Scattering and Metric Lines

Richard Montgomery
UC Santa Cruz (*)

in `Geometry, Mechanics and Dynamics’
organized by

Paula Balseiro  Francesco Fassò  Luis García-Naranjo  Tudor Ratiu  Nicola Sansonetto

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(*) : am retiring, July 1, 2020:
- keep me in mind for post
C-virus longish term invites in 2021 or 2022
Def. A **metric line** in a metric space \((Q,d)\) is an isometric image of the real line:

\[
\gamma : \mathbb{R} \to Q, \ d(\gamma(t), \gamma(s)) = |t - s|, \ (\forall t, s \in \mathbb{R})
\]

**equivalently:** a globally minimizing geodesic.

Def. A **metric ray** : isometric image of the closed half-line \([0, \infty)\):

Def. A **minimizing geodesic** : isometric image of a compact interval \([a,b]\).
1. What are the metric lines for the N-body problem?
   
   *using the Jacobi-Maupertuis metric formulation of dynamics to measure distances*

2. What are the metric lines for homogeneous subRiemannian geometries?

   *commonality: like those of Riemannian geometries, the geodesics of these geometries are generated by Hamiltonian flows*

3. Scattering in the N-body problem: how do asymptotic (Euclidean) rays at $t = -\infty$ get mapped to asymptotic lines at $t = +\infty$?
Why care? ...
1. **What are the metric lines for the N-body problem?**
   - use the Jacobi-Maupertuis metric formulation of its dynamics
   
   *report on work of E Maderna and A Venturelli*
   
   *thanks A Albouy, V Barutello, H Sanchez, Maderna, Venturelli*

2. **What are the metric lines for homogeneous subRiemannian geometries?**
   - commonality: like those of Riemannian geometries, the geodesics of these geometries are generated by Hamiltonian flows
   
   *report on work of A. Anzaldo-Meneses and F. Monroy-Perez; A Doddoli*

3. **Scattering in the N-body problem: how do asymptotic (Euclidean) lines at $t = -\infty$ get mapped to asymptotic lines at $t = +\infty$?**
   
   *with Nathan Duignan, Rick Moeckel and Guowei Yu*  
   
   *thanks A Knauf, J Fejoz, T Seara, A Delshams, M Zworski, R Mazzeo*
Warm-up: Kepler problem = 2-body problem

\[ \ddot{q} = -\frac{q}{|q|^2} \]

\[ E(q, \dot{q}) = \frac{1}{2} |\dot{q}|^2 - \frac{1}{|q|} = h \]

Jac.-Maup. metric:

\[ ds_h^2 = 2 \left( h + \frac{1}{|q|} \right) |dq|^2 \]

on domain \( \Omega_h = \{ q \in \mathbb{R}^2 : h + \frac{1}{|q|} \geq 0 \} = \text{`Hill region'} \)

geodesics = solutions having energy \( h \), up to a reparam. = Kepler conics
Metric properties.
Ωₜ is a complete metric space.
Riem., except at the Hill boundary ∂Ωₜ
and collision q=0. Solutions are metric geodesics
up until they hit the Hill boundary or collision (hit the `Sun’)
beyond which instant they cannot be continued as geodesics.

The conformal factor vanishes at the Hill boundary and is infinite at collision

\[ ds_h^2 = 2 \left( h + \frac{1}{|q|} \right) |dq|^2 \]

**h < 0.** \( \Omega_h = B(2a) \)
\( h = -1/2a. \) No metric lines!

**h > 0, (or h =0).** \( \Omega_h = \mathbb{R}^2 \) still **no metric lines.**
many **metric rays:** all the Kepler hyperbolas (or parabolas)
up to aphelion (closest approach to `sun’)

¿ Why .. ?
cut point/ reflection argument
N-bodies, $i = 1, 2, .. N.$

$$F_{13} = G m_1 m_3 (q_3 - q_1) / r_{13}^3$$

N=3 or greater: **Conjecture:** there are no metric lines for the JM metric (which depends on energy $h$).
What's known? JM metric depends on energy $h$:

$h = 0$: [da Luz-Maderna] **No metric lines.** Many **metric rays**
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Set-up and eqns.

\[ q = (q_1, \ldots, q_N) \in \mathbb{E} := \mathbb{R}^{Nd} \quad q_a \in \mathbb{R}^d, \ a = 1, \ldots, N \]

Conserved energy

\[
E(q, \dot{q}) = \frac{1}{2} \langle \dot{q}, \dot{q} \rangle_m - G \sum \frac{m_a m_b}{r_{ab}}
\]

\[ = h. \]

\[ = K(\dot{q}) - U(q) \]

where

\[ 2K(\dot{q}) = \langle \dot{q}, \dot{q} \rangle_m = \sum m_i \|\dot{q}_i\|^2 = \]

and

\[ U(q) = G \sum \frac{m_a m_b}{r_{ab}} \]
Newton’s eqns: \[ \ddot{q} = \nabla_m U(q) \]

where \[ \langle \nabla_m U(q), w \rangle_m = dU(q)(w) \]

Solutions for fixed \( E = h \) are reparam’s of geodesics for the JM -metric:

\[ ds_h^2 = 2(h + U(q))|dq|_m^2 \quad \text{on} \quad \Omega_h = \{ q : h + U(q) \geq 0 \} \]
\( \Omega_h \) is a complete metric space.

Riemannian except at the Hill boundary \( h + U(q) = 0 \) and at the collision locus \( h + U(q) = + \infty \)

Solutions to Newton at energy \( h \) are metric geodesics up until they hit the Hill boundary or the collision locus beyond which instant they cannot be continued as geodesics.

\[
\begin{align*}
    h \geq 0 \quad & \implies \quad \Omega_h = \mathbb{R}^{Nd}
\end{align*}
\]
Dynamical implications of positive energy.

\[ I(q) = \| q \|^2_m \]

\[ h \geq 0 \implies \Omega_h = \mathbb{R}^{Nd} \]

A solution is **bounded** iff \( I(q(t)) \) is bounded.

\[ \dot{I} = 2 \langle q, \dot{q} \rangle_m \]

\[ \ddot{I} = 2 \langle \dot{q}, \dot{q} \rangle_m + 2 \langle q, \ddot{q} \rangle_m \]

\[ = 4K - 2U(q) \]

\[ = 4h + 2U(q) \]

Since \( U > 0 \):

\[ h \geq 0 \implies \ddot{I} > 0 \text{ along a solution.} \]

\( h \geq 0 \) and defined for \( t \in [0, \infty) \implies \) unbounded

( periodic \( \implies \) bounded \( \implies h < 0 \) )
Def a solution is **hyperbolic** iff

\[ r_{ij}(t) \sim C(ij)t \to +\infty, \]

equivalently:

\[ \frac{q_i(t)}{t} \to a_i \quad \text{or} \quad \dot{q}_i(t) \to a_i \neq 0 \]

\[ a_i \neq a_j, \; i \neq j \]

Note: then \( h = K(a) > 0 \).
Thm: [Chazy, 1920s]: any hyperbolic solution $q(t)$ satisfies

$$q(t) = at + (\nabla_m U(a)) \log t + c + f(t) \quad \text{as } t \to \infty$$

with $f(t) = O(\log(t)/t)$, and $f(t) = g(1/t, \log(t))$, $g$ analytic in its two variables.

and $a \in \mathbb{R}^{Nd} \setminus \{ \text{collisions} \}$

Think of $a$ as an asymptotic position at infinity.

Question: Given $a, q_0$ in $\mathbb{R}^{Nd}$ with $a$ not a collision configuration. Does there exist a hyperbolic solution connecting $q_0$ at time 0 to $a$ at time $\infty$?

Thm [Maderna-Venturelli; 2019]. YES. Moreover this solution is a metric ray for the JM metric with energy $h = K(a) = (1/2) |a|^2$. 
Method of proof: Weak KAM, a la Fathi for
\[ H(q, dS(q)) = h \]

so: calculus of variations + some PDE

Metric input: Buseman, Buseman functions as solutions to the (weak) Hamilton-Jacobi eqns some Gromov ideas re the boundary at infinity
change gears

subRiemannian geometry
2. **SubRiemannian geometry**

\[
X = \sum X^\mu(q) \frac{\partial}{\partial q^\mu} \quad \quad Y = \sum Y^\mu(q) \frac{\partial}{\partial q^\mu}
\]

smooth vector fields on an n-dim. manifold $Q$.

**Def.** A path $q(t)$ in $Q$ is "horizontal" if

\[
\dot{q} = u_1(t)X(q(t)) + u_2(t)Y(q(t))
\]

**sR Geodesic problem:** find the shortest horizontal path $q(t)$ joining $q_0$ to $q_1$.

where

\[
\ell(q(\cdot)) = \int \sqrt{u_1(t)^2 + u_2(t)^2} \, dt
\]

Such a path, if it exists, is a **sR geodesic**.
If $X$, $Y$, $[X,Y]$, $[X, [X, Y]]$, ... eventually span $TQ$ and if $Q$ is connected then any two points are joined by a horiz. curve and the corresponding distance function:

$$d(q_0, q_1) = \inf \{ \ell(q(\cdot)) : q(t) \text{ horizontal } q \text{ joins } q_0 \text{ to } q_1 \}$$

gives $Q$ the same topology as the manifold topology.

and $sR$ geodesics exist, at least locally

**Geodesics:** (most) are generated by

$$H = \frac{1}{2}(P_1^2 + P_2^2) : T^*Q \to \mathbb{R}$$

$$P_1 = P_X = \sum p_\mu X^\mu(q) \quad P_2 = P_Y = \sum p_\mu Y^\mu(q)$$
Example: \[ Q = \mathbb{R}^3 \]

\[ X = \frac{\partial}{\partial x} + A_1(x, y) \frac{\partial}{\partial z} \]

\[ Y = \frac{\partial}{\partial y} + A_2(x, y) \frac{\partial}{\partial z} \]

\[ (A_1(x, y) = 0, \quad A_2(x, y) = x : \text{standard contact distribution.}) \]
Then

\[ H = \frac{1}{2} \left\{ (p_x + A_1(x, y)p_z)^2 + (p_x + A_2(x, y)p_z)^2 \right\} \]

no z's

so \[ \dot{p}_z = 0 \]

View the const. parameter \( P_z \) as electric \textbf{charge}

Then \( H \) is the Hamiltonian of a particle of mass 1 and this charge moving in the xy plane under the influence of the magnetic field \( B(x,y) \) where

\[ B(x, y) = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \]

\( \{ P_1, P_2 \} = -B(x, y)p_z \)
Eqns of motion:

1) For plane curve part:
\[ c(t) = (x(t), y(t)) = \pi(x(t), y(t), z(t)); \pi : \mathbb{Q} = \mathbb{R}^3 \rightarrow \mathbb{R}^2 \]

\[ \kappa(s) = \lambda B(x(s), y(s)) \]

- \( s = \text{arc length} \)
- \( \lambda = p_z \) = "charge"
- \( \kappa(s) = \text{plane curvature of } c(s) \)

2) \( z(t) \) determined from \( c(t) \) by horizontality
(by being tangent to distribution \( D = \text{span}(X,Y) \):

\[ z(s) = z(0) + \int_{c([0,s])} A_1(x, y) \, dx + A_2(x, y) \, dy \]

(call \( (x(s), y(s), z(s)) = \text{"horizontal lift" of } c(s) = (x(s), y(s)) \) ).
Observe:

Straight lines in the plane are solutions (with charge 0), for any $B(x,y)$

Their horizontal lifts are always metric lines since

$\pi : Q = \mathbb{R}^3 \rightarrow \mathbb{R}^2$

satisfies $\ell(\gamma) = \ell_{\mathbb{R}^2}(\pi \circ \gamma)$

for any horizontal curve $\gamma$

Question [LeDonne]: are there any other metric lines besides those whose projections are straight lines?
Theorem. No: The only metric lines for the Heisenberg group are those projecting onto Euclidean lines in the plane.
`Martinet case’: \( B(x,y) = x \).

Geod eqns: \( \kappa = \lambda x \)

Theorem. **[Ardentov-Sachkov]** Yes.

The Euler kinks correspond to the other metric lines.

All but the kink are periodic in the \( x \) direction

These are the full list of projected geodesics. They are the Euler elastica aligned to have y-axis (\( x = 0 \)) as directrix.
Elastica also arise as the projections of the geodesics for:

- Rolling a ball (sphere) on the plane
  - [Jurdjevic-Zimmerman,..]

- Rolling a hyperbolic plane on the Euclidean plane
  - [Jurdjevic-Zimmerman,..]

- Bicycling
  - [Ardentov-Bor-LeDonne-M., Sachkov]

and two Carnot groups:

- Engel: (2,3,4)  - [Ardentov-Sachkov]
- ‘Cartan’: (2,3,5)  - [Moiseev-Sachkov]
For all of these: \( Q = \mathbb{R}^2 \times G \)

\[
X = \frac{\partial}{\partial x} + \xi_1(x, y) \quad Y = \frac{\partial}{\partial y} + \xi_2(x, y)
\]

\( \xi_1, \xi_2 : \mathbb{R}^2 \to \mathfrak{g} \)

so \( \pi : Q \to \mathbb{R}^2 \)

satisfies \( \ell(\gamma) = \ell_{\mathbb{R}^2}(\pi \circ \gamma) \) for any horizontal curve \( \gamma \)

**In all these**, only Euclidean lines and Euler kinks correspond to metric lines.

(For the rolling ball not all kinks that arise as projections correspond to metric lines upstairs)
Why do the kinks give the only additional lines?

a) exclude all the other Elastica
b) verify kinks are metric lines

a) Prop. [Hakavuouri-LeDonne]: any curve which is the horizontal lift of a planar curve periodic in one direction, cannot be a metric line unless the planar curve is a Euclidean line.

`periodic in x direction': means (x(s), y(s)) satisfies
x(s + L) = x(s),  y(s + L) = a + y(s)

All non-kink elastica are periodic in the direction orthogonal to their directrix, ie. in the x direction for the Martinet case.

Pf of Prop.: metric blow-downs.

b): By hand [`optimal synthesis’] sorting out all cut and conjugate points in all case except bicycling, where we have a simple conceptual proof inspired by `bicycling mathematics’:
  by Ardentov, Bor, LeD, M-, Sachkov
Why?
   a) excluding all the other Elastica
   b) verifying the kinks

   a) **Prop. [Hakavuouri-LeDonne]** A plane curve which is periodic in one direction cannot be the projection of a metric line unless that plane curve is a line

   \[ c(s) = (x(s), y(s)) \] is periodic in \( x \) means that there is a constant \( L > 0 \) [the x-period] such that
   \[ x(s + L) = x(s), \quad y(s + L) = a + y(s) \]

   **All elastica except the kink are periodic in the direction orthogonal to their directrix.**

   Pf of prop. : metric blow-downs. The blow-down of a periodic in-one-direction curve is a line NOT parameterized by arc length…

   b): by hand [`optimal synthesis’]
   in all case **except** bicycling where the proof is simple and inspired by `bicycling mathematics’:
   Ardentov, Bor, LeD, M-, Sachkov
Geodesics and metric for the simplest jet spaces

A. Anzaldo-Meneses-Felipe Monroy Perez, 2005;
B. Doddoli, 2019-2020
change gears

Scattering in the N body problem
Def a solution is \textit{hyperbolic} iff

\[ r_{ij}(t) \sim C(ij)t \to +\infty, \]

equivalently:

\[ \frac{q_i(t)}{t} \to a_i \quad \text{or} \quad \dot{q}_i(t) \to a_i \neq 0 \]

\[ a_i \neq a_j, \ i \neq j \]
Think of $a = (a_1, \ldots, a_N)$ as initial “positions” at infinity.

**Question:** Can we join a given $a$ for $t = -\infty$ to a given $b$ for $t = +\infty$ by a collision-free hyperbolic solution?

Necessary conditions:

- $K(a) = K(b)$ [conservation of energy],
- $P(a) = P(b)$ [conservation of linear momentum]

and $a, b$ collision-free.

**Kepler case (N=2):** Yes! as long as $a \neq \pm b$.

**General case:** ??.

Thm. [Duignan, Moeckel, M-, Yu]

Yes, provided $b$ lies in a small open punctured nbhd of $a$. 

Fig. 11. Geodesic of a scattering metric.
‘Spherical ’ change of var’s: 

\[ s \in \mathbb{S} \cong S^{Nd-1} \]

ENERGY: \[ \frac{1}{2} v^2 + \frac{1}{2} \|w\|^2 - \rho U(s) = h. \]

\[ r^2 = I(q) = \|q\|_m^2 \]
\[ \rho = 1/r \]
\[ dt = rd\tau \]
\[ q = rs \]
\[ \dot{q} = vs + w, \ w \perp s \]

Newton’s eqns \[ \iff \]
\[ \rho' = -v\rho \]
\[ s' = w \]
\[ v' = |w|^2 - \rho U(s) \]
\[ w' = \rho \nabla U(s) - vw - |w|^2 s \]

Spatial Infinity : \[ \iff \rho = 0 \]
, an invariant submanifold
Flow at infinity. Set $\rho = 0$. 

\begin{align*}
    s' &= w \\
    w' &= -vw - \|w\|^2 s \\
    v' &= \|w\|^2 \\
\end{align*}

$s \in \mathbb{S} \cong S^{dN-1}$

$v \in \mathbb{R}, v \neq 0$

Energy at infinity: $\frac{1}{2}v^2 + \frac{1}{2}\|w\|^2 = h$. 
Flow at infinity is independent of $U$.

Set $U = 0$ to understand the dynamics at infinity. Flow = reparam. of free motion! :

$s, -s$ become equilibria! ; flow is gradient like between them…
Equilibria!
Only at infinity.
Given by:

\[(\rho, s, v, w) = (0, s, v, 0)\]
\[s \in S \cong S^{dN-1}\]
\[v \in \mathbb{R}, v \neq 0\]

Energy of an equilibrium:

\[h = \frac{1}{2}v^2\]
so
\[v = \pm \sqrt{2h}\]

Equilibria \(= \Sigma_- \cup \Sigma_+\)

- branch: \(v < 0\). Incoming. LINEARLY UNSTABLE mfd of fixed points.
+ branch: \(v > 0\). Outgoing: LINEARLY STABLE mfd of fixed points

eigenvalues: 0 in the \(s\) and \(v\) directions, ie along Equilibria
-\(v\) in the \(w\)-direction.

... in the \(\rho\) direction...?
Push in to "bulk" — the real N-body phase space by turning on $\rho > 0$.

$\Sigma_{\pm}$ are **normally hyperbolic**!

Generalized eigenvector corresponding to $\delta \rho$
Summarizing:

**Linearization at an equilibrium** $e$:

$$e = (\rho, s, v, w) = (0, s, v, 0)$$

Spectrum of linearization at $e$: $0, -v$

Un/stable manifold of an equilibrium $e$

$$W_{\mp}(e), e \in \Sigma_{\pm}$$

is Lagrangian in the bulk, transverse to the equilibrium manifold

its tangent space at $e$ is the nonzero generalized eigenspace for $-v$ at $e$,

$$\text{dim}(W_{\mp}(e)) = \text{dim}(\Sigma_{\pm}) = dN = \text{half dim of phase space}.$$
Normally hyperbolic submanifolds of equilibria.

\[ S_+ \sim S_- \]

\[ W_u(S_-, V_0) \cap W_s(S_+, V_0) \neq \emptyset \]
Our scattering map is the same as their `scattering map’ ! except that their stable/unstable intersections are (1) typically homoclinic and (2) they have a center manifold with a slow dynamics in place of our manifold of equilibria.

A. Delshams, Tere Seara, R de la Llave, M Gidea, ....
Fini !