A Magnetic Playground for SubRiemannian Geodesics

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Enrico’s Int’l sR seminar
via Zoom, April 29, 2020

(*) : am retiring, July 1, 2020:
so - keep me in mind for post
C-virus longish term invites, eg, for 2021..
Two-plane fields in 3-space: \[ \{dz - A_1(x, y)dx - A_2(x, y)dy = 0\} \]

\( \text{distribution', } D \)

\( \text{one-form, } \theta \)

(provided: the two-planes don’t go vertical: \( \frac{\partial}{\partial z} \notin D(x, y, z) \))

and they are invariant under z-translations
Getting there: **PROBLEM**: to join \((x_0, y_0, z_0)\) to \((x_1, y_1, z_1)\) by a horizontal path. \`horizontal\' = tangent to \(D\).

Write horiz. paths as control system:

\[
\begin{align*}
\dot{x} &= u_1 \\
\dot{y} &= u_2 \\
\dot{z} &= u_1(t)A_1(x, y) + u_2(t)A_2(x, y),
\end{align*}
\]

`controls'

or:

\[
\dot{q} = u_1(t)X(q(t)) + u_2(t)Y(q(t))
\]

with:

\[
X = \frac{\partial}{\partial x} + A_1(x, y)\frac{\partial}{\partial z} \\
Y = \frac{\partial}{\partial y} + A_2(x, y)\frac{\partial}{\partial z}
\]
Strategy:
1. Line up x and y coordinates, using a line segment
2. Fiddle around at the final \((x_1, y_1)\) using planar loops \(c\).

**Step 1.**

\[ u_1(t) = x_1 - x_0 = \text{const} \]
\[ u_2(t) = y_1 - y_0, \]

\(0 < t < 1\).

With i.c.: \(x(0) = x_0, y(0) = y_0, z(0) = z_0\).

Yields: \(x(1) = x_1, y(1) = y_1\) but

\[ z(1) = \int_{\ell} A_1 \, dx + A_2 \, dy \neq z_1 \]

**Step 2.** Try moving around in a planar loop \(c\) based at \((x_1, y_1)\).

Then our height \(z\) changes according to:

\[ \dot{z} = A_1(x, y) \dot{x} + A_2(x, y) \dot{y} \]

or…
\[ \Delta z = \int_{c} A_1 \, dx + A_2 \, dy = \int \int_{D} B(x, y) \, dx \, dy = \text{Flux of Magnetic Field} \]

\[ B(x, y) = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \]

= Magnetic Field

So, choose \( c \) so that flux = \( z_1 - z(1) \).

DONE!

( via Lie brackets: \([X, Y] = B(x,y) \, Z; \quad Z = \frac{\partial}{\partial z}\))
Next. Getting there **optimally**: 

Join \((x_0, y_0, z_0)\) to \((x_1, y_1, z_1)\) by the **shortest** horizontal path connecting them.
`Shortest?`: Let the length of a horiz. path = length of its proj. to xy plane.

(we need that D does not go vertical for this def. to work)

\[ \iff \ell_{sR}(\gamma) = \ell_{\mathbb{R}^2}(c); \ c = \pi \circ \gamma, \pi(x, y, z) = (x, y) \]

\[ \iff sR \text{ struc. is: } D = \{dz - A_1 dx - A_2 dy = 0\}, \]

and \[ \langle \cdot, \cdot \rangle = (dx^2 + dy^2)|_D \]

\[ \iff X = \frac{\partial}{\partial x} + A_1 \frac{\partial}{\partial z} \quad \text{form an orthonormal frame for D} \]

\[ , Y = \frac{\partial}{\partial y} + A_2 \frac{\partial}{\partial z} \]

complete this frame:

\[ Z = \frac{\partial}{\partial z} \]
Deriving sR geodesics. Use \( \theta = dz - A_1(x, y)dx - A_2(x, y)dy \)

Riem. structure (\`penalty metric\') tending to
\[
ds^2_\varepsilon = dx^2 + dy^2 + \frac{1}{\varepsilon^2} \theta^2 \quad \rightarrow \varepsilon \rightarrow 0
\]
our sR structure;

\[
dx, dy, \theta \quad \leftrightarrow_{\text{dual}} \quad X, Y, Z
\]
so dually:
\[
X^2 + Y^2 + \varepsilon^2 Z^2 \rightarrow X^2 + Y^2 \quad \text{encodes sR structure.}
\]
viewed as:
- 2nd order diff'1 operators
- co-metric [symm. bilinear form on T*]
- fiber-quadratic f'n (\`Hamiltonian\') on cotangent bundle
Symbol of $X$: $X$, thought of as a fiber-linear Hamiltonian on $T^*$

$$X = \frac{\partial}{\partial x} + A_1(x, y) \frac{\partial}{\partial z} \quad \rightarrow \quad P_X = p_x + A_1(x, y)p_z$$

$$Y = \frac{\partial}{\partial y} + A_2(x, y) \frac{\partial}{\partial z} \quad \rightarrow \quad P_Y = p_y + A_2(x, y)p_z$$

$$Z = \frac{\partial}{\partial z} \quad \rightarrow \quad P_Z = p_z$$

$(x, y, z, p_x, p_y, p_z)$ coord. on $T^*\mathbb{R}^3$

$$p = p_x dx + p_y dy + p_z dz \in T^*_{(x,y,z)}\mathbb{R}^3$$

$$H_\epsilon = \frac{1}{2}(P_X^2 + P_Y^2 + \epsilon^2 P_Z^2) \rightarrow \frac{1}{2}(P_X^2 + P_Y^2)$$

governs (normal) geodesics.
Full disclosure:
up till now, right out of a review of the book
`A Comprehensive Guide to subRiemannian Geometry’
- by Agrachev, Barilari, and Boscain
which I wrote for the Bulletin of the AMS.

out in a year?
Geod eqns = Ham’ns eqns =

\[ \dot{f} = \{f, H\} \]

f runs over fns on \( T^* \); \( f = x, y, z, P_X, P_Y, P_Z = p_z \), good enough

\[ \dot{x} = P_X \]
\[ \dot{y} = P_Y \]
\[ \dot{z} = A_1 P_X + A_2 P_Y + \epsilon P_Z \]
\[ \dot{P}_X = -(B(x, y) P_Z) P_Y \]
\[ \dot{P}_Y = (B(x, y) P_Z) P_X \]
\[ \dot{P}_Z = 0 \]

\( P_Z = \text{const.} = '\text{charge}' = \lambda \), later
Details of computation:

\[
\{f, gh\} = g\{f, h\} + h\{f, g\}
\]

\[\implies \{f, H_\epsilon\} = P_X\{f, P_X\} + P_Y\{f, P_Y\} + \epsilon^2\{f, P_Z\}\]

\[
\{x, p_x\} = \{y, p_y\} = \{z, p_z\} = 1; \quad \{x, y\} = \ldots = 0 = \{p_x, p_y\} = \ldots = 0
\]

\[\implies \text{for } f = f(x, y, z), \{f, P_X\} = X[f] := df(X); \{f, P_Y\} = Y[f]; \ldots
\]

\[\implies \{f(x, y, z), H\} = P_X X[f] + P_Y Y[f] \implies u_1(t) = P_X(t), u_2(t) = P_Y(t)
\]

\[\text{of } \dot{q} = u_1(t)X(q(t)) + u_2(t)Y(q(t))\]

Finally:

\[
\{P_Z, P_X\} = \{P_Z, P_Y\} = 0; \{P_X, P_Y\} = -B(x, y)P_Z
\]
The \( x, y, P_X, P_Y \) eqns decouple from \( z \); \( P_Z = \lambda \), parameter:

\[
\frac{d}{dt} (x, y) = (P_X, P_Y)
\]

\[
\frac{d}{dt} (P_X, P_Y) = \lambda B(x, y) \mathcal{J}(P_X, P_Y)
\]

where \( \mathcal{J}(P_X, P_Y) = (-P_Y, P_X) = 90 \text{ deg. rotation of } (P_X, P_Y) \) regardless of \( \epsilon \)!
These planar ODES are the eqns of charged particle traveling in the plane under the influence of a magnetic field of strength \( B(x,y) \) `pointing out of the plane’

WLOG: \( H = 1/2 \), so \( (P_X)^2 + (P_Y)^2 = 1 \), which says that the plane curve is parameterized by arc length \( s \), i.e. \( t = s \).

Riem case: \( P_X^2 + P_Y^2 + \epsilon^2 P_Z^2 = 1; P_Z = \lambda \)

which in turn are equivalent to the geometric eqns:

\[
\kappa(s) = \lambda B(x(s), y(s))
\]

where: \( \kappa = \) plane curvature of curve \( (x(s), y(s)) \)

Recall \( \kappa \)

\[
\vec{q}(s) = (x(s), y(s)), \quad \frac{d}{ds} \vec{q}(s) = \vec{T}(s); \quad \frac{d}{ds} \vec{T}(s) = \kappa(s) \parallel \vec{T}'(s)
\]
Heisenberg Group Case
\[ B(x,y) = \text{const.} = 1 \]

Projected geodesic eqs: \[ \kappa(s) = \lambda \]

Solutions: circles and lines

Geodesics: (rough) helices, and lines

for the circles: rate of climbing
\[ \Delta z / \text{cycle} = \text{(signed) area of circle} \]
A nice surprise:

The set of planar curves arising as projections to the xy plane of geodesics is the same for all the Riemannian [penalty] metrics \( H_\epsilon \)

and for the sR case

\[
H = \lim_{\epsilon \to 0} H_\epsilon
\]
Horiz. lift of nondeg. zero locus of magnetic field = $C^1$- rigid curve (sense of Bryant-Hsu)

**Thm.** These curves are geodesics (= loc. length minimizers)

**repaired geodesic eq:**

$$\lambda_0 \kappa(s) = \lambda B(x(s), y(s))$$

multiplier for `cost' in Max princ.

zero for these abnormal good.

accounts for all possible geod's.
as charge ($\lambda$) $\rightarrow$ infinity
normal geod $C^0$-converge to
abnormal geod

so lines have
inflection points when
cross $B = 0$
($y = xB$)
Straighten out zero locus:
\[ B(x,y) = x; \]

Martinet model for \( D(x,y,z) \):
\[ dz - (x^2) dy = 0 \]

Abnormal geod is also Normal
\[ \lambda_0 k(s) = \lambda B(x(s), y(s)) \]
reads `0 = 0' for all choices of multipliers

Other geodesics:
Euler elastica, given by elliptic fns
thank you Levien ; Ardentov
BRANCHING GEODESICS.
(Meitton-Rizzi, 2019)

\[
B = \begin{cases} 
  x, y < 0 \\
  1, y > 1 
\end{cases}
\]

interpolates between flat Martinet and Heisenberg

A simpler model (for me)

\[
B = x, y < 0 \\
\text{and} \\
B(0, y) > 0, y > 0
\]
sR exponential map
`explanation’ of branching
Big open problem:
Are all sR geodesics smooth?

Idea for counterexamples:
look at situations where the zero-locus of $B(x,y)$
is not smooth.

Eg: a) $B(x,y) = xy$ [`normal crossing’]

Eero Hakavouri & Enrico Le Donne shoot down this example.
with their ``No corners theorem’’ [2016]
Eg (b): Tacnode

\[ B = y^2 - x^4 \]

a certain warm crowded cafe across the railroad tracks. Crossing the frozen Mississippi, with hexagonal patterns of ice;
(youngest daughter, heart problem appears..)
Agrachev… looking looking looking for counterexamples…

so..

subject: returning to the beginning

Richard Montgomery <rmont@ucsc.edu>
Wed, Apr 22, 9:15 PM (7 days ago)
to sussmann, Andrei…
[..] I hope you are well [...] 

I have been thinking about old things, and realize that you two have likely already pursued these things and with high likelihood to the bitter end - and found it a dead end. ... Hence this letter, so either I do not repeat your dead end, or you give me some nuggets of hope. [ ....] 

Take a magnetic field $B$ whose zero locus is a tacnode or its higher degree generalizations: $B(x,y) = y^2 - x^{2k}$; Its zero locus - [...] consists of two branches $y = x^k$, and $y = -x^k$ with order $k-1$ of contact at the origin. 

[...] Either branch, following until the origin will be a locally minimizing geodesic of Martinet type. So, follow the + branch to the origin, then switch to the other branch.. [to get ] a horizontal curve which is $C^{k-1}$ but not $C^k$.

QUESTION: is this concatenation a minimizer for any positive integer $k$?
Sat, Apr 25, 2020 at 3:32 AM Andrei Agrachev wrote:
Dear Richard,

I am fine, thank you very much.
I do not know the answer but may be there is a way to reduce this case to the Hakavuori - Le Donne theorem by taking a jet prolongation in the spirit of your and Misha Monster?

What do you think?
With kindest regards,
Andrei
Points and Curves in the Monster Tower

Richard Montgomery
Michail Zhitomirskii
**Thm:** [Agrachev-M-;][2020]

*The tacnode example does not minimize.***

**More generally** any piecewise smooth (or piecewise $C^k$)

$sR$ minimizer is smooth ($C^k$)

**Pf.** By induction, starting with Hakavouri-LeDonne’s $k = 1$.

**Tool:** Prolongation of distributions AND their curves.

**Key facts:**

1) the sR struc. also prolongs,

2) *the prolongation of a geod is a geod.*
Prolonging a distribution and its horizontal curves

old space: sR manifold $Q$, w distribution $D$, inner prod on...

new space: points are rays in $D$ downstairs.

\[ \tilde{Q} = \mathbb{P}D \]

so: points in new space:
(pt, line): $(q, \ell), \ell \subset D(q), q \in Q$

new distribution:

def a) : $\tilde{D}(q, \ell) = d\pi_q^{-1}(\ell)$
def b): horiz. curves are curves s.t.

\[ (q(t), \ell(t)) : dq/dt \in \ell(t) \]
**Prolonging** horizontal curves

\[ q(t) \sim (q(t), \ell(t)) = (q(t), \text{span}(dq/dt)) \]

Tacnode case: \[ x = t, y = \pm t^2, z = 0; \]

Prolong: introduce fiber variable \( u \)

\[ [dx, dy] = [1, u] = [\cos(\theta), \sin(\theta)] \]

\[ u = \tan(\theta) = dy/dx \]

\[ x = t; u = \begin{cases} 2t, & t < 0 \\ -2t, & t > 0 \end{cases} \]

NOW A CORNER!

want to invoke Hakavouri-LeDonne…
Prolonging sR struc.:
use that fiber inherits metric from inner
prod on \( D \). Eg: rank 2 case: identify proj line
w unit circle [doubled]

\[
X, Y \text{ o.n. for } D,
\]

\[
X_\theta = \cos(\theta)X + \sin(\theta)Y; \quad \frac{\partial}{\partial \theta}
\]

Exer: The proj. map \( \pi : \tilde{Q} \rightarrow Q \) is distance decreasing
(actually ``non-increasing"")

Exer: The prolongation of a geodesic is a geodesic.
Proof of theorem: \( k = 2 \):
Say a geodesic is p.w. \( C^2 \).
Prolong it, and its sR struc.:
Result: a \( C^1 \) geod in the prolongation
WITH corners!

Contradicts `no corners’ theorem of H-LeD.

So the original curve must have been \( C^2 \)

General \( k \). Similar. Use induction on \( k \).

fini. Thank you all, esp. Enrico,
for the opportunity to talk and inspiration to prove a theorem
in base
circle radius $r = 1/\lambda$
in space:
roughly, helices:
lie on cylinder of radius,
in sR case: they climb
area of circle/ revolution
in Riem case: increase height
from sR case by $\epsilon*\lambda$
per rev.