angle is always bounded by $4\pi$. On the other hand, $\frac{1}{\sqrt{K}}f > \frac{1}{2\sqrt{K}}c(a)$. Consequently, if we let the curve $C$ approach the lobe of the homoclinic orbit which contains it, then its period $T(K)$ tends to $\infty$. We now see that the dynamic term of eq. (2.6.2) tends to $+\infty$. Thus, the corollary is proved once we have established the validity of the Berry phase type formula (2.6.2).

### 2.6.1 Proof of Berry phase formula

We begin the proof of eq. (2.6.2) by recalling by summarizing our situation, and applying the discussion of (2.4.2.4) for relating the reduced dynamics to dynamics in $T^*(SU(2)\times S^1)$ and curves in $T^*SU(2)$. We have fixed $J = (L, P)$ to equal the value $\mu = (r_0 e_3, 1) \in R^3 \times \mathbb{R}$ where $r_0 \neq 0$. The values of the Casimirs which characterize our reduced curve $C$ are then $K = r_0^2$, and $P = 1$. The Fefferman light-like geodesics $C_F$ associated to $C$ and our choice of $\mu$ must lie on the manifold $J^{-1}(\mu) \cap \pi^{-1}(C)$ which is a three three-torus inside $T^*(SU(2) \times S^1)$. Project this three torus into $T^*S^3$ via the the product structure induced projection: $pr_2 : T^*(S^3 \times S^1) = T^*S^3 \times T^*S^1 \to T^*S^3$ and in this way arrive at a two-torus $X(C) = pr_2(J^{-1}(\mu)) \cap \pi^{-1}(C)) \subset T^*SU(2) \times \{1\}$ which projects onto $C$ via the canonical projection $T^*(SU(2)) \times \{1\} \to \mathbb{R}^3 \times \{1\}$. We will soon need that $X(C) \subset L^{-1}(r_0 e_3) \times \{1\}$ which follows from the fact that $J = (L, P)$

so that $pr_2(J^{-1}(\mu)) = L^{-1}(r_0 e_3) \times \{1\}$. The canonical projection just refered to is that of the quotient map $T^*(SU(2)) \to \mathbb{R}^3$ for the (lifted) left action of of $SU(2)$ on itself. The momentum map associated to this map is $L$. We will also use that the canonical projection, $T^*(SU(2)) \to \mathbb{R}^3$ restricted to level sets of $L$ corresponds to symplectic reduction for $T^*SU(2)$. The chains $ch$ associated to the reduced solution $C$ and our choice of momentum axis $e_3$ lie in the two-torus $X(C)$. To coordinatize $X(C)$ choose any global section $\hat{C} : C \to X(C)$ and let $\phi$ be an angular coordinate around $C$ so that $\hat{C}$ is a closed curve in $X(C)$ parameterized by $\phi$ and projecting onto $C$. Now act on $\hat{C}$ by the one-parameter subgroup $exp(\theta e_3) = SU(2)_L$. Then any point of $X(C)$ can be written as $exp(\theta e_3) \cdot \hat{C}(\phi) \in X(C)$ where $\theta, \phi$ are global angular coordinates. (The multiplication "$\cdot$" of "$exp(\theta e_3) \cdot \hat{C}(\phi)$" denotes the action of the group element $exp(\theta e_3) \in SU(2)$ on $T^*SU(2)$ by cotangent lift.)

Every cotangent bundle $T^*Q$ is endowed with a canonical one-form. Let $\Theta$ be the canonical one-form on $T^*SU(2)$. Our Berry phase formula (2.6.2) will be proved by applying Stoke's theorem to the integral of $\Theta$ around a well-chosen closed curve $c$ in $X(C)$.

This curve curve $c \subset X(C) \subset T^*SU(2) \times \{1\}$ is the concatenation of two curves. One curve is any one of the chains $ch$ corresponding $C$ –which is to say – the projection by $pr_2$ of any one of the Fefferman geodesics $C_F \subset J^{-1}(\mu) \cap \pi^{-1}(C)$ covering $C$. We parameterize $ch$ by the Fefferman dynamical time, $0 \leq t \leq T$ making sure to stop when, upon projection,we have gone once round $C$, so that $C(0) = C(T)$. Having gone once round $C$, we must have $ch(T) = exp(\Delta\theta e_3) \cdot c(0)$. The "holonomy" $\Delta\theta$ is

THE ANGLE WE ARE TRYING TO COMPUTE. For the other curve $c_{group}$ we simply move backwards in the group direction to close up the curve: $c_{group}(s) = exp(-se_3) \cdot ch(T)$. Our curve $c$ is then the concatenation $+$ of these two smooth curves:

$$c = c_{group} + ch.$$

The curve $c$ is a closed curve lying in the two-torus $X(C)$. Not all closed curves in the two-torus bound discs, but $X(C) \subset L^{-1}(r_0 e_3) \times \{1\} \cong SU(2)$ which is simply connected, so that $c$ does bound a disc $\tilde{D} \subset L^{-1}(r_0 e_3) \times \{1\}$. Apply Stoke's formula:

$$\int_{\tilde{D}} d\Theta = \int_{c_{group}} \Theta + \int_{ch} \Theta. \tag{2.6.4}$$

The proof of (2.6.2) proceeds by evaluating each term in eq (2.6.4) separately.

Write $S^2$ for the two-sphere $K = r_0^2$, $P = 1$. Write $\pi_{r_0} : L^{-1}(r_0 e_3) \to S^2$ for the restriction of the canonical reduction map $\pi : T^*SU(2) \times \{P = 1\} \to \mathbb{R}^3 \times \{1\}$. Under $\pi_{r_0}$ the disc $\tilde{D}$ projects onto a topological disc $D \subset S^2$ which bounds our reduced curve $C$. $S^2$ is the symplectic reduced space of $T^*SU(2)$ by the left action of $SU(2)$, reduced at the value $L = r_0 e_3$. A basic result from symplectic reduction, essentially its definition, asserts that as a symplectic reduced space $S^2$ is endowed with a 2-form $\omega_{r_0}$ (the reduced symplectic form) defined by $\pi_{r_0}^* \omega_{r_0} = i^*(-d\Theta)$, where $i : L^{-1}(r_0 e_3) \to T^*SU(2)$ is the inclusion. Let $d\Omega$ denote the unique rotationally invariant two-form on the two sphere, normalized so that its integral over the entire sphere is $4\pi$. (The form $d\Omega$ is not closed, but the notation is standard, and suggestively helpful, so we use it.) It is well-known that $\omega_{r_0} = -r_0 d\Omega$, which is to say, that

$$r_0(\pi_{r_0}^*(d\Omega)) = i_{r_0}^*(d\Theta).$$

(See [AM78] for the standard "high-tech" computation, and [Mon91] for an elementary computation of this well-known fact.) Thus

$$\int_{\tilde{D}} (d\Theta) = \int_D r_0 d\Omega = r_0(\text{ solid angle enclosed by } C) \tag{2.6.5}$$

It is worth noting that this area is a signed area, positive or negative depending on the orientation of the bounding curve $C$ of $D$.

It follows from the definition of the momentum map on the cotangent bundle that $\Theta(\frac{d}{ds}(exp(se_3)(p)) = r_0$ for any point $p \in L^{-1}(r_0 e_3)$. It follows that

$$\Theta = r_0 d\theta \text{ along } c_{group},$$

and thus

$$\int_{c_{group}} \Theta = -r_0 \Delta\theta. \tag{2.6.6}$$

where the minus sign arises because in travelling along $c_{group}$ we moved backwards in the $e_3$-direction.

It remains to compute $\int_{ch} \Theta$. For this computation we will have to work on $T^*(SU(2) \times S^1)$. There we have the canonical one form

$$\Theta_F = \Theta + P d\gamma. \tag{2.6.7}$$

Now relative to any coordinates $x^a$ for $SU(2) \times S^1$, where $p_a$ are the corresponding momentum coordinates we have

$$\Theta_F = \Sigma p_a dx^a.$$

Plugging in along one of the light-like Fefferman geodesics and using the metric relation $p_a = \Sigma g_{ab}\dot{x}^a$ where $g_{ab}$ are the metric components we see that

$$\Theta_F(\dot{C}_F(t)) = 2H = 0$$

where the last equality arises becasue the Fefferman geodesic is light-like. Since $pr_2 \circ C_F = ch$ where $pr_2 : T^*SU(2) \times T^*S^1 \to T^*SU(2)$ is the projection, we have, from (2.6.7),

$$\Theta(\frac{d}{dt}ch) = -P\dot{\gamma} = -\dot{\gamma},$$

where we used $P = 1$. It follows that

$$\int_{ch} \Theta = -\int_0^T \dot{\gamma} dt$$

Now $\dot{\gamma} = \frac{\partial H}{\partial P}$. Referring back to the equation for the Hamiltonian, and remembering that we set $P = 1$ after differentiating we see that

$$\dot{\gamma} = -\frac{3}{2}M_3 - c(a).$$

Now using the formula for $M_3$ in terms of $M_1, M_2$ and a bit of algebra we see that

$$-\dot{\gamma} = f,$$

where $f$ is as in the eq (2.6.3). Thus:

$$\int_{ch} \Theta = \int_0^T f dt. \tag{2.6.8}$$

Putting together the pieces (2.6.5), (2.6.6), (2.6.8) into Stokes' formula (2.6.4) and some algebra yields the Berry phase formula (2.6.2).

# Part II

# Singular Curves and Monsters

# Chapter 3

# Introduction

"A colheita é comum, mas o capinar é sozinho."

Grande sertão: veredas - Página 57, de João Guimarães Rosa - Publicado por J. Olympio, 1958 - 571 páginas

# Chapter 4

# Our results: Curve singularities and monster towers

This is a formatted reproduction of our paper which has been submitted to the *Journal of the London Mathematical Society.*

# Part III

# Conclusion

# Chapter 5

# Conclusion

# Appendix A

# Some Ancillary Stuff About Chains

## A.1 The dynamics when $a = 1$.

The chains for the standard structure on $S^3$ are formed by intersecting $S^3 \subset \mathbb{C}^2$ with complex lines in $\mathbb{C}^2$. See [Gol99]. In this appendix we verify that the Fefferman metric description of chains when $a = 1$ yields these circles.

The key to our verification is the observation that when $a = 1$ the Fefferman Hamiltonian (2.3.3) splits into two commuting pieces $H = H_0 - H_1$ with $\{H_0, H_1\} = 0$. This observation and the following method of computation is the same one which led to explicit formulae for subRiemannian geodesic flows in chapter 11 of [Mon02], formulae identical to that of Lemma 1 below. We have $H_0 = \frac{1}{2}K = \frac{1}{2}(M_1^2 + M_2^2 + M_3^2)$ and $H_1 = \frac{1}{2}(M_3 - \frac{3}{2}P)^2$. Since the two Hamiltonians commute, their flows up on the cotangent bundles commute. This observation leads to the explicit formula for the chains through the identity:

$$ch(t) = exp\left[t(M_1e_1 + M_2e_2 + M_3e_3)\right] exp\left[-t(M_3 - \frac{3}{2}P)e_3)\right] \tag{A.1.1}$$

The $M_i, P$ are constants which satisfy the $H = 0$ condition

$$(M_1^2 + M_2^2 + M_3^2) = (M_3 - \frac{3}{2}P)^2.$$

In this formula (A.1.1) for the chains, the first factor corresponds to the flow of $H_0$, whose integral curves correspond to one-parameter subgroups in $SU(2)$, and the second factor corresponds to the projection to $SU(2)$ of solutions to the Hamilton's equation for $-H_1$.

To verify that the chains computed via Fefferman's metric are the circles described above we use two lemmas from linear algebra.

**Lemma 1.** ( circles in SU(2)) *Every geometric circle in $SU(2) = S^3$ through the identity can be parameterized as $\gamma(t) = exp(\alpha t)exp(-\beta t)$ where $\alpha, \beta \in su(2)$ are Lie algebra elements of the same length.*

**Lemma 2.** *When $\beta = ce_3$ as in equation (A.1.1) then these circles sit on complex lines.*

**Remark.** The condition $|\alpha| = |\beta|$ in lemma 1 is a $1 : 1$ resonance condition.

The proofs rely on identifying the quaternions $\mathbb{H}$ with $\mathbb{C}^2$ and hence the group of unit quaternions with $SU(2)$ and $S^3$. Since the contact plane is annihilated by $\omega_3$, and is to correspond with the $T_xS^3 \cap J(T_xS^3)$, we must take the identification $\mathbb{C}^2 \cong \mathbb{H}^2$ such that the complex structure on $\mathbb{C}^2$ corresponds to right multiplication by $k$, where $k$ is to correspond to $e_3$ in $\mathfrak{su}(2)$.

**Proof of lemma 1**. In a Euclidean vector space, (such as $\mathbb{H}$) the circles are described by $c(t) = P + r(\cos(\omega t)e_1 + \sin(\omega t)e_2)$ where $P$ is the center of the circle, $r$ its radius, and where $e_1, e_2$ are an orthonormal basis for the plane through $P$ containing the circle. Now use the fact that for a unit quaternion $n$ we have $exp(nt) = \cos(t)1 + (\sin(t)n$. Thus $\gamma(t)$ of lemma 1 is equal to $(\cos(t) + \sin(t)\alpha)((\cos(t) - \sin(t)\beta)$. Algebra and trigonometry identities yield

$$\gamma(t) = \frac{1}{2}[(1 - \alpha\beta) + \cos(2t)(1 + \alpha\beta) + \sin(2t)(\alpha - \beta)]$$

which we can rewrite as

$$\gamma(t) = P + \cos(2t)v + \sin(2t)w,$$

with $P = \frac{1}{2}[(1-\alpha\beta)$, $v = \frac{1}{2}(1+\alpha\beta)$ and $w = \frac{1}{2}(\alpha - \beta)$. It remains to show that $v$ and $w$ have the same length and are orthogonal. Using $\bar{\alpha} = -\alpha$ and remembing that $\alpha$ is unit length we see that we have $v = -\alpha w$ and so indeed $|v| = |w|$. Their common length is the radius $r$ of the circle. Since the Euclidean inner product is given by $Re(v\bar{w})$ the fact that $v = -\alpha w$ also shows that $v$ and $w$ are orthogonal. Q.E.D.

**Proof of lemma 2.** Let $v, w$ be as in the proof of lemma 1. We must show that the real 2-plane spanned by $v$ and $w$ is a complex line when $\beta = k$. Recall that under our identification of $\mathbb{C}^2$ with $\mathbb{H}$ the complex structure corresponds to multiplication on the right by $k$. Now compute $wk = v$, to see that the span of $v$ and $w$ is indeed a complex line. Q.E.D.

## A.2 Relation to the Rossi example.

Rossi [Ros65] constructed a much-cited example of a family of non-embeddable CR-structures on $S^3$. The purpose of this appendix is to show that Rossi's family is isomorphic to our left-invariant CR family with $a \neq 1$. This isomorphism is well-known to experts. We include it here for completeness. We use the description of CR manifolds to be found in the remark towards the beginning of section 2.2.1. In that construction a CR structure is defined as the span of complex vector field. Let $Z$ be the complex vector field corresponding to the standard CR structure. In terms of our left invariant frame, the *Lewy operator* of the standard structure on $S^3$ can be written as:

$$Z = e_1 - ie_2. \tag{A.2.1}$$

See equation (2.2.2) of section 2.2.1. Then Rossi's perturbed CR structure is defined by

$$Z_\mu = Z - \mu \bar{Z}$$

with $\mu$ a real parameter. On the other hand, we saw (again, cf. equation 2.2.2) that our left-invariant CR structures correspond to the span of

$$Z_a = e_1 - \frac{i}{a} e_2.$$

Set $a = 1 + \epsilon$ and expand out: $Z_a = e_1 - i(1+\epsilon)e_2 = e_1 - ie_2 - i\epsilon e_2 = Z + \frac{1}{2}\epsilon(Z - \bar{Z})$. Upon rescaling $Z_a$ by dividing by $(1 + \frac{1}{2}\epsilon)$ we see that $span(Z_a) = Span(Z - \mu(\epsilon)\bar{Z})$, where $\mu(\epsilon) = \frac{\frac{1}{2}\epsilon}{1+\frac{1}{2}\epsilon}$. This shows that the left-invariant structure for $a$ corresponds to Rossi's structure for $\mu = \mu(\epsilon)$.

The important facts concerning Rossi's structures for $\mu \neq 0$ is that every CR-function for one of these structures on $S^3$ is *even* with respect to the antipodal map $(x, y, z) \mapsto (-x, -y, -z)$. We recommend Burns' [Bur79] for the proof. This forced evenness implies that there is no CR embedding of our left-invariant structures for $a \neq 1$ into $\mathbb{C}^n$ for any $n$. The structures do however, have explicit $2:1$ immersions into $\mathbb{C}^3$ which can be found in Rossi. See also Burns ([Bur79]) or Falbel [Fal92]. Upon taking the quotient by the antipodal map each $a \neq 1$ structure induces a left-invariant CR structure on $\mathbb{R}P^3 = SO(3)$ which does embed into $\mathbb{C}^3$.

This embedded image bounds a domain within an explicit Stein manifold $S \subset \mathbb{C}^3$. We shall return to this point in the next section.

### A.2.1 Representation theory behind the scenes

If we view $S^3$ as embedded in $\mathbb{C}^3$ as the unit sphere there : $\{|z|^2 + |w|^2 = 1\}$, the associated Lewy operator (A.2.1) can be expressed as:

$$Z_0 = w\frac{\partial}{\partial \bar{z}} - z\frac{\partial}{\partial \bar{w}}. \tag{A.2.2}$$

In these coordinates we have:

$$Z_t = Z_0 + t\overline{Z_0}, t \in R.$$

Using purely representation-theoretic arguments we demonstrate the Burns-Rossi structures cannot embed globally, fulfilling a claim in the previous paragraph. We also show that these $CR$ structures, after dividing by an appropriate discrete kernel, embed though in $\mathbb{C}^3$ as a $SO(3)$ invariant structure. This reproduces in part Burns' original argument, and completes Gil's suggestion to re-derive Burns embedding map through purely algebraic means.

We start with a lemma.

**Lemma A.2.1** *Every CR-function for one of these structures on $S^3$ is even with respect to the antipodal map $(z, w) \mapsto (-z, -w)$.*

**Corollary A.2.1** *Since CR-functions **do not separate antipodal points**, we cannot have a global embedding.*

#### A.2.1.1 Proof of Lemma

First we recall that the space of square integrable functions on the three-sphere decomposes, topologically, as

$$L^2(S^3, dg) = \widehat{\bigoplus_{p,q}} H^{p,q}, \tag{A.2.3}$$

where $H^{p,q}$ are *harmonic*[1] homogeneous polynomials of type $(p,q)$ and $dg$ is the Haar measure. By that we mean that any polynomial in $H^{p,q}$ is a linear combination of monomials of the form $z^{k_1}\bar{z}^{k_2}w^{k_3}\bar{w}^{k_4}$, where $k_1+k_2=p$ and $k_3+k_4=q$. (Restricted to $S^3$.) Recall that the space of polynomials $P=\mathbb{C}[z,\bar{z},w,\bar{w}]$ breaks into homogeneous components ('Newton's theorem') and we can write

$$P=\bigoplus P^n,$$

$P^n=\{$ homogeneous polynomials of degree n $\}$. Let us call $C[S^3]$ the restriction[2] of $P$ to $S^3$. It follows from Weierstrass' theorem $C[S]$ is a dense subset of $L^2(S^3,dg)$. Each component $H^n$ breaks further into $H^n=\bigoplus_{p+q=n}H^{p,q}$, with $H^{p,q}$ described as above. The action of $SU(2)$ in these polynomial spaces is the *contravariant* one, $(gh)(m):=h(g^{-1}m)$. In fact, for a given $n$ all the $H^{p,q}$ with $p+q=n$ are irreducible representations of $SU(2)$. We give a proof of this fact this below. It follows from a simple computation on the bi-degrees that

$$\overline{Z_0}(H^{p,q})\subset H^{p-1,q+1}.$$

To verify these maps are non-trivial we must remark that if $z^n\in H^{n,0}$, then $\overline{Z_0}^n(z^n)=n!w^n\neq 0$ in $H^{0,p}$. Therefore $H^{p,q}\overset{\overline{Z_0}}{\to}H^{p-1,q+1}$ is non-trivial as desired. Then, by Shur's lemma, the intertwining operator induced by $\overline{Z_0}$ must be an isomorphism.[3] A similar argument applied to $Z_0$ also shows that $H^{p,q}\overset{Z_0}{\to}H^{p+1,q-1}$ is an intertwining operator.

[1]The kernel of $\triangle_{\mathbb{R}^4}$ is invariant under the action of $SO(4)$ and consequently invariant under $SU(2)$ embedded in $SO(4)$ as the diagonal subgroup of $SU(2)_R\times SU(2)_L/\{\pm1\}$. Consequently, within each subspace of homogeneous polynomial of degree $n$, called here $P^n$, the kernel $\ker\triangle_{P^n}$ is also $SU(2)$ invariant.

[2]It is easy to check that the restriction map commutes with the action of $SU(2)$.

[3]Provided that $p-1\geq 0, q\geq 0$.

Now take a solution $u$ of the perturbed Lewy operator $Z_t : Z_t(u) = 0$. Such functions would serve as coordinates functions for a potential embedding. Using (A.2.3) we consider a Hilbert space decomposition of $u$ in terms of $u^k \in H^k$:

$$u = \sum_{k \geq 0} u^k.$$

Then, $Z_t(u^k) = 0, \forall k$. Assume $u^k = u^{2l+1} = \sum_{p+q=2l+1} u^{p,q}$, and write out $\overline{Z_t}(u^k) = 0$ in $H^{p,q}$ components. Therefore,

$$\begin{aligned}
(k,0):& \qquad 0 = Z_0(u^{k-1,1}) \\
(k,1):& \qquad -t\overline{Z_0}u^{k,0} = Z_0(u^{k-2,2}) \\
(k-2,2):& \qquad -t\overline{Z_0}u^{k-1,1} = Z_0(u^{k-3,3}) \\
&\vdots \\
(1,k-1):& \qquad -t\overline{Z_0}u^{2,k-2} = Z_0(u^{0,k}) \\
(0,k):& \qquad -t\overline{Z_0}u^{1,k-1} = 0.
\end{aligned} \tag{A.2.4}$$

These series of equations together with the injectivity of each one of the $Z_0, \overline{Z_0}$ on the respective spaces, and the assumption that $k = 2l+1$ implies that $u^{2l+1} = 0$, as desired. Q.E.D.

#### A.2.1.2 Proof that the $H^{p,q}$ spaces are irrep's of $SU(2)$

Representations of $SU(2)$ are the same as representations of $\mathfrak{su}(2)$ which in turn, by complexification, are the same as representations of $\mathfrak{sl}(2, \mathbb{C})$. We are going to describe the action of $\mathfrak{su}(2)$ on $H^n$, the space of homogeneous harmonic polynomials of degree $n$, in terms of maximal weight representations [ES02]. Let $h \in \mathfrak{sl}(2,\mathbb{C})$ be the element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The the action of $h$ on polynomials can be described as follows:

$$L_h p(\underbrace{z, \bar{z}, w, \bar{w}}_{=x\in\mathbb{R}^4}) = \frac{d}{dt}|_{t=0} p(\exp(-th)x) =$$

$$= \langle \partial_z p dz + \partial_{\bar{z}} p d\bar{z} + \partial_w p dw + \partial_{\bar{w}} p d\bar{w}, -z\partial_z - \bar{z}\partial_{\bar{z}} + w\partial_w + \bar{w}\partial_{\bar{w}} \rangle,$$

where the "angled brackets" denote the natural pairing of a vector and a covector. In particular, $h(z^{k_1}\bar{z}^{k_2}w^{k_3}\bar{w}^{k_4}) = (k_3 + k_4) - (k_2 + k_1)$. Now using the structure theory of $\mathfrak{sl}(2,\mathbb{C})$ we decompose $H^n$ into weight spaces:

$$H^n = \bigoplus_k V[k],$$

where

$$V[k] = \{p \in V[k] \iff h(p) = kp\}.$$

By direct inspection, one verifies that the maximal weight $k$ of this representation is $k = n$, and that $V[n]$ is generated by $w^p\bar{w}^q$, $p + q = n$. Since any representation of $\mathfrak{sl}(2,\mathbb{C})$ is completely reducible, we must have:

$$H^n \cong \bigoplus d_k V_k,$$

where $V_k$ is isomorphic to the irrep $S^k(\mathbb{C}^2)$.

Let us denote by $e$ (resp. $f$) the raising (resp. lowering) operator [ES02] associated to the $\mathfrak{sl}(2,C)$ action on $V$. Since $V[n]$ is the subspace of maximal weight we must have $e(V[n]) = 0$. By fixing a basis $\{b_1, b_2, \ldots, b_{n+1}\}$ on $V[n]$, each basis vector $b_i$

will generate an irrep of $\mathfrak{sl}(2, C)$ span by $b_i, f(b_i), f^2(b_i), \ldots$. Now recall that $V[n]$ itself has dimension $(n+1)$. On the other hand, each $H^{p,q}$ itself is an invariant subspace.

**Lemma A.2.2** *(Dimension of the kernel of the Laplacian)*

$$\ker(\triangle)|_{P^n} = dim(H^n) = dim(S^n(\mathbb{R}^4)) - dim(S^{n-2}(\mathbb{R}^4)),$$

*where $S^n(\mathbb{R}^4)$ is the space of homogeneous polynomials in four variables of degree $n$.*

The proof is an induction on the degree $n$. See [Arn04] for more details on spherical harmonics.

**Corollary A.2.2**

$$dim(H^n) = C_3^{n+3} - C_3^{n+1} = (n+1)^2,$$

*where in the formula $C_k^{n+k}$ is the number of polynomials of degree $n$ in $(k+1)$ variables.*

Since each subspace $H^{p,q}$ is invariant, and contains an element of maximal weight $n$, namely $w^p \bar{w}^q$, they all must contain a copy of $S^n(\mathbb{C}^2)$ embedded as the irrep generated by $w^p \bar{w}^q$. But according to lemma (A.2.2), $\dim(H^n) = (n+1)^2$ and $H^n = \bigoplus H^{p,q}$ where each $H^{p,q}$ contains a subspace of dimension $(n+1)$. We have thus proved that, by dimensionality, that each $H^{p,q}$ is itself a degree $(n+1)$ irrep of $SU(2)$. Q.E.D.

**Remark.** Permit us to say it would be interesting to analyze from this same point view, i.e. representation theory, spherical harmonics etc., if the left-invariant structures of $SL_2(\mathbb{R})$ or $SE(\mathbb{R}^2)$ are also globally realizable. We will leave however these prospects of exploration to the interested reader.

#### A.2.1.3 Proof of Burns' embedding formula

**Theorem A.2.1** *The space of solutions of $Z_t(u) = 0$ in $H^2$ is three-dimensional, and spanned by the "almost-embeddings":*

$$\begin{aligned} X &= \frac{\sqrt{2}}{2i}\left[z^2 + w^2 + t(\overline{z}^2 + \overline{w}^2\right], \\ Y &= \frac{\sqrt{2}}{2}\left[z^2 - w^2 - t(\overline{z}^2 - \overline{w}^2)\right], \\ Z &= \sqrt{2}\left[zw - t\overline{zw}\right]. \end{aligned} \tag{A.2.5}$$

One can easily verify that the coordinate functions are invariant under composition under the antipodal map, and in fact provide a global embedding of $SU(2)/\{\pm I\} \cong SO(3)$.

*Proof of the theorem.* Let us denote the irrep of $G = SU(2)$ on $Sym^2(\mathbb{C}^2)$ by $V$. Now consider $H = SO(3) \cong SU(2)/\{\pm I\}$. In general, when $H$ is a subgroup of index two in a group $G$, there is a close relationship between their representations [FH91].

Let $U$ and $U'$ denote the trivial and nontrivial representation of $G$ obtained from the two representations of $G/H$[4]. For any representation $V$ of $G$, let $V' = V \bigotimes U'$. In our case, we have that $V \cong V'$ since up to isomorphism $SU(2)$ has only one three-dimensional irrep. Next if we consider $W$ to be the restriction to $SO(3)$ of the standard representation of $SU(2)$ on $V$ it follows from a result in ([FH91], pp. 64) that $W$ splits as $W = W' \bigoplus W''$ where $W'$ is conjugate to $W''$, but non-isomorphic. This splitting of the restriction $W$ is the group-theoretic justification for Burns' embedding formula. Q.E.D.

[4]Since $G/H$ has order two, the only two irreps $G/H$ the trivial one and the 'antipodal' irrep which sends a vector to its antipodal image.

**Remark A.2.1** *Another more "straightforward" explanation for this fact is as follows. Let $v^2, v \odot w, w^2$ be a basis for $V = Sym^2(\mathbb{C}^2)$ viewed as a complex vector space, where $v, w$ are the canonical basis of $\mathbb{C}^2$. There is a natural action of $SO(3)$ on $V$ obtained by restriction of the $SO(3, \mathbb{C})$ action there, and this latter representation is irreducible. Since $SO(3)$ also preserves a nondegenerate symmetric bilinear form on $V$ it must be real [FH91]. This implies that there exists a complex linear $SO(3)$-module homomorphism $\phi$ such that $\phi^2 = Id$ and which splits $V$ into a sum of real eigenspaces $V_- \bigoplus V_+$, also $SO(3)$ invariant subspaces.*

#### A.2.1.4 What is so interesting about these embeddings?

A little bit algebra shows that under Burns' embedding A.2.5, the source space $\mathbb{C}^2$ gets mapped to

$$(1 + t^2)(X^2 + Y^2 + Z^2) = 2t(|X|^2 + |Y|^2 + |Z|^2) \subset \mathbb{C}^3, \tag{A.2.6}$$

which is not a complex manifold but the original three-sphere gets mapped to

$$Q_t \cap S_t^5 \cong SO(3),^5 \tag{A.2.7}$$

where $S_t^5 = \{|X|^2 + |Y|^2 + |Z|^2 = (1 + t^2)\}$ and $Q_t = \left\{X^2 + Y^2 + Z^2 = 2t\right\}$. We are assuming the target complex ambient space $\mathbb{C}^3$ has coordinates $X, Y, Z \in C$. From now on, and to give our problem yet another geometric twist, we can think of the CR three manifolds obtained via Burns' embedding as boundaries of the portion of quadric $Q_t$ sitting inside the 5-sphere $S_t^5$. Permit us to denote the corresponding complex manifold

[5] The resulting quadric is isometrically invariant under $SU(2)$ action.

by $\Sigma_t^2$. *It is worthwhile pointing out that the resulting manifold in the critical case $t \to 0$ has a conical singularity at the origin.*

Based on previous geometric observation, and the purely geometric *synthetic* characterization of the chains for the standard structure on the 3-sphere, Dan Burns asked us the following question:

> Find a "synthetic" construction of the chains for the left-invariant (image) structures, in the spirit of the construction of the chains for the standard structure, but now using complex curves in $\Sigma_t^2$.

Robin Graham upon seeing this challenge suggested to us first trying to equip the open complex surface $\Sigma_t^2$ with a canonical complex hyperbolic metric, such as the Bergman metric on the unit ball in $\mathbb{C}^2$, in the spirit of geometric function theory from which one could try construct Riemann surfaces which would intersect the boundary at "infinity" along chains. Though we did not entirely succeed in providing a synthetic construction for the left-invariant chains, our quest for a possible family of complex hyperbolic metrics on the $\Sigma_t^2$ proved to be very illuminating.

Such metrics, though in an entirely different context, were first predicted[6] by Andrew S. Dancer and Ian A. B. Strachan [DS94]. We dedicate our next session to shortly explain the Dancer-Strachan proof.

## A.3 K$\ddot{a}$hler-Einstein metrics with $SU(2)$ action

Dancer and Strachan (op. cit.) took the challenge of analyzing Kähler-Einstein metrics in real dimension four admitting an isometric action of $SU(2)$ with generi-

[6] An infelicitous surprise, since your present author had ambitions to construct these metrics himself!

cally three-dimensional orbits. On the other hand, if there exists a family of equivariant Kähler-Einstein metrics in $\Sigma_t^2$ and conformally asymptotic to a left-invariant CR strucutre then as suggested by Robin Graham these metrics are expected to have a $SO(3)$ isometry. It is therefore expected that *if* there is a metric solving Graham's problem such metric should lie within the two families constructed by Dancer and Strachan. As it happens, our left-invariant CR structures can be read off from the asymptotic expansion at infinity of their metrics.

### A.3.1 Generalities. Cohomogeneity one Kähler-Einstein metrics

Suppose that we have a metric $g$ in four real dimensions with an isometric action of $SU(2)$. We also suppose that the generic orbit is three-dimensional. The union of the principal orbits will form an open dense set in $M$. On this set we can write the metric as

$$g = dt^2 + g_t, \tag{A.3.1}$$

where $t$ is the arc-length parameter along a geodesic orthogonal to the group orbits, and $g_t$ is homogeneous metric on the orbits. In general the homogeneous part will look like:

$$g_t = a(t)^2\sigma_1^2 + b(t)^2\sigma_2^2 + c(t)\sigma_3^2,$$

where the $\sigma_i$ are invariant one-forms satisfying $d\sigma_1 = \sigma_2 \wedge \sigma_3$ and cyclically . For the purpose of our analyze, it is convenient to introduce a new transversal coordinate denoted by $u$ and such that $dt = (abc)du$. To avoid additional bookkeeping we rewrite the metric as:

$$g = (abc)^2 dt^2 + a(t)^2\sigma_1^2 + b(t)^2\sigma_2^2 + c(t)\sigma_3^2, \tag{A.3.2}$$

where by abuse of notation we are also denoting the new parameter $u$ by $t$.

From now on derivatives with respect to $t$ shall be denoted by a prime, ' .

We have an orthonormal coframe for $g$ given by $e_0 = (abc)dt$, $e_1 = a\sigma_1$, $e_2 = b\sigma_2$ and $e_3 = c\sigma_3$ and one makes a choice of orientation so that

$$\Omega_1^+ = e_0 \wedge e_1 + e_2 \wedge e_3,$$

$$\Omega_2^+ = e_0 \wedge e_2 + e_3 \wedge e_1,$$

$$\Omega_3^+ = e_0 \wedge e_3 + e_1 \wedge e_2,$$

are self-dual two-forms. Assuming that the metric is Kähler with respect to some complex structure, and that there is a complex structure inducing this orientation. Now the Kähler form $\Omega$ on a complex surface is always self-dual and therefore, $\Omega = A(t)\Omega_1^+ + B(t)\Omega_2^+ + C(t)\Omega_3^+$. The Kähler condition imposed to $\Omega$ implies:

$$(Abc)' = Aa^2bc, \tag{A.3.3}$$

$$(aBc)' = Bab^2c,$$

$$(abC)' = Cabc^2.$$

By a sleeky change of dependent variables: $w_1 = bc, w_2 = ac, w_3 = ab$, the original system A.3.3 becomes:

$$w_1' = w_2w_3 + \alpha w_1,$$

$$w_2' = w_3w_1 + \beta w_2,$$

$$w_3' = w_1 w_2 + \alpha w_3,$$

where the functions $\alpha, \ldots$ etc. satisfy $A' = -\alpha A$ etc. Thus for each metric $g$ there is a three-dimensional space of closed, self-dual, $SU(2)$ invariant two-forms, which are candidates for Kähler forms. Then there comes the necessity for checking which of these two-forms are in fact Kähler forms for some choice of complex structure. Given such a form $\Omega$ we can use the metric to define an endomorphism $I$ of the tangent bundle by

$$g(IX, Y) = \Omega(X, Y).$$

Moreover $I$ is an almost complex structure if and only if $A^2 + B^2 + C^2 = 1$. Dancer and Strachan then go on to check for which $A, B, C$'s the almost complex structure is *integrable* and their computations result in the following theorem.

**Theorem A.3.1** *(op. cit., pp. 517) If the metric A.3.2 is Kähler and not hyper-Kähler the one of the following three conditions hold:*

1. $\alpha = 0, \beta = \gamma$,
2. $\beta = 0, \gamma = \alpha$,
3. $\gamma = 0, \alpha = \beta$.

*And conversely, if* (1), (2) *or* (3) *is true then the metric is Kähler.*

We now consider what form the Einstein equations take for such metrics. Let us denote the Einstein constant by $\Lambda$.

**Theorem A.3.2** *(op. cit., pp. 517) In terms of the variables $w_1, w_2, w_3$, and assuming that* (3) *in theorem A.3.1 holds the Einstein equations reduce to:*

$$\alpha = -\Lambda w_2^3.$$

**Corollary A.3.1** *The Kähler-Einstein metrics of the form A.3.2 which are not hyper-Kähler are given, up to permutations, by solutions of the equations:*

$$\begin{aligned} a' &= \frac{1}{2}a(b^2 + c^2 - a^2), \\ b' &= \frac{1}{2}b(c^2 + a^2 - b^2), \\ c' &= \frac{1}{2}c(a^2 + b^2 - c^2 - 2\Lambda a^2 b^2). \end{aligned} \tag{A.3.4}$$

Let us just say that in what concern us here only the $\Lambda < 0$ is of interest [7]. By rescaling $t, a, b, c$ we can set $\Lambda$ equal to $-1$.

### A.3.2 Complete metrics

Let us summarize Dancer and Strachan's main results. Since our expressions for the metric as well as the equations are invariant under changes of sign of $a, b$, or $c$ we shall take $a, b, c$ to be positive.

**Main results:**

- The critical points of the equations A.3.4 *'with $a, b, c \geq 0$'* are the points satisfying $a = b, c = 0$ and cyclically. The linearization about a critical point which is not the origin has one positive, one negative and one zero eigenvalue. *Hence there is at least one unstable curve for such critical point.*

[7]In [DS94] they describe that all such metrics for $\Lambda$ have already been studied.

- Complete metrics must be defined on a semi-infinite interval $(-\infty, \eta)$, where the upper limit is *finite*.

- Solutions to A.3.4 (with $a \geq b$ and $a, b, c \geq 0$) defined on $(-\infty, \eta)$ are unstable curves of critical points $(q, 0, q)$ or $(q, q, 0)$ where $q \geq 0$.

- If the critical point is $(q, 0, q)$ then we have $a > b$. Otherwise $a \equiv b$. In the case $(q, q, 0)$ there is only a discrete family of complete metrics, correponding to the quantization condition $q^2 =$ half-integer. [8]

- The trajectories of solutions to A.3.4 which give complete metrics with $a > b \geq 0$ are precisely the unstable curves with $b \geq 0$ are precisely the unstable curves with $b \geq 0$ of the critical points $(q, 0, q)$ where $q$ is positive.

From now on we shall refer to $\eta$ as the *blow-up time* of the corresponding solution, and the two families of metrics corresponding to the families of critical points $(q, q, 0)$ and $(q, 0, q)$ shall be referred to as families (I) and (II), respectively. We are particularly interested in family (II), i.e., the family of complete metrics corresponding to the unstable manifolds of $(q, 0, q)$ for $q \geq 0$, the main reason being the following. Let us find the asymptotics of the metric as $t \to \eta$. If we let $r = 2\sqrt{ab}$ the metric A.3.2 becomes

$$g = W^{-1}dr^2 + \frac{1}{4}r^2(V\sigma_1^2 + V^{-1}\sigma_2^2 + W\sigma_3^2), \tag{A.3.5}$$

[8] These metrics have in fact an even bigger symmetry group, namely, $U(2)$.

where $W = c^2/ab$ and $V = a/b$. (This is a permissible change of coordinates because one can show $(ab)' > 0$. ) Moreover $r \to \infty$ as $t \to \eta$. Now

$$\frac{dW}{dr} = \frac{dW}{dt}\frac{dt}{dr} = \frac{2}{r}(\frac{a}{b} + \frac{b}{a}) + r - \frac{4}{r}W.$$

Now asymptotically $a/b \to L \geq 1$ and so,

$$\frac{dW}{dr} = \frac{2}{r}(L + L^{-1}) + r - \frac{4}{r}W.$$

Therefore the metric is asymptotically

$$g \sim W^{-1}dr^2 + \frac{1}{4}r^2(L\sigma_1^2 + L^{-1}\sigma_2^2 + W\sigma_3^2), \tag{A.3.6}$$

where $W = (\frac{1}{2}(L + L^{-1}) + \frac{r^2}{6} + \frac{\kappa}{r^4})$ and $\kappa$ is some constant of integration. Notice that the *conformal asymptotic structure* [Biq06] is not that of a conformal metric, but of a *CR-structure* given by the contact form $\sigma_3 = 0$ and carrying a Levi form $\sigma_1^2 + a\sigma_2^2$ , where $a = L^{-2}$. To see it, notice that as $r \to \infty$ the metric 'blows-up' faster in the direction transverse to the plane $\sigma_3 = 0$ since $r^2 W = O(r^4)$.

As $t \to -\infty$ we have,

$$\begin{aligned} a &\sim q, \\ b &\sim k\exp(q^2 t), \\ c &\sim q \end{aligned}$$

for some constant of integration $k$, so the metric is asymptotically

$$\sim q^4 k^2 \exp(2q^2 t)dt^2 + q^2(\sigma_1^2 + \sigma_3^2) + k^2\exp(2q^2 t)\sigma_2^2.$$

Putting $v = k\exp(q^2 t)$ brings the metric asymptotically into the form

$$\sim dv^2 + q^2(\sigma_1^2 + \sigma_3^2) + v^2\sigma_3^2$$

as $v \to 0$, so we obtain a complete metric by adding a 'bolt'(physicists), or a sphere(mathematicians).[9] *The form of the metric on the bolt means that the underlying manifold is topologically the total space of the tangent bundle over $S^2$.* The underlying complex complex structure is that of a *Grauert tube.*

### A.3.3 Tubes and Sz*ö*ke's rigidity theorem

We need to explain what we mean by a (Grauert) tube on $TS^2$. For the round sphere $S^2$ in $\mathbb{R}^3$, the tangent tangent bundle $TS^2$ can be identified [Sző99] with a submanifold of $\mathbb{R}^3 \times \mathbb{R}^3$ as follows

$$TS^2 \cong \{(x, v) : ||x||^2 = 1, \langle x, v\rangle = 0\}.$$

Now let $Q^2$ be affine quadric $X^2+Y^2+Z^2 = 1$ in $\mathbb{C}^3$. When we identify $\mathbb{C}^3$ with $\mathbb{R}^3 \bigoplus \mathbb{R}^3$ under the map $(X, Y, Z) \mapsto v_{\text{real}} + v_{\text{imaginary}}$, where $v_{\text{real}}$ is the three-dimensional vector of the real parts of $X, Y, Z$ etc. $Q_1$ becomes:

$$\{v_1, v_2 \in \mathbb{R}^3 : \langle v_1, v_1\rangle - \langle v_2, v_2\rangle = 1, \langle v_1, v_2\rangle = 0\}.$$

We can now define a map $\delta : TS^2 \to Q^2$ as follows:

$$(x, v) \mapsto \cosh(||v||)x + i\frac{1}{||v||}\sinh(||v||)v. \tag{A.3.7}$$

[9] The asymptotic metric as we approach $v = 0$ is known in general relativity as *Eguchi-Hanson* metric.

Moreover this map has a nice group theoretic property. The diffeomorphism $\delta$ is equivariant with respect the $SO(3)$ action in both $TS^2$ and $Q_1$. To see it, in the target $SO(3)$ is identified with a subgroup of $SO(3,\mathbb{C})$ preserving $Q_1$. In the source, the identification goes as follows. Given a 2-frame in $\mathbb{R}^3$, i.e. a pair of orthonormal vectors, let the first vector define a point in $S^2$ and the second a unit tangent vector to the sphere at that point. In this way, the Stiefel manifold $V_2(R^3) \cong SO(3)$ may be identified with the unit tangent bundle to $S^2$. Therefore the tangent bundle $TS^2$ is a cohomogeneity-one space for the extended $SO(3)$ action. Relative to A.2.7, notice how by rescaling both $x, v$ in the definition of $\delta$ by a factor of $\sqrt{2t}$ we get a map from $TS^2_{\sqrt{2t}}$ to $Q_t$, and if we take $||v||$ small enough we can get $||\delta(x,v)|| \leq (1+t^2)$ since

$$||\delta(x,v)|| = \cosh(2||v||) = \frac{1}{2}(\exp(2||v||) + \exp(-2||v||)). \tag{A.3.8}$$

Our model of $S^2$ as a Riemannian manifold is that of the unit sphere equipped with the induced round metric from $\mathbb{R}^3$. The corresponding Riemannian manifold shall be denoted by $(S^2, \mathrm{can})$ from now on. We let $T^rS^2$ denote the open disk bundle in $TS^2$ consisting of tangent vectors of norm less than $r$ with respect to this metric. (Note that $r$ can also be infinite.)

**Definition A.3.1** *We call $T^rS^2$ a* tube of radius $r$.

It follows from our arguments in the previous paragraph that $\delta$ *provides an equivariant map between a tube of radius $T^{r(t)}S^2$ of radius*[10] *$r(t)$ and the family complex surfaces $\Sigma^2_t$.* The complex structure on $T^{r(t)}S^2$ is obtained by pulling back the canonical complex

[10] The function $r(t)$ is strictly monotonic with respect to $t$.

structure on $Q^2$ by $\delta$. Denote this complex structure by $I$. This is a special type complex structure on $TS^2$, since it interacts with the metric structure. We follow Szöke's ([Sző99]) paper to explain the properties of this complex structure.

Take $v_1, v_2 \in S^2$ with $\langle v_1, v_2 \rangle = 0$. Then the map $c : S^1 \to S^2$ defined by $c(\theta) = \cos(\theta)v_1 + \sin(\theta)v_2$ is a unit speed geodesic of $S^2$. We can continue this map analytically to the entire complex plane and induce a map $\hat{c} : \mathbb{C} \to \mathbb{C}^3$, $\hat{c}(z) = \cos(z)v_1 + \sin(z)v_2$. The image of $\hat{c}$ lives in the quadric $Q_1$. One can now check by hand that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{C} & \stackrel{\hat{c}}{\to} & Q_1 \\ \uparrow i & & \uparrow \delta \\ T\mathbb{R} & \stackrel{dc}{\to} & TS^2 \end{array}$$

The resulting complex structure on $TS^2$ is called an *adapted complex* structure, *adapted with respect to $g$*. If the tube $T^rS^2$ has an adapted complex structure then it is called a *Grauert tube.*

If $T^rS^2$ has an adapted complex structure, then each for each orthonormal 2-frame $v_1, v_2$ the associated map $\hat{c}$ is holomorphic on $\hat{c}^{-1}(T^rS^2)$ and the resulting image is a Riemann surface away from the zero section of $T^rS^2$.

Robert Szöke and Leslie Lempert [LS91] have shown that

$$|| \bullet || : TS^2 \to \mathbb{R}$$

is *strictly plurisubharmonic* [Sha76] with respect to the adapted complex structure above, and therefore it follows that the relatively compact domain $(T^rS^2, I)$ is strictly pseudoconvex. We say that $|| \bullet ||$ is a strictly plurisubharmonic exhaustion of $TS^2$. Now it follows from the work of Cheng and Yau [CY80] that a strictly pseudoconvex relatively compact domain $D$ in a Stein manifold $X$ admits a complete Kähler-Einstein metric. This metric is *unique* if we normalize the Einstein constant to be $-1$. But $(T^r S^2, I)$ is also Stein. To see it, use the map $\delta$. Let us denote this unique Cheng-Yau-Kähler-Einstein metric of $T^r S^2$ by $h_r$. Szöke's main theorem, a so-called *rigidity theorem*, is the following:

**Theorem A.3.3** *([Sző01], pp. 2916) If $r, R$ are distinct positive finite numbers then the Kähler-Einstein manifolds $(T^r S^2, h_r)$ and $(T^R S^2, h_R)$ are* not isometric.

## A.4 Dancer and Strachan predictions

Besides strong existence results proved in ([DS94, DS02]) one can also find some "formulas" predicting:

1. how to determine the equivalence class of left-invariant CR structures at the conformal infinity (our moduli parameter $a$),
2. estimates for the radius of the Grauert tube for different metrics.

By summoning formula A.3.5, one sees that the equivalence class of left-invariant CR structures at infinity is characterized by the parameter $L$ which *a priori* could be computed as

$$L(q) = \lim_{t \to t^*} a(t)/b(t), \tag{A.4.1}$$

where $a(t), b(t)$ are the coordinate functions of the unstable manifold of the critical point $(q, 0, q)$ and $t^*$ the upper limit of the maximal interval of existence of the corresponding solution, or blow-up time if you wish. Analogously, after Dancer and Strachan's conclusion (cf. op. cit., remark on pp. 524) that the topology of the underlying manifolds is that of a Grauert tube of finite radius, one could also *a priori* compute the radius of these tubes by the formula

$$R(q) = \int_{-\infty}^{t^*} a(t)b(t)dt, \tag{A.4.2}$$

using again our "knowledge" of the unstable manifolds. However an explicit determination of the unstable manifolds is in practice impossible. There is evidence the system A.3.4 is not integrable using Painlevé analysis and also due to the lack of a twistor interpretation of the problem in hand. But if one contents himself or herself with numerical evidence, then simple numerical integration can provides us with a good deal of information concerning the behavior of A.3.4.

At each critical point $(q, 0, q)$ the unstable direction is $(0, 1, 0)$ with eigenvalue $q^2$. Since most solutions go off to infinity in finite time, we re-parameterize our system by multiplying the right-hand side of

$$\begin{aligned} \dot{a} &= p_1(a, b, c), \\ \dot{b} &= p_2(a, b, c), \\ \dot{c} &= p_3(a, b, c), \end{aligned} \tag{A.4.3}$$

by $1/s = 1/\sqrt{1+p_1^2+p_2^2+p_3^2}$ and append a new equation to system A.4.3 written as $\frac{d}{dt}\hat{t} = 1/s$. Our new system is the following,

$$
\begin{aligned}
\dot{a} &= p_1(a,b,c)/s, \\
\dot{b} &= p_2(a,b,c)/s, \\
\dot{c} &= p_3(a,b,c)/s, \\
\frac{d}{dt}\hat{t} &= 1/s.
\end{aligned}
\tag{A.4.4}
$$

We borrowed this renormalization idea from Hirota and Ozawa [HO06]. In their work they use the term "arc-length transform" for the above reparameterization of A.3.4. Let us call a solution to A.4.4 a *regularized solution.* An important consequence of this "regularization" procedure is that the solutions of A.4.3 no longer blow up in finite time, and moreover *we can predict the blow-up time of the old system by studying the asymptotic behavior of* $\hat{t}(t)$. Our integration scheme can be briefly summarized as follows:

1. At each critical point $(q,0,q)$ we consider a initial condition of the form $(q,\epsilon,q)$ , $\epsilon \ll 1$ and shoot A.4.4 from there.

2. We let the solutions run for a fixed time interval $T$ and monitor the terminal value of $\hat{t}$. Starting at $a(T), b(T), c(T), \hat{t}(T)$ we run another round of numeric integration for an extra $\triangle T$ and compare $|\hat{t}(T+\triangle T) - \hat{t}(T)|$ to some predetermined precision parameter $\delta$. If $|\hat{t}(T+\triangle T) - \hat{t}(T)| < \delta$ we stop the simulation and call $\hat{t}(T+\triangle T)$ the asymptotic value of $\hat{t}$ or the blow-up time $t^*$ of $(q,0,q)$. The parameters $L$ (eqn. A.4.1) and $R$ ( eqn. A.4.2) can also be computed in this step.

3. If we have not reached the desired precision for the terminal value of $\hat{t}$ we repeat the first part of step (2).

Using Mathematica©, we set our simulation parameters as follows:

- The parameter $q$ identifying the critical point A.3.4 was varied from 0.01 to 10 at intervals of $\triangle q = 0.05$ .
- We initialize the time variable $\hat{t}$ at zero, $\hat{t}(0) = 0$ and as specified above we shoot the remaining variables in the direction of the unstable manifold at $(q, 0, q)$ which coincidentally is $(0, 1, 0), \forall q \neq 0$. Our initial run is $T = 10$ and $\triangle T = 5$. For each numerical integration we set $(a(0), b(0), c(0)) = (q, 0, q) + 10^{-2}(0, 1, 0)$.
- For every value of $q$, once we have determined the terminal time $t_q^*$ we perform a numerical quadrature to determine the parameters $L$ and $R$ according to the formulas (A.4.2) and (A.4.1), respectively.
- Each value $t_q^*, R_q$ and $L_q$ is stored and at the end we plot our results as a function of $q$.

For values of $q$ near zero, since $(0, 0, 0)$ is a *degenerate* critical point of A.3.4 we expect that the blow up time of the unstable manifolds should drastically increase since the singularity at the origin "slows" them down and therefore their permanence time near the origin is bigger. In figure A.1 display a semi-log plot of the blow-up, or terminal time, as a function of $q$. Figures A.3 and A.2 show how the predicted CR parameter $L$ and the tube radius $R$ varies with $q$. We conclude this section by mentioning here that

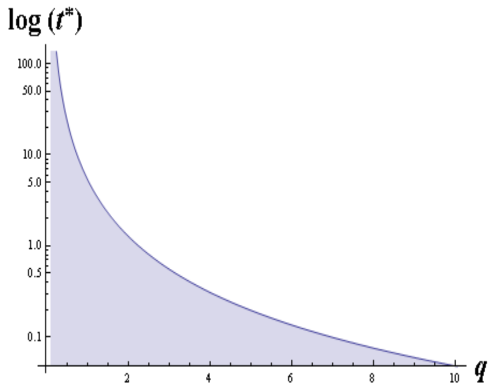


Figure A.1: Semi-log plot of blow time $t^*$ versus critical point parameter $q$.

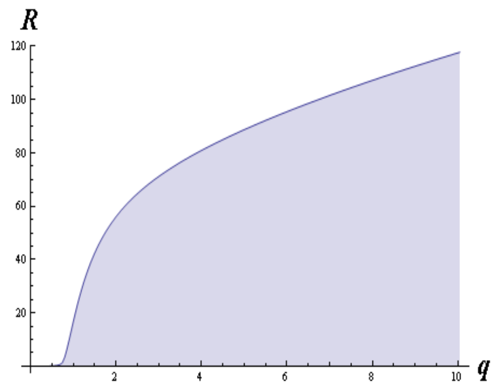


Figure A.2: Plot of Grauert radius $R$ versus critical point parameter $q$.

that relation between the parameter $L$ in A.3.6 and the moduli parameter $a$ is

$$a = L^{-2}.$$

But looking the graph of $L$ (A.3) we notice that $L$ range's is $[1, \infty)$, and therefore $a$ should theoretically be in the interval $(0, 1]$. We remarked earlier that the map $a \mapsto 1/a$ is a symmetry of left-invariant CR structures, and therefore according to our last remark by letting $L$ vary we obtain every single CR structure in our family.

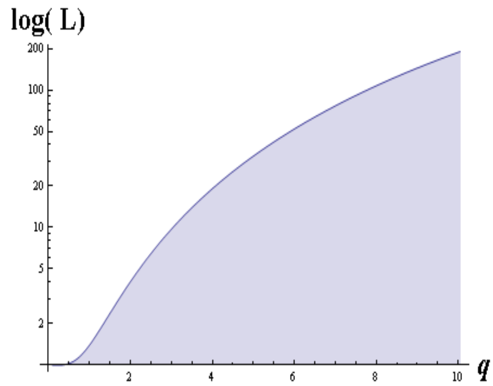


Figure A.3: Semi-log plot of asymptotic CR parameter $L$ versus parameter $q$.

If we invoke Szöke's rigidity theorem for tubes, and the Lewy-Fefferman[11] theorem then according to our numerical experiments left-invariant CR structures corresponding to different values of the moduli should be inequivalent. An analytic proof of all the statements above remain elusive.

## A.5 Equivariant Kähler geometry. Stenzel and Kan's work.

Family (II) should correspond to some complete invariant metrics in Grauert tubes of finite or infinite radius. These metrics have been in fact more-or-less explicitly described in the works of M. Stenzel and Su-Jen Kan [Ste93, Kan07]. Stenzel, and later Kan, use the $SO(3)$ symmetry present to reduce the highly-nonlinear complex Monge-Ampére equation in $T^r S^2$ to a simple second-order ordinary differential equation for a potential of the form $\psi = f(u)$ , and $u$ is related to some invariant plurisubharmonic exhaustion of $T^r S^2$ in the sense of several complex variables. *In what follows we provide*

[11]Two strictly pseudoconvex domains on a Stein manifold are biholomorphically equivalent if and only if their boundaries are CR equivalent.

*an elementary derivation of the (non-homogenous) equivariant Monge-Ampére ODE's for $T^rS^2$.*

The precise statement of our problem can be stated as follows.

> Find on a given strictly pseudoconvex domain $D \subset \Sigma^2$ of a Stein surface a complete Kähler-Einstein metric.
> $$ds^2 = \sum g_{j\bar{k}}(z) dz_j d\overline{z_k}.$$

A Kähler metric $ds^2$ is called Kähler-Einstein if

$$Ric_{j\bar{k}} = -k g_{j\bar{k}}. \tag{A.5.1}$$

There are three cases to consider in equation A.5.1: $k > 0, k = 0, k < 0$. The case $k = 0$ ("Ricci-flat") was extensively investigated by Stenzel in [Ste93]. It follows from S. B. Meyers' theorem that there is no complete Kähler-Einstein metric of positive Ricci curvature on a non-compact manifold. Therefore we can restrict our attention to the $k < 0$ case.

In terms of a local potential $\psi$ for the Kähler metric we have $(g_{i\bar{j}}) = \sqrt{-1}\partial\overline{\partial}(\bullet, J\bullet)$ and

$$(Ric_{j\bar{k}}) = -\sqrt{-1}\partial\overline{\partial}\log\det(\frac{\partial^2\psi}{\partial z_j \overline{\partial z_k}})(\bullet, J\bullet).$$

By rescaling the potential, or equivalently the metric, $\psi \mapsto \lambda\psi$ the Ricci tensor does not change, showing we can suppose that the Einstein constant $k$ is $-1$. Permit us to remark again that the general theory of symmetric Kähler-Einstein metrics on rank-one symmetric spaces have been studied by Stenzel and Kan in (op. cit.), so to save paper we deal only with a single example pertinent to our purposes, namely the "complexification" of the two-sphere.

We have identified already the cotangent bundle of $S^2$ with the affine quadric

$$Q^2 = \{X, Y, Z \in \mathbb{C} : X^2 + Y^2 + Z^2 = 1\} \subset \mathbb{C}^3$$

via the map A.3.7. Using Stenzel's notation let $\tau$ be the restriction to $Q^2$ of the function $X\overline{X} + Y\overline{Y} + Z\overline{Z}$. *We are searching for a Kähler potential of the form* $\psi = f \circ \tau$. Fixing $Z \neq 0$, $X, Y$ become coordinate functions of an open dense chart in $Q^2 \setminus \{Z = 0\}$. For sake of convenience and the tensorial nature of our present calculations we rename our variables as follows: $X \mapsto z_1, Y \mapsto z_2, Z \mapsto z_3$. All derivatives of $\tau$ will be with respect to $z_1, z_2$ where conjugate indices indicate derivatives with respect to $\overline{z_1}, \overline{z_2}$. By implicit differentiation we can take partial derivatives of $z_3$ if necessary too.

A straightforward computation leads to

$$\det(f \circ \tau)_{i\bar{j}} = ((f' \circ \tau)^2 + (f' \circ \tau)(f'' \circ \tau)\tau^{i\bar{j}}\tau_i\tau_{\bar{j}}) \det(\tau_{i\bar{j}}). \tag{A.5.2}$$

Now $\det(\tau_{i\bar{j}}) = |z_3|^{-2}\tau$. Our choice for $\tau$ above now becomes clear. From A.3.8 it follows that given $p \in Q^2$, $\cosh^{-1}(\tau(p)) = \cosh^{-1}(||\delta(x, v)||) = 2||v||$, where $v \in T_xS^2$. But by a result of Szöke and Lempert [LS91] the function $|| \bullet ||$ on $TS^2$ satisfies the homogeneous complex Monge-Ampère equation:

$$(\partial\overline{\partial} \cosh^{-1} \tau) \wedge (\partial\overline{\partial} \cosh^{-1} \tau) = 0,$$

where $\partial$ and $\overline{\partial}$ satisfy $d = \partial + \overline{\partial}$. Using this fact we can also explicitly compute that

$$\tau^{i\bar{j}}\tau_i\tau_{\bar{j}} = \tau^{-1}(\tau^2 - 1).$$

Therefore we can rewrite A.5.2 as

$$\det(f \circ \tau)_{i\bar{j}} = (\tau(f' \circ \tau)^2 + (f' \circ \tau)(f'' \circ \tau)(\tau^2 - 1))|z_3|^{-2}. \tag{A.5.3}$$

Thus the Kähler-Einstein equation for our *ansatz* potential $\psi = f \circ \tau$ becomes

$$\partial\overline{\partial} \log \det(f \circ \tau)_{i\bar{j}} = \partial\overline{\partial} f \circ \tau \tag{A.5.4}$$

$$\Rightarrow \partial\overline{\partial}[\log[\tau(f' \circ \tau)^2 + (f' \circ \tau)(f'' \circ \tau)(\tau^2 - 1)] - f] \equiv 0.$$

(Recall that $\partial, \overline{\partial}$ only involve derivations with respect to $z_1, z_2, \overline{z_1}, \overline{z_2}$. Notice also that $\log|z_3|^{-2} = -\log z_3 - \log \overline{z_3} \Rightarrow \partial\overline{\partial} \log |z_3|^{-2}$ =0, implying this last term do not interfere in the metric components.)[12] Since our differential operators are second order, we make an "intelligent guess" that

$$\log[\tau(f' \circ \tau)^2 + (f' \circ \tau)(f'' \circ \tau)(\tau^2 - 1)] - f = d + c\tau.$$

But since $\partial\overline{\partial}\tau > 0$ we are forced to assume that $c \equiv 0$. (This is a type of local $(1,1)$ Poincaré lemma.) Henceforth the equation for the potential becomes,

$$\tau(f')^2 + f'f''(\tau^2 - 1) = \kappa \exp(f),$$

where $\kappa = \exp(d)$ depending on $r$ and we are viewing now $f$ as a function of the "variable" $\tau$. Changing the independent variable to $w = \cosh^{-1} \tau$ brings us to the following normal form:

$$\frac{d}{dw}(f'(w))^2 = 2\kappa(\sinh w). \tag{A.5.5}$$

[12]In fact if multiply any Kähler potential by an analytic function $g(z_3, \overline{z_3})$ we do not affect the metric components. *This corresponds to a gauge freedom in the choice of the potential.*

In Kan's paper one can find 5 infinite families of ODE's generating Kähler-Einstein potentials in compact rank-one symmetric spaces.[13] Equation A.5.5 can be found in ([Kan07], pp. 646 with $n = 2$).

**Remark A.5.1** *In Stenzel's and Kan's language the energy function*

$$\rho = (\cosh^{-1} \circ \tau)^2 \sim (|| \bullet ||^2 : TS^2 \to [0, \infty))$$

*is a $SO(3)$-invariant, real analytic,* ***strictly plurisubharmonic exhaustion*** *of $TS^2$ [Sző01]. Therefore the level sets $\rho^{-1}(r), r \neq 0$ are invariant strictly pseudoconvex hypersurfaces, or a CR manifold in $TS^2$ and coinciding with the $SO(3)$ orbits there. Each orbit, as expected, is diffeomorphic to the sphere bundle in $TS^2$, and there is an* ***exceptional or singular*** *orbits diffeomorphic to $S^2$ corresponding to the zero level set of $\rho$. (One says the center is an isotropic manifold.) Finally, the values of $|| \bullet ||$ provide so-called* ***slice-coordinates*** *for the orbit structure of the SO(3) action.*

In conclusion a (complete) solution to equation A.5.5 can be interpreted as an equivariant Kähler-Einstein potential and consequently a $SO(3)$ symmetric Kähler-Einstein metric. *These metrics should be isometric to metrics predicted by Dancer and Strachan in ([DS94, DS02]).* By the uniqueness of analytic continuation, if there exists a potential function for a Kähler-Einstein metric in a Grauert tube which depends solely on the Monge-Ampère solution $|| \bullet ||$ then the potential satisfying the ODE A.5.5 is defined in the whole tube. By a theorem of Cheng-Yau, later specialized by Szöke to

[13] These are $S^n, \mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n$, and the Cayley projective plane.

tubes, we are assured the existence of a unique normalized Kähler-Einstein metric on $T^r S^2$ (cf. [Sző01]). These two observations together form the content of Kan's theorem summarized below[14]:

**Theorem A.5.1 ([Kan07], pp. 645)** *Let $(g_{i\bar{j}})$ be the unique normalized complete Cheng-Yau-Kähler-Einstein metric in $T^r S^2$. Then the metric has a unique globally defined real analytic Kähler-Einstein potential solving A.5.5 and satisfying the boundary conditions $\psi(0) = \psi'(0) = 0, \psi''(0) > 0$ and $\psi(r^2) = \infty$.*

**Remark A.5.2** *Back to Dan Burns' initial question, according to another theorem of Stenzel [Ste02] the "continued geodesics"*

$$\hat{c}(z) = \cos(z)v_1 + \sin(z)v_2, \forall z \in \mathbb{C} \subset Q^2(= TS^2),$$

*where $\langle v_1, v_1\rangle = 1, \langle v_1, v_2\rangle = 0$, intersect the CR hypersurfaces $||v|| = r, r > 0$ along chains.* **It is natural to ask if every chain comes from a "Stenzel chain."**

*The sphere bundle $S(T^*S^2)$ is three dimensional, and the geodesic flow on $T^*S^2$ defines a foliation $F$ of $S(T^*S^2)$ by one-dimensional leaves. Therefore the space of geodesics $S(T^*S^2)/F$ for $(S^2, can)$ is two-dimensional and by analytic continuation there can be only a 2-parameter family of "continued" geodesics on any tube $T^r S^2$. On the other hand, according to Jacobowitz' theorem [Jac85] any two sufficiently nearby points on a CR-manifold can be joined by a chain. But it is also known that on a Riemannian manifold any two nearby points can be joined by a unique geodesic. Heuristically speaking, there must be at least as many chains as there is geodesics on $S^3$. It*

[14] We thank Prof. Matthew Stenzel to provide us with this reference.

*not hard to convince ourselves that the "space of geodesics on $S^3$" is a four parameter space.* This seems to suggest that not every chain comes from a Stenzel chain.

**Remark A.5.3** *It remains open the integrability question of the nonlinear implicit ODE A.5.5.*

# Appendix B

# More Ancillary Stuff, This Time About The Monster

# Bibliography


[AM78] Ralph Abraham and Jerrold E. Marsden. *Foundations of mechanics*. Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, Mass., 1978. Second edition, revised and enlarged, With the assistance of Tudor Raţiu and Richard Cushman.

[Arn99] V. I. Arnol′d. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 199? Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein, Corrected reprint of the second (1989) edition.

[Arn04] Vladimir I. Arnold. *Lectures on partial differential equations*. Universitext. Springer-Verlag, Berlin, 2004. Translated from the second Russian edition by Roger Cooke.

[Biq06] Olivier Biquard. Cauchy-Riemann 3-manifolds and Einstein fillings. In *Perspectives in Riemannian geometry*, volume 40 of *CRM Proc. Lecture Notes*, pages 27–46. Amer. Math. Soc., Providence, RI, 2006.

[Bur79] Daniel M. Burns, Jr. Global behavior of some tangential Cauchy-Riemann equations. In *Partial differential equations and geometry (Proc. Conf., Park City, Utah, 1977)*, volume 48 of *Lecture Notes in Pure and Appl. Math.*, pages 51–56. Dekker, New York, 1979.

[Čap06] Andreas Čap. On left invariant CR structures on SU(2). *Arch. Math. (Brno)*, 42(suppl.):185–195, 2006.

[Car33] Elie Cartan. Sur la géométrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes. *Ann. Mat. Pura Appl.*, 11(1):17–90, 1933.

[Chê88] Jih Hsin Chêng. Chain-preserving diffeomorphisms and CR equivalence. *Proc. Amer. Math. Soc.*, 103(1):75–80, 1988.

[CM83] S. S. Chern and J. K. Moser. Erratum: "Real hypersurfaces in complex manifolds" [Acta Math. **133** (1974), 219–271; MR0425155 (54 #13112)]. *Acta Math.*, 150(3-4):297, 1983.

[CY80] Shiu Yuen Cheng and Shing Tung Yau. On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation. *Comm. Pure Appl. Math.*, 33(4):507–544, 1980.

[DFN92] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov. *Modern geometry—methods and applications. Part I*, volume 93 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1992. The geometry of surfaces, transformation groups, and fields, Translated from the Russian by Robert G. Burns.

[DS94] Andrew S. Dancer and Ian A. B. Strachan. Kähler-Einstein metrics with SU(2) action. *Math. Proc. Cambridge Philos. Soc.*, 115(3):513–525, 1994.

[DS02] Andrew S. Dancer and Ian A. B. Strachan. Einstein metrics on tangent bundles of spheres. *Classical Quantum Gravity*, 19(18):4663–4670, 2002.

[ES02] Michael Eastwood and Justin Sawon. The Borel-Weil theorem for complex projective space. In *Invitations to geometry and topology*, volume 7 of *Oxf. Grad. Texts Math.*, pages 126–145. Oxford Univ. Press, Oxford, 2002.

[Fal92] Elisha Falbel. Nonembeddable CR-manifolds and surface singularities. *Invent. Math.*, 108(1):49–65, 1992.

[Far86] Frank A. Farris. An intrinsic construction of Fefferman's CR metric. *Pacific J. Math.*, 123(1):33–45, 1986.

[Fef76] Charles L. Fefferman. Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains. *Ann. of Math. (2)*, 103(2):395–416, 1976.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[Gol99] William M. Goldman. *Complex hyperbolic geometry*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1999. Oxford Science Publications.

[Hit95] N. J. Hitchin. Twistor spaces, Einstein metrics and isomonodromic deformations. *J. Differential Geom.*, 42(1):30–112, 1995.

[HO06] Chiaki Hirota and Kazufumi Ozawa. Numerical method of estimating the blow-up time and rate of the solution of ordinary differential equations—an application to the blow-up problems of partial differential equations. *J. Comput. Appl. Math.*, 193(2):614–637, 2006.

[Jac85] Howard Jacobowitz. Chains in CR geometry. *J. Differential Geom.*, 21(2):163–194, 1985.

[Jac90] Howard Jacobowitz. *An introduction to CR structures*, volume 32 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[Kan07] Su-Jen Kan. Some complete invariant metrics in Grauert tubes. *Math. Res. Lett.*, 14(4):633–648, 2007.

[Koc88] Lisa K. Koch. Chains on CR manifolds and Lorentz geometry. *Trans. Amer. Math. Soc.*, 307(2):827–841, 1988.

[Lee86] John M. Lee. The Fefferman metric and pseudo-Hermitian invariants. *Trans. Amer. Math. Soc.*, 296(1):411–429, 1986.

[Lev09] Mark Levi. *The mathematical mechanic: using physical reasoning to solve problems*. Princeton University Press, Princeton, NJ, 2009.

[Lew57] Hans Lewy. An example of a smooth linear partial differential equation without solution. *Ann. of Math. (2)*, 66:155–158, 1957.

[LS91] László Lempert and Róbert Szőke. Global solutions of the homogeneous complex Monge-Ampère equation and complex structures on the tangent bundle of Riemannian manifolds. *Math. Ann.*, 290(4):689–712, 1991.

[Mon91] Richard Montgomery. How much does the rigid body rotate? A Berry's phase from the 18th century. *Amer. J. Phys.*, 59(5):394–398, 1991.

[Mon02] Richard Montgomery. *A tour of subriemannian geometries, their geodesics and applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[NS03] Pawel Nurowski and George A. Sparling. Three-dimensional Cauchy-Riemann structures and second-order ordinary differential equations. *Classical Quantum Gravity*, 20(23):4995–5016, 2003.

[Pen83] Roger Penrose. Physical space-time and nonrealizable CR-structures. *Bull. Amer. Math. Soc. (N.S.)*, 8(3):427–448, 1983.

[Ros65] H. Rossi. Attaching analytic spaces to an analytic space along a pseudoconcave boundary. In *Proc. Conf. Complex Analysis (Minneapolis, 1964)*, pages 242–256. Springer, Berlin, 1965.

[Sha76] B. V. Shabat. *Vvedenie v kompleksnyi analiz. Chast II. Funktsii neskolkikh* *peremennykh*. Izdat. "Nauka", Moscow, 1976. Second edition, revised and augmented.

[Ste93] Matthew B. Stenzel. Ricci-flat metrics on the complexification of a compact rank one symmetric space. *Manuscripta Math.*, 80(2):151–163, 1993.

[Ste02] Matthew B. Stenzel. Orbits of the geodesic flow and chains on the boundary of a Grauert tube. *Math. Ann.*, 322(2):383–399, 2002.

[Sző99] Róbert Szőke. Adapted complex structures and geometric quantization. *Nagoya Math. J.*, 154:171–183, 1999.

[Sző01] Róbert Szőke. A family of Kähler-Einstein manifolds and metric rigidity of Grauert tubes. *Proc. Amer. Math. Soc.*, 129(10):2913–2917 (electronic), 2001.