M. Zhitomirskii

I met Misha Zhitomirskii through singularity theory, specifically the singularity theory of *distributions* - sub-bundles of the tangent bundle of a manifold.

He took a sabbatical at my university and there he challenged me to understand why the singularities of a certain class of distributions, the "Goursat distributions" seem to grow like the Fibonacci numbers

We largely completed the classification of the Goursat distributions. To do so we introduced the "Monster tower". The whole area was enough off the beaten track that it yielded good thesis topics, at least 5.

Alex de Castro was one of these 5. He uncovered work of Gary K. and Susan C. on the same tower, under another name, the "Semple tower', and for reasons internal to enumerative alg. geometry.

$$\begin{array}{ll}
 & \vdots \\
 & \Big\downarrow \mathbb{P}^1 \\
k+3\text{-fold} & M_{k+1} \\
 & \Big\downarrow \mathbb{P}^1 \\
k+2\text{-fold} & M_k \\
 & \Big\downarrow \mathbb{P}^1 \\
 & \vdots \\
 & \Big\downarrow \mathbb{P}^1 \\
\text{Engel 4-fold} & M_2 \\
 & \Big\downarrow \mathbb{P}^1 \\
\text{contact 3-fold} & M_1 \\
 & \Big\downarrow \mathbb{P}^1 \\
\text{surface} & M_0
\end{array}$$

Each M_i is endowed with a rank 2 distribution Δ_i

(M_{k+1}, Δ_{k+1}) is constructed from (M_k, Δ_k) by the process of 'prolongation'

$$
\begin{array}{c}
\downarrow \mathbb{P}^1 \\
M_{k+1} \\
\downarrow \mathbb{P}^1 \\
M_k \\
\downarrow \mathbb{P}^1 \\
\vdots \\
\downarrow \mathbb{P}^1 \\
M_2 \\
\downarrow \mathbb{P}^1 \\
M_1 \\
\downarrow \mathbb{P}^1 \\
M_0
\end{array}
$$

Points in the Tower $\longrightarrow$ Legendrian curve germs in 3-space $\longrightarrow$ Plane curve germs

RVT code $\longrightarrow$ Puiseux characteristic

?? Value set (Zariski; Hefez-Hernandes)

Gary K. to Justin Lake: Answer “??”:
What invariant(s) or subclasses in the Tower,
does the value set of a curve repesent?

Each M_i is endowed with a rank 2 distribution Δ_i

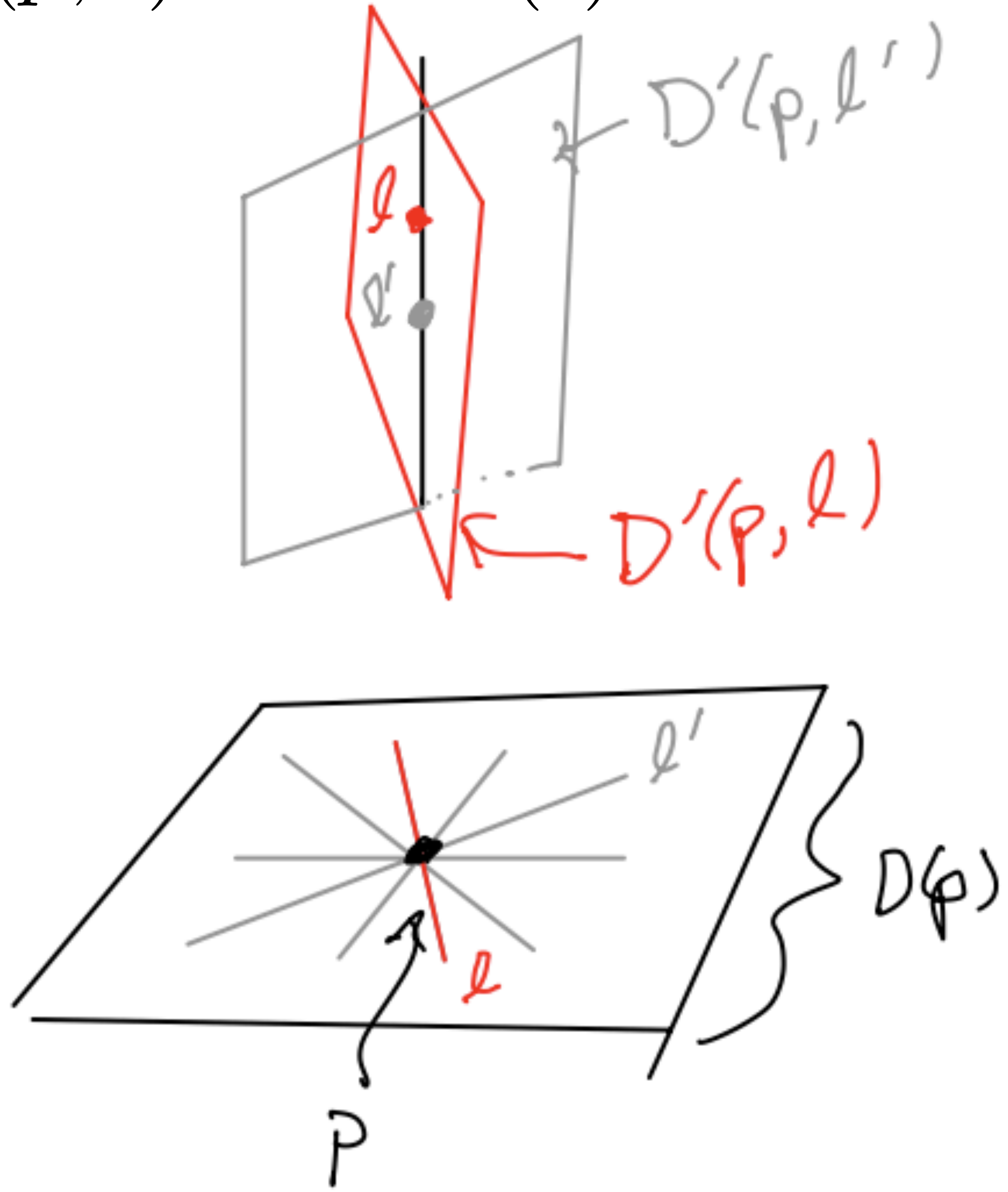
$$\begin{tikzcd}
 & \vdots \arrow[d, "\mathbb{P}^1"] \\
k+3\text{-fold} & M_{k+1} \arrow[d, "\mathbb{P}^1"] \\
k+2\text{-fold} & M_k \arrow[d, "\mathbb{P}^1"] \\
 & \vdots \arrow[d, "\mathbb{P}^1"] \\
\text{Engel 4-fold} & M_2 \arrow[d, "\mathbb{P}^1"] \\
\text{contact 3-fold} & M_1 \arrow[d, "\mathbb{P}^1"] \\
\text{surface} & M_0
\end{tikzcd}$$

(M_{k+1}, Δ_{k+1}) is constructed from (M_k, Δ_k)
by the process of 'prolongation'

Prolongation.

Start: (M, D), a manifold-with-distribution ,
so $D \subset TM$ is a linear sub-bundle. Assume rank $D = 2$.
End product : a new manifold-with-distribution (M', D')
which fibers over M with fiber $\mathbb{P}^1$.

$$D'(p, \ell) = d\pi^{-1}(\ell)$$



$$\begin{tikzcd} M' \arrow[d, "\mathbb{P}^1"] \\ M \end{tikzcd}$$

$$M' = \mathbb{P}D$$

$(M_0, \Delta_0) = $ (surface, tangent space of surface)
$(M_{k+1}, \Delta_{k+1}) = $ prolongation of (M_k, Δ_k)

$(M_1, \Delta_1) = $ (contact 3-fold, contact distribution)

Symmetries of M_k: $=$ (local) diffeos of M_k preserving Δ_k

prolongation acts on symmetries giving an injective homomorphism
prol: $Symm(M_k) \to Symm(M_{k+1})$

Thm. [Bäcklund, 1880s]. $prol : Symm(M_k) \cong Sym(M_{k+1})$ when $k \geq 1$.

$Symm(M_0) = Diff(M_0)$,
$Symm(M_1) = $ contact diffeos of (M_1, Δ_1)
$Symm(M_1)$ is much larger than (the image under $prol$ of)$Symm(M_0)$
$Symm(M_k) = $ (the image under $prol$ of) $Symm(M_1), k \geq 1$

PROBLEM: Classify the orbits in the Tower under the action of the full (contact) symmetry group.

Contactomorphisms act on the Tower

after deleting the zeroth level

$$\begin{array}{c}
\downarrow{\scriptstyle \mathbb{P}^1} \\
M_{k+1} \\
\downarrow{\scriptstyle \mathbb{P}^1} \\
M_k \\
\downarrow{\scriptstyle \mathbb{P}^1} \\
\vdots \\
\downarrow{\scriptstyle \mathbb{P}^1} \\
M_2 \\
\downarrow{\scriptstyle \mathbb{P}^1} \\
M_1 \\
\downarrow{\scriptstyle \mathbb{P}^1} \\
\cancel{M_0}
\end{array}$$

PROBLEM: Classify the orbits in the Tower under the action of the full (contact) symmetry group.

Contactomorphisms act on the Tower
after deleting the zeroth level

A coarser equivalence relation on the M_k's than lying on the same contact orbit: having the same RVT class

$$\begin{array}{c} \big\downarrow \mathbb{P}^1 \\ M_{k+1} \\ \big\downarrow \mathbb{P}^1 \\ M_k \\ \big\downarrow \mathbb{P}^1 \\ \vdots \\ \big\downarrow \mathbb{P}^1 \\ M_2 \\ \big\downarrow \mathbb{P}^1 \\ M_1 \\ \big\downarrow \mathbb{P}^1 \\ \cancel{M_0} \end{array}$$

BUILDING RVT classes

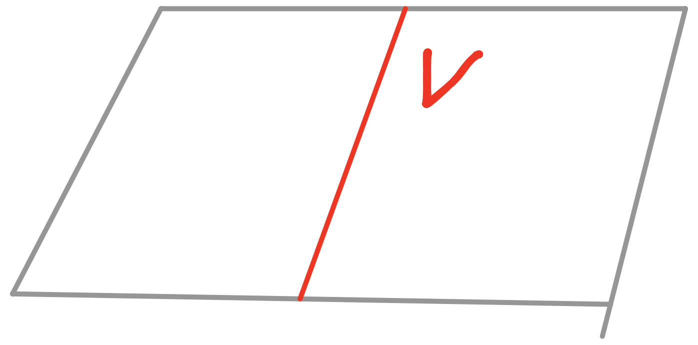
This action preserves levels and maps fibers to fibers.
So the action leaves invariant two families of curves within each level $k > 1$

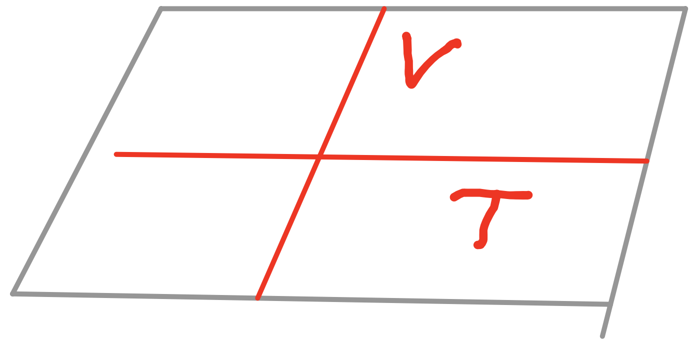
- the fibers (V's)

- the prolongations (*) of fibers from lower levels (T's) – for $k > 2$

The action preserves the tangent line field to these curve families, –
a line sub-bundle $V \subset \Delta_k$ and
a line sub-bundle $T \subset \Delta_k|_{crit}$ only defined along a hypersurface '*crit*'

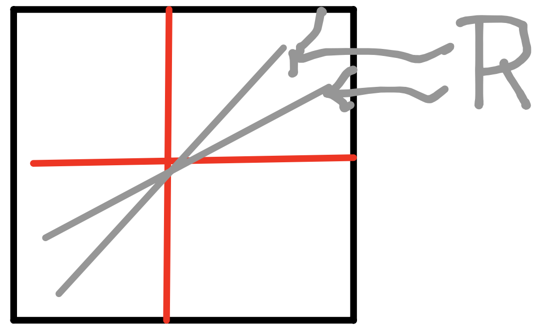
Through every 2-plane $\Delta_k(p)$, $k>1$ there are either 1 or 2 distinguished lines





p regular

p critical



?

$\bullet \; p = (p_{k-1}, \ell \quad)$

p is $\begin{cases} \underline{\text{critical}} & \text{if } \ell \in \Delta_{k-1}(p_{k-1}) \text{ is a V or T} \\ \underline{\text{regular}} & \text{'' '' '' '' not} \end{cases}$

V: tangent line to fiber thru p

T: '' '' to a prolongation to a fiber below p

Mark $p_K = (p_{K-1}, \ell_{K-1})$ as
R, V, or T
dep. on whether $\ell_{K-1} \in \Delta_{K-1}(p_{K-1})$ is
R, V, or T

Repeat: for $p_{K-1} = (p_{K-2}, \ell_{K-2})$

Call $p_1 \in M_1$, $p_2 \in M_2$ 'R' regular

get: a word in alphabet $\{R, V, T\}$

RRRVVTRRV....VR
1 2 3 4 5 6 7 8 9 K-1 K

one spelling rule: the letter immediately precedi

Words for level k have length $k-2$.
There are F_{2k-3} RVT words of length $k-2$.

$\Longrightarrow$ M_K decomposes into the disjoint union of F_{2k-3} RVT classes. Each class is a union of orbits

level k	1	2	3	4	5	6	7	8
dim M_k	3	4	5	6	7	8	9	10
$\#(RVT$ classes in $M_k)$	1	1	2	5	13	34	89	233
$\#($orbits in $M_k)$	1	1	2	5	13	34	93	∞

Here are the first 10 Fibonacci numbers with every other one crossed out so that just the odd-indexed ones remain.

$$\cancel{0},\ 1,\ \cancel{1},\ 2,\ \cancel{3},\ 5,\ \cancel{8},\ 13,\ \cancel{21},\ 34, \cancel{55},\ 89$$

#(distinct Goursat germs in dim k+2)

(Piotr Mormul !)

RVT classes = Contact Orbits UP TO LEVEL 6.

$89 \to 93$ At level $k = 7$ four RVT classes break up, each into two orbits for $93 = 89 + 4$

$233 \to \infty$ At level $k = 8$ at least one RVT class breaks up into a continuum of orbits
Which ones? How?

$F_{2k-3} \to ??$ At the general level k there are F_{2k-3} RVT classes.

Which ones break up into more orbits? How do they break up?

HISTORICAL COMMERCIAL

Our original motivation for climbing up the Tower:

to CLASSIFY GOURSAT DISTRIBUTIONS

(From Susan Colley's slides)

- Differential geometers and control theorists are interested in noninvolutive distributions. $\mathcal{D}$ or D
- They consider the *Lie square*

$$\mathcal{D}_2 = \mathcal{D} + [\mathcal{D}, \mathcal{D}].$$

In general this won't be a distribution, as the rank may vary from point to point.

- But suppose that it is, and let's iterate the construction to get a sequence of subbundles of the tangent bundle

$$\mathcal{D} = \mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}_3 \subset \cdots ,$$

where each bundle is the Lie square of the previous bundle. Call this the *Lie squares sequence*.

Goursat Distributions

- The distribution $\mathcal{D}$ is said to be *Goursat* if in the associated Lie squares sequence

$$\mathcal{D} = \mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}_3 \subset \cdots$$

all the $\mathcal{D}_i$'s have constant rank that increases by one at each step, until we reach the tangent bundle. Δ_k is Goursat

- Montgomery & Zhitomirskii showed that every rank 2 Goursat distribution is locally equivalent to the focal distribution on ~~some~~ the monster space. (over n'importe quelle surface M_0; or contact manifold M_1 or Engel manifold M_2)

Every Goursat germ on a $k+2$ dimensions is diffeomorphic to $(p, U, \Delta_k|U)$ for some $p \in M_k$ with U some nbhd of p

Two such Goursat germs at different points $p, p' \in M_k$ are loc. diffeo $\iff$ $\exists$ a contact diffeo $\Phi : M_k \to M_k$ with $\Phi(p) = p'$

Problem of classifying Goursat germs up to diffeo

$\iff$

Problem of decomposing Tower into orbits w.r.t contact symmetry group action

END. HISTORICAL COMMERCIAL

CONNECTIONS TO COLLEY-KENNEDY

Case of $M_0 = \mathbb{P}^2$. Then $M_1 = \mathbb{P}TM_0$ and

$\mathbb{P}GL(3) \subset Diff(\mathbb{P}^2) \subset Cont(M_1)$

	Montgomery-Zhitomirskii	Colley-Kennedy	symbol
	distribution	focal bundle	Δ_k
	fiber	divisor at infinity	$V_k; I_k; \pi^{-1}(p_{k-1})$
symmetries	contact diffeos	proj. transf's or diffeos of surface	$Cont(M_1)$; $\mathbb{P}GL(3)$ or $Diff(M_0)$

RETURN TO MAIN STORY LINE

$89 \to 93$

At level $k = 7$ four RVT classes
break up, each into two orbits for $93 = 89 + 4$

$233 \to \infty$

At level $k = 8$ at least one RVT class
breaks up into a continuum of orbits
Which ones? How?

$F_{2k-3} \to ??$

At the general level k there are F_{2k-3} RVT classes.

Which ones break up into more orbits? How do they break up?

TABLE of BREAK-UP OF LEVEL 7 RVT CLASSES.

RVT class	jet ID. #	Lag. Puis. Char	Puis. Char	rep. curve classes
$RVTRR$	8	$[3,7]$	$[3,10]$	$(t^3,t^7),(t^3,t^7+t^8)$
$VRVVR$	11	$[6,9,11]$	$[6,15,17]$	$(t^6,t^9\pm t^{11})$
$VTVRR$	8	$[5,7]$	$[5,12]$	$(t^5,t^7),(t^5,t^7+t^8)$
$VVVRR$	9	$[5,8]$	$[5,13]$	$(t^5,t^8),(t^5,t^8+t^9)$

Meaning of 'representative curve class':
Example: (t^3, t^7+t^8) represents the Legendrian curve

$$x = t^3, u = t^7 + t^8, y = \frac{3}{10}t^{10} + \frac{3}{11}t^{11}.$$

The $y(t)$ is obtained from $u(t)$ by integrating $dy = udx$. The (x,u,y) are Darboux coordinates for M_1 endowed with contact form $dy - udx$.

Every point in the class RVTRR is realized as the 6-fold prolongation (@ $t=0$) of a Legendrian curve equivalent to either $(t^3, t^7, \frac{3}{10}t^{10})$ or its cousin $(t^3, t^7, \frac{3}{10}t^{10} + \frac{3}{11}t^{11})$

MODULI AT LEVEL 8 (one class with..)

RVT class	jet ID. #	Lag. Puis. Char	Puis. Char	rep. curve classes
$VRVRVR$	15	$[8,12,14,15]$	$[8,20,22,23]$	$(t^8, t^{12}+t^{14}+at^{15})$

A CLASS OF MULTIPLICITY 4 with MODULI which occurs at level 9

RVT class	jet ID. #	Lag. Puis. Char	Puis. Char	rep. curve classes
$RVRVRRR$	13	$[4,10,11]$	$[4,14,15]$	$(t^4, t^{10}+t^{11}+bt^{13})$

$$\gamma_b(t) : x = t^4, u = t^{10}+t^{11}+bt^{13}, y = 4\left(\frac{1}{14}t^{14}+\frac{1}{15}t^{15}+\frac{b}{17}t^{17}\right)$$

GUIDING PRINCIPLE

Since our group is the group of all (germs of) contact transformations, our questions are questions in contact geometry.

The Contact Creed: Everything is Legendrian

–paraphrasing Alan Weinstein

Our classification problem must be encoded into a problem about Legendrian curve germs

LEGENDRIAN CURVES

(M_1, Δ_1) is a contact manifold.

Any two contact manifold are locally diffeo.

Local model: $\underbrace{dy - udx}_{\alpha} = 0.$

$(x, u, y) \in \mathbb{K}^3$ are loc. coord for M_1,
$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Legendrian curve: analytic curve germ in M_1 everywhere tangent to Δ_1.

Thus $\gamma^* \alpha = 0$ or $\gamma(t) = (x(t), u(t), y(t)$ with $\frac{dy}{dt} = u(t)\frac{dx}{dt}$.

Every point $p \in M_k$ is realized as the evaluation at $t = 0$ of the $k-1$-fold prolongation of some Legendrian curve germ γ from level 1. This prolonged curve may be taken to be IMMERSED at level k and tangent to a regular direction at $t = 0$.

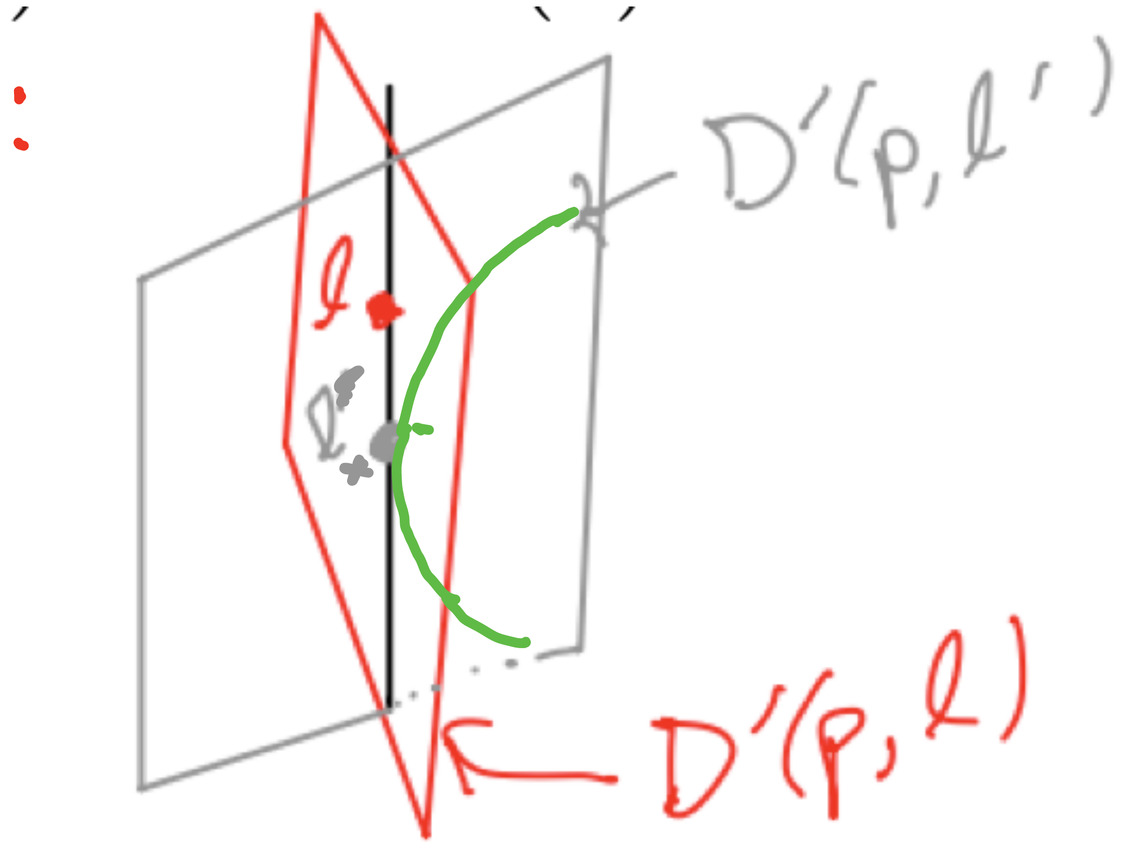
In symbols: $p = \gamma^{(k-1)}(0)$

$Leg(p)$ = set of such Legendrian curve germs.

PROLONGATION OF CURVES

'Curve' in an analytic mfd-w-distribution :=
analytic well-parameterized curve everywhere tgt to distribution

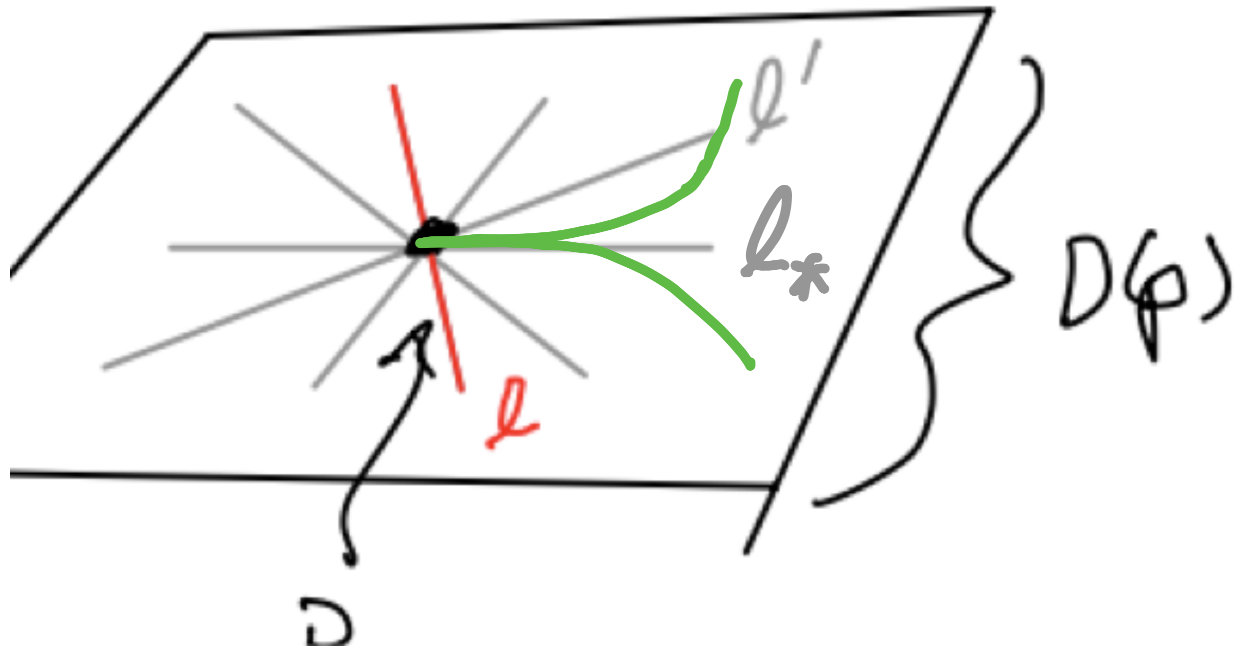
D′: prolongation of (M, D):



prolongs to curve $(p(t), \ell(t))$ thru ℓ_*; $\ell(t) = \mathrm{span}\left(\frac{dp}{dt}(t)\right)$, $t \neq 0$

$\ell_* = \ell(0) = \lim_{t \to 0} \mathrm{Span}\left(\frac{dp}{dt}(t)\right)$

prolonging a curve



curve $p(t)$ in (M, D) thru $p = p(0)$.

$\left(t = 0: \quad \frac{dp}{dt}(0) = 0, \text{ isolated crit. pt.}\right)$

DEF: Leg. curves $\gamma, \gamma' : (\mathbb{K}, 0) \to (\mathbb{K}^3, 0)$ are called RL-contact equivalent if $\exists$ contactomorphism $g : (\mathbb{K}^3, 0) \to (\mathbb{K}^3, 0)$ and reparam $\phi : (\mathbb{K}, 0) \to (\mathbb{K}, 0)$ (so $\phi'(0) \neq 0$) such that

$$\gamma' = g \circ \gamma \circ \phi$$

($\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$)

Thm: $p' = g \cdot p \iff Leg(p)$ is RL-contact equivalent to $Leg(p')$

Here g a contact diffeo

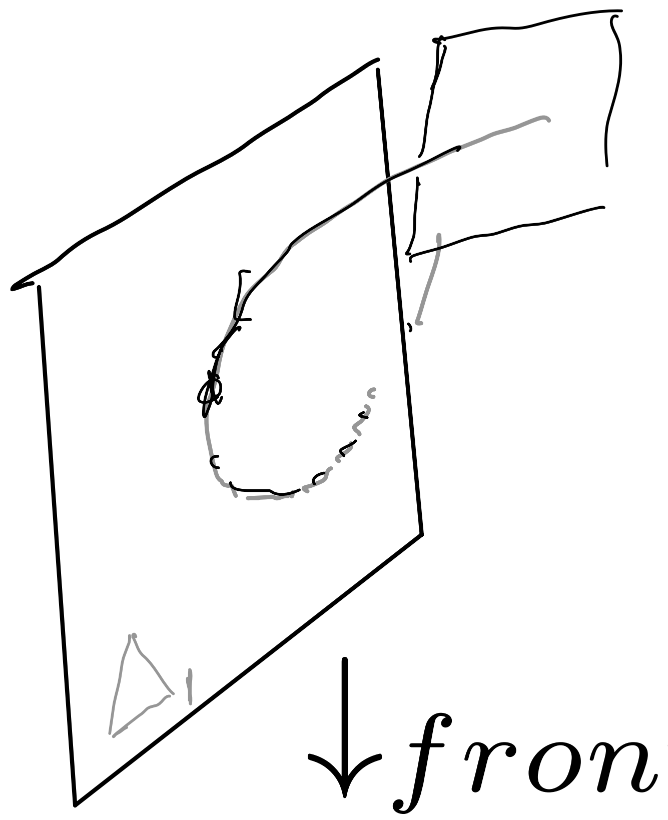
...reduces our orbit classification problem in the Tower to the better-understood problem of classifying LEGENDRIAN CURVE SINGULARITIES

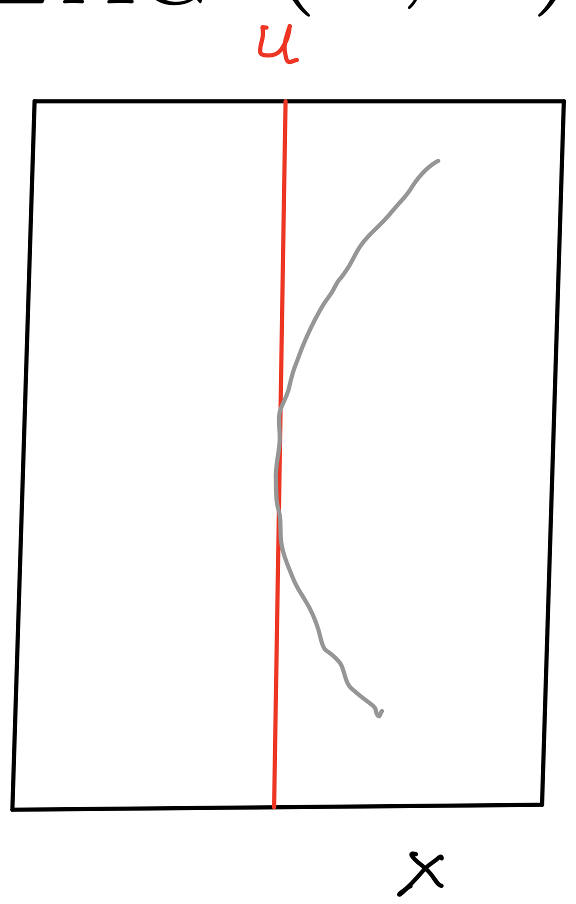
...which in turn can be partially reduced to the classical problem of classifying PLANE CURVE SINGULARITIES

Lagrangian Projection

Front Projection

$$(x, u, y) \to_{LAG} (x, u)$$





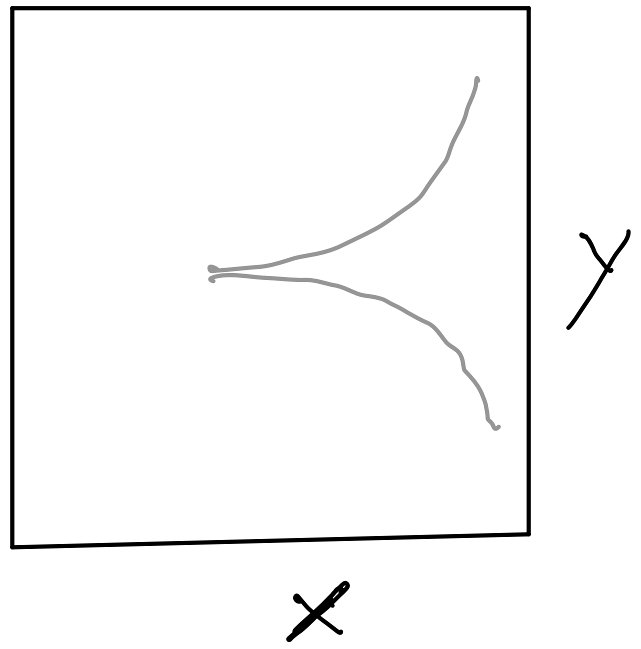
$d\pi_{Lag}|\Delta_1 : \Delta_1 \cong \mathbb{K}^2$

$ker(d\pi_{front}) = V \cong \mathbb{K} \subset \Delta_1$

$$\downarrow front$$

π_{front} = proj onto zeroth level of Tower

$$(x, y)$$



$(\underset{x}{t}, \underset{u}{t^2}, \underset{y}{\tfrac{2}{3}t^3})$

Facts:

1) The Lag. proj of all curves in $Leg(p)$ lie in the same 'equisingularity' class

NOTATION. Call this class $LPC(p)$.
Encode this equisingularity class as the Puiseux char. or semigroup of the plane curve

2) $LPC(p)$ only depends on the RVT class of p

NOTATION: If $\alpha = RV..$ is an RVT word, write $LPC(\alpha)$ for $LPC(p)$, $p \in [\alpha]$.

3) Indeed LPC only depends on the STABILIZED RVT class:
if $\beta = \alpha R$ then $LPC(\beta) = LPC(\alpha)$.

N.B.: In our book, and in Colley-Kennedy
the equisingularity class of the FRONT PROJECTIONS of
curves in $Leg(p)$ is used. Encode it as the Puiseux charac.
Write it as $PC(p)$
Requires that the Leg. curves be "properly aligned".
When they are, we have the relation:

$$LPC(p) = [m, \lambda_1, \ldots, \lambda_k] \iff PC(p) = [m, \lambda_1 + m, \ldots, \lambda_k + m]$$

'Properly alligned' Leg curves:

tgt @ $t=0$ is x-axis

$\Rightarrow \operatorname{ord}(x(t)) < \operatorname{ord}(u(t)) < \operatorname{ord}(y(t))$

& from

$dy - u\,dx = 0$ along curve

$$u = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

or $y(t) = \int_0^t u(s) \frac{dx}{ds}(s)\, ds$

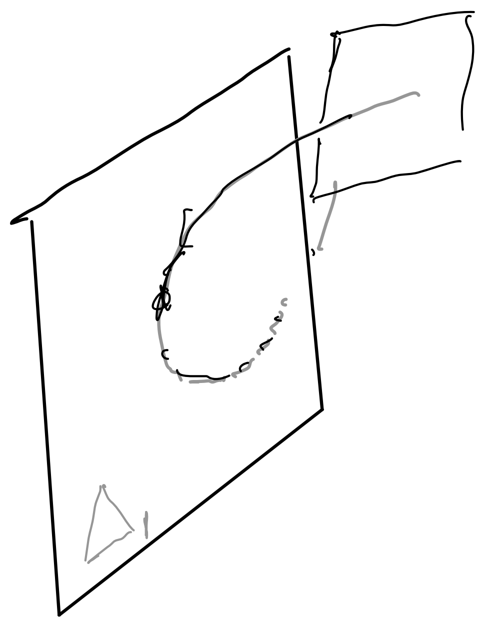
$$\begin{array}{ccc} (x,u,y) & \xmapsto[\text{proj.}]{\text{Lag}} & (x,u) \\ \big\downarrow{\scriptstyle \text{front proj.}} & & \\ (x,y) & & \end{array}$$

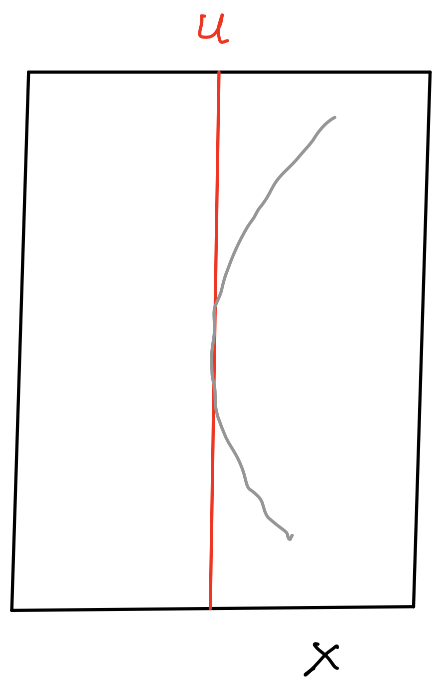
Any one of: Leg. curve $(x(t), u(t), y(t))$

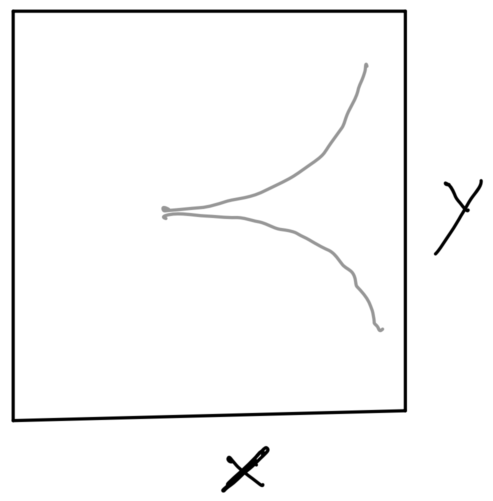
Lag curve $(x(t), u(t))$

Leg curve $(x(t), y(t))$

determines the other two curves



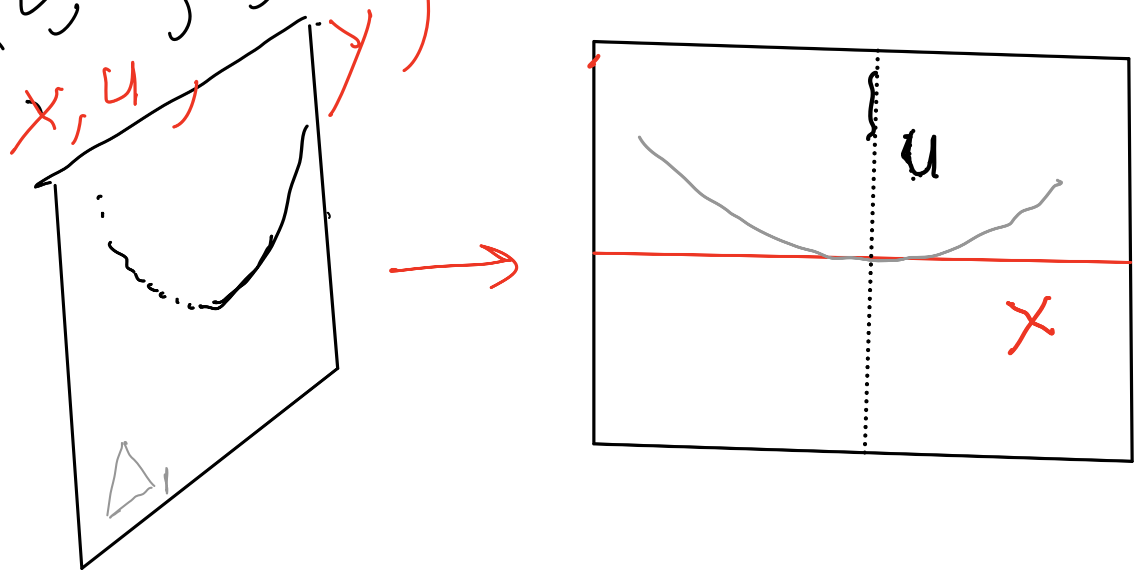


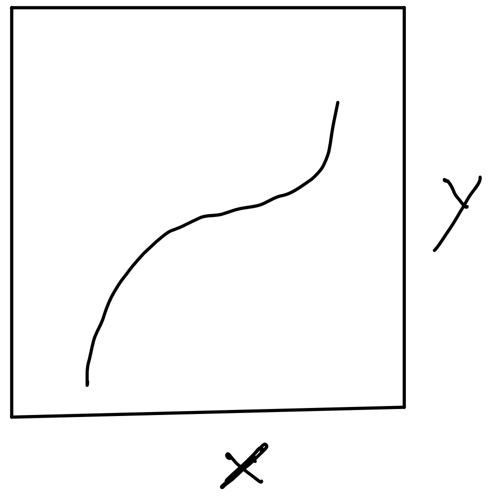


$$(x(t), u(t), y(t))$$
$$= \left(t^2, t, \tfrac{2}{3}t^3\right)$$

$(t, t^2, \frac{2}{3}t^3)$

(x, u, y)





$(x, u, y) \to (u, x, y)$

$(x(t), u(t), y(t))$

$= (t^2, t, \frac{2}{3}t^3)$

$\to (t, t^2, \frac{2}{3}t^3)$

Some examples.

$LPC(R^q) = [1]$.

Meaning: All immersed Legendrian curve germs are Locally RL-contact equivalent.
Their Lag projections are immersed.
Use [1] for the Puiseux char. of an immersed curve germ.
Normal form $(t, 0)$
. The class R^q is the unique open dense orbit inside $M_k, k = q + 2$

$LPC(R^s V R^q) = [2, 2s + 1]$.

$Leg(p)$ for $p \in [R^s V R^q]$ is realized by the " A_{2s} singularity"
Normal form: $(t^2, t^{2s+1}, \frac{2}{2s+3} t^{2s+3}) = (x(t), u(t), y(t))$
. These classes exhaust the codim 1 RVT classes.

The LPC of $[p, q]$ is realized by $(t^p, t^q, \frac{p}{q+p} t^{q+p})$
Its corresponding RVT class consists of a word with a single critical block:
$R^s \beta R^q$ with $\beta = V \ldots$ consisting ENTIRELY of Vs and Ts.

If $LPC(R^s \beta R^q) = [p, q]$ then $LPC(R^{s+1} \beta R^q) = [p, q + p]$

Thm: $p' = g \cdot p \iff Leg(p)$ is RL-contact equivalent to $Leg(p')$

Here g a contact diffeo

DEF: Leg. curves $\gamma, \gamma' : (\mathbb{K}, 0) \to (\mathbb{K}^3, 0)$ are called RL-contact equivalent if $\exists$ contactomorphism $g : (\mathbb{K}^3, 0) \to (\mathbb{K}^3, 0)$ and reparam $\phi : (\mathbb{K}, 0) \to (\mathbb{K}, 0)$ (so $\phi'(0) \neq 0$) such that

$$\gamma' = g \circ \gamma \circ \phi$$

$(\mathbb{K} = \mathbb{C}$ or $\mathbb{R})$

THM. Suppose p is 'regular' i.e. its RVT code α ends in an 'R'. $\exists!$ pos. integer $r = r(\alpha(p))$ with the following property. For $\gamma_* \in Leg(p)$ and γ an arb. Leg passing through $p_1 = \pi_{k,1}(p)$, we have $\gamma \in Leg(p) \iff \exists$ reparam ϕ such that $j^r(\gamma) = j^r(\gamma_* \circ \phi)$.

In other words, we can identify p with the rth order projective jet of a (typically singular!) Legendrian curve.

JET ID

Formula:

$$r = \lambda_k + (q-1)$$

where the RVT class of p is αR^q, $q > 0$ and where α ends in a V or T and where $[m, \lambda_1, \lambda_2, \ldots, \lambda_k]$ is the LAGRANGIAN PUISEUX CHARACTERISTIC associated to α.

[LPC]

Special Case: if p has RVT class R^q then $r = q + 1$.
(word indexing starts at level 3 so $(R) \subset M_3$.)

Terminology: r = jet ID number of p.
NOTE: Points in the same RVT class have the same jet ID numbers .

Semigroup $\mathbb{S}$ of a curve $\gamma : (\mathbb{K}, 0) \to (\mathbb{K}^n, 0)$

$\mathcal{O}_n :=$ ring of germs of $\mathbb{K}$-analytic functions $(\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$.
$\gamma^* : \mathcal{O}_n \to \mathcal{O}_1$ by $\gamma^* f := f \circ \gamma$
$ord : \mathcal{O}_1 \to \mathbb{N} \cup \infty$ order of vanishing;
so $ord(t^k) = k$ for $k \in \mathbb{N}$, $ord(fg) = ord(g) + ord(g)$, etc

$$\mathbb{S} = \{ord(\gamma^* f) : f \in \mathcal{O}_n, f \neq 0\}$$

$ord(fg) = ord(f) + ord(g)$ yields semigroup law: $s_1, s_2 \in \mathbb{S} \implies s_1 + s_2 \in \mathbb{S}$

Def (or theorem). The curve γ is 'well parameterized" its semigroup is co-finite in $\mathbb{N}$, i.e $\#\mathbb{N} \setminus \mathbb{S}$ is finite.

We take all curves taken to be well-parameterized

Value set $\mathbb{V}$ of a curve γ.

Extend 'ord to differential forms $f(t)dt$ on the line by using $ord(dt) = 1$,and $ord(f(t)dt) = ord(f(t)) + ord(dt)$.
Find: $ord(dg) = ord(g)$.

Ω^1_n = germs of analytic differential forms on $\mathbb{K}^n$ defined near 0.
$\gamma^* : \Omega^1_n \to \Omega_1$

$$\mathbb{V} = \{ord(\gamma^*\omega) : \omega \in \Omega^1_n\} \cup \{0\}.$$

Our conventions imply that $ord(\gamma^*(f\omega)) = ord(\gamma^* f) + ord(\gamma^*\omega)$, showing that $\mathbb{V}$ is indeed a module over $\mathbb{S}$. We throw 0 into $\mathbb{V}$ simply to insure that $\mathbb{V} \supset \mathbb{S}$.

C

- Equivalent def of the prolonged D:

Use a distribution determines & is uniquely determined by its integral curves

· Curve in ~~B~~ M'

$\gamma(t) = (p(t), \ell(t))$. $\ell(t) \subset D(p(t))$

Declare γ integral for D'

$\iff \frac{dp}{dt} \in \ell(t)$ holds in M.

D

Properties of prolongation.

1) Any analytic curve for (M, D) has a ! prolongation
- this is an analytic integral curve on (M', D') which projects to the original.

2) Any ~~ge~~ symmetry of (M, D) prolongs to yield a symmetry of (M', D').

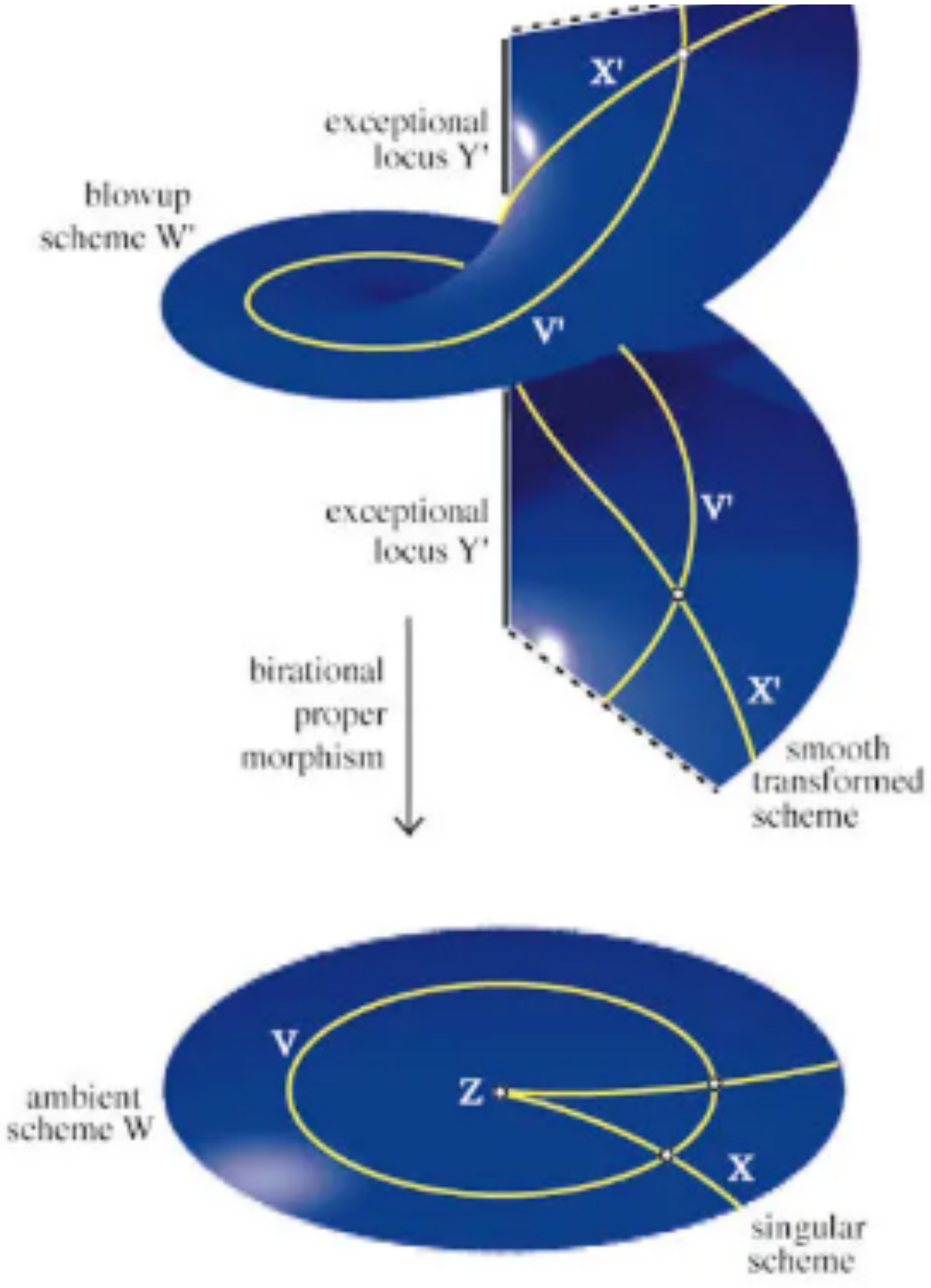
"symmetry": germ of analytic diffeo preserving the distribution.

Construction: $\Phi: M \dashrightarrow M$.

$$\phi'(p, \ell) = (\Phi(p), d\Phi_p(\ell)).$$

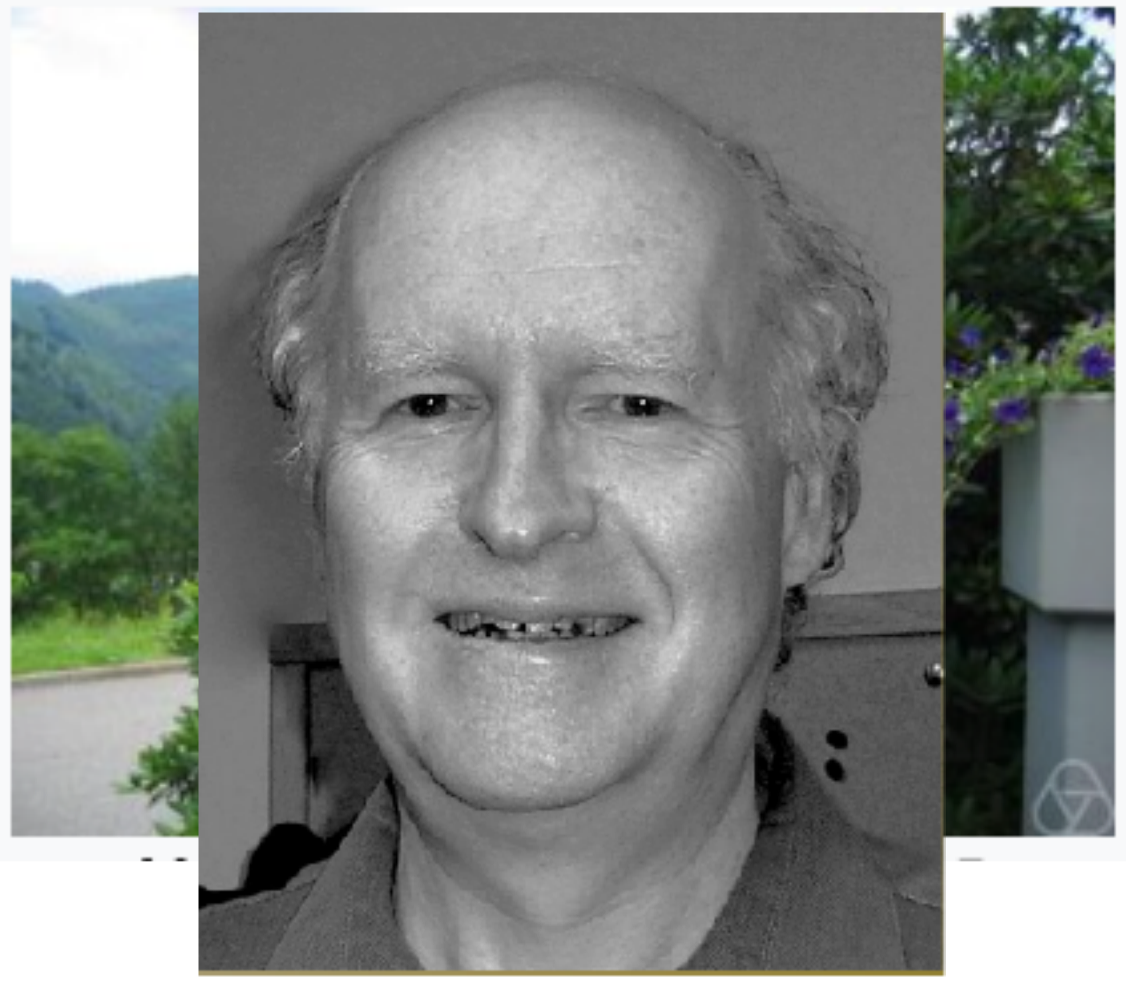
$$\phi': M' \to M'$$

Heisuke Hironaka



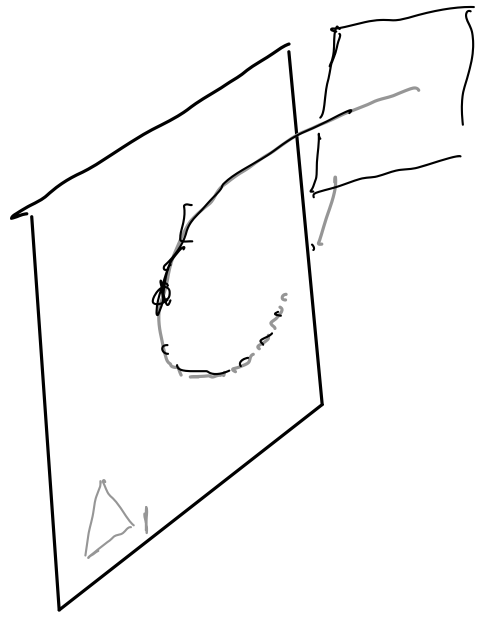
V. Arnold

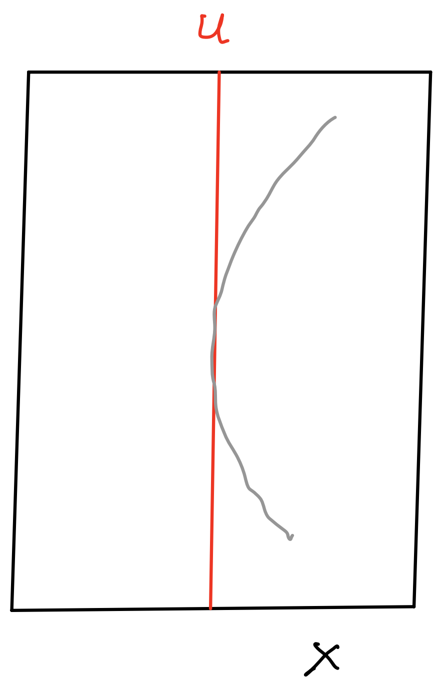
John N. Mather

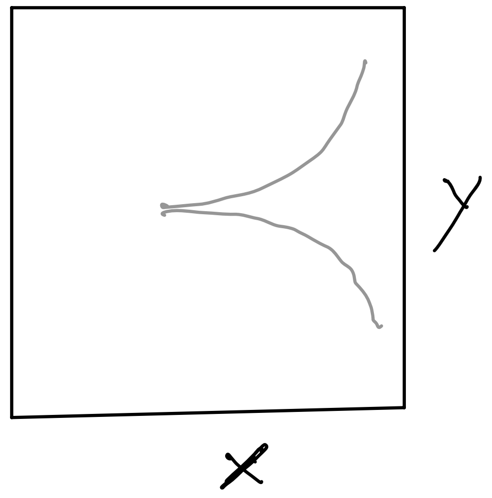


René Thom

Photograph by Paul Halmos





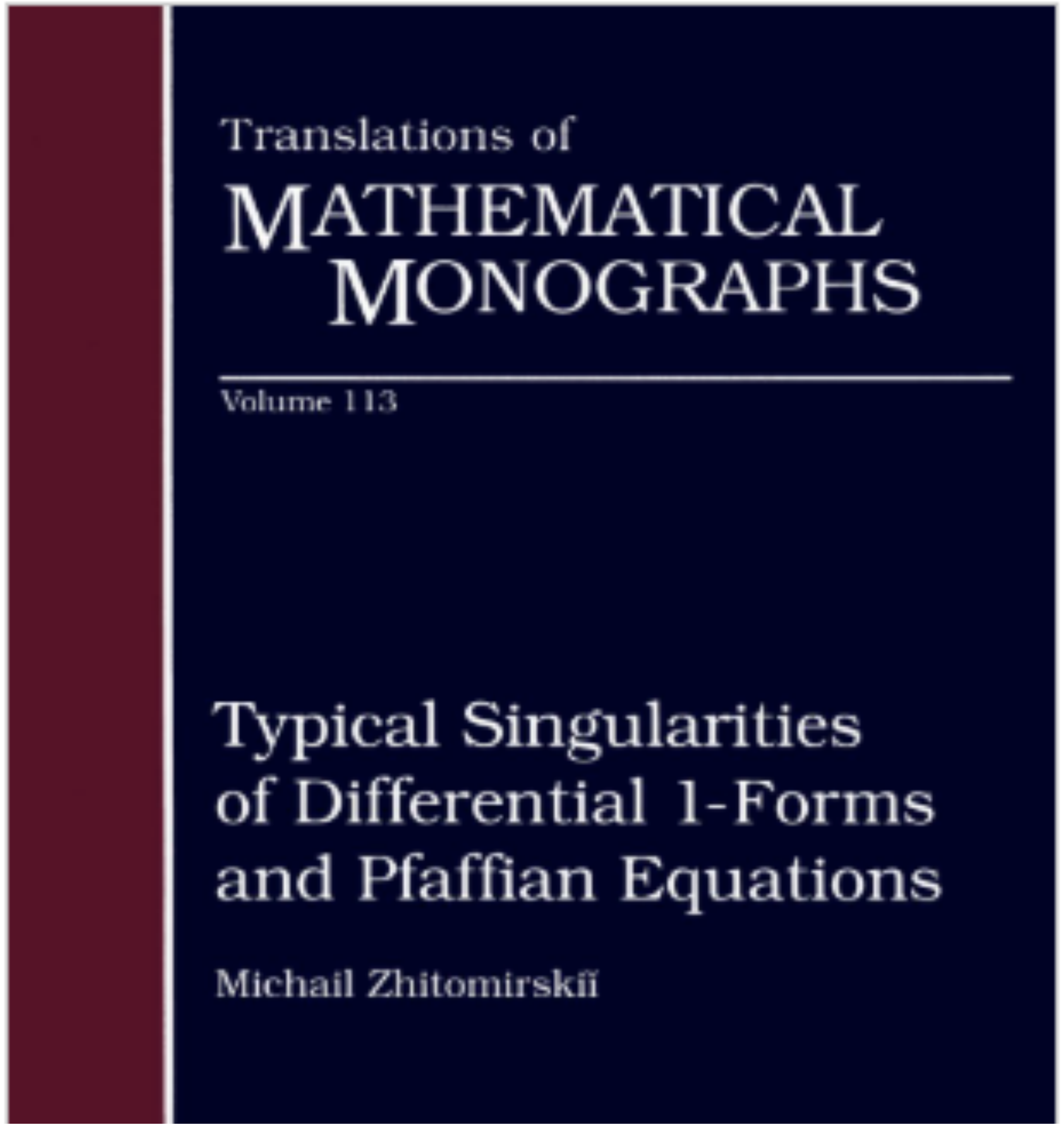


$$(x(t), u(t), y(t))$$
$$= (t^2, t, \tfrac{2}{3}t^3)$$

Misha:

Go classify

Singularities



Model example: the A-D-E classification of (scalar) functions

$$f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0) \quad \text{or} \quad f : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$$

$$f \sim g \iff g = \Phi \circ f \circ \psi^{-1}$$

Open dense germ:
$f(x, y, \dots) = x$

criteria: if $dg(0) \neq 0$ then $g \sim f = x$

So insist $df(0) = 0$.

Morse lemma: If $d^2 f(0)$ has full rank n then

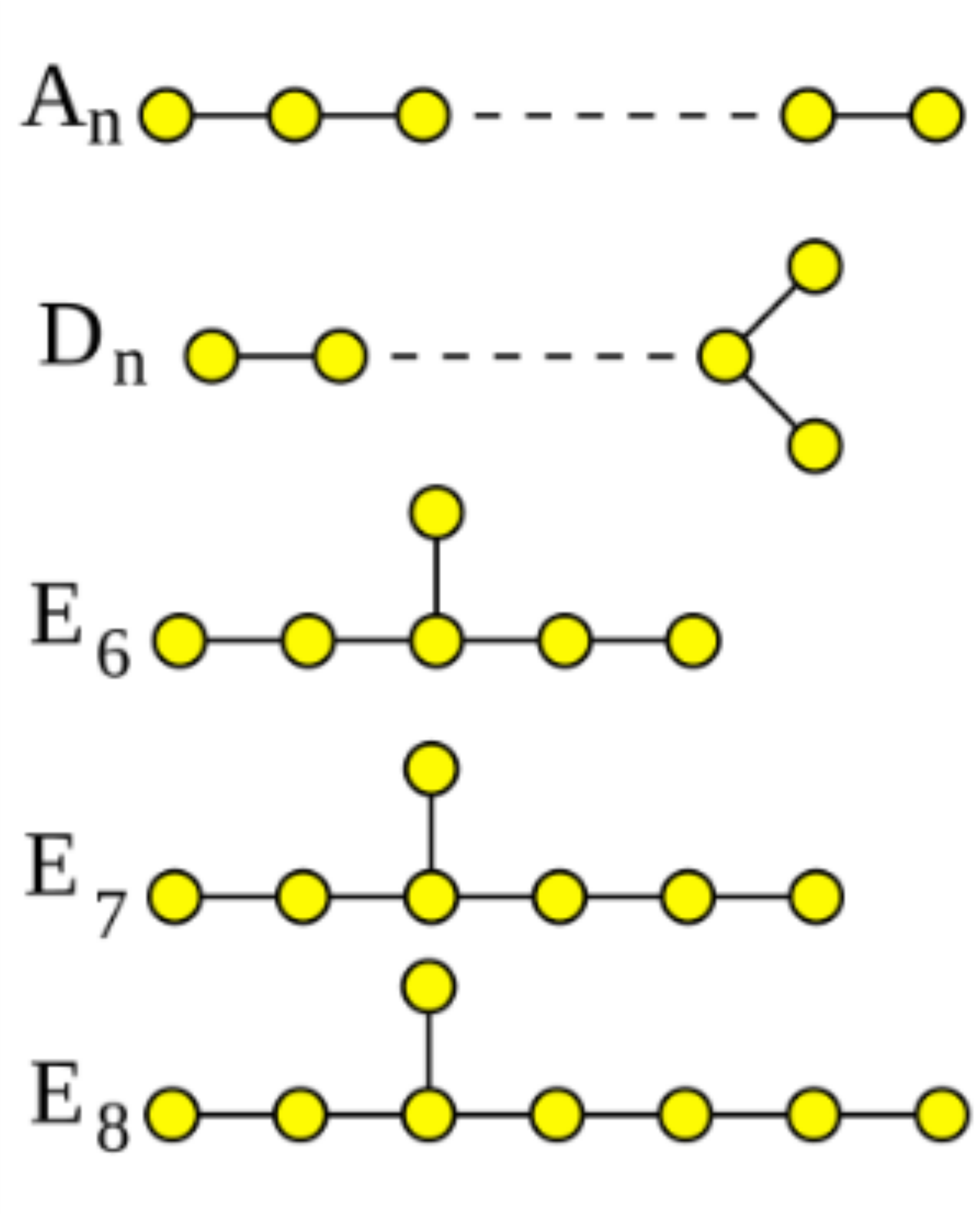
$$f(x, y, \dots, z) = \pm x^2 \pm y^2 \pm \dots \pm z^2$$

Two germs are said to be *stably equivalent* if they become R-equivalent* after the addition of quadratic forms in an appropriate number of variables.

Theorem 1 (see [9], [10]): *Simple germs of holomorphic functions (germs with $m = 0$) are given, up to stable equivalence, by the following list:*

$$A_k\colon f(x) = x^{k+1},\ k \geqq 1;$$
$$D_k\colon f(x, y) = x^2y + y^{k-1},\ k \geqq 4;$$
$$E_6\colon f(x, y) = x^3 + y^4;$$
$$E_7\colon f(x, y) = x^3 + xy^3;$$
$$E_8\colon f(x, y) = x^3 + y^5.$$

The connection between these singularities and the simple Lie algebras or groups generated by reflections, denoted by the same symbols, is discussed in [11]. These singularities may also be obtained from the regular polyhedra in three-dimensional Euclidean space or more precisely from the discrete subgroups



Unfolding the D4 singularity

The D4 singularity is the first one where the Coxeter-Dynkin diagram has a branch:  .

3D Model: unfolded D4 singularity

s	2	3	4	5	6	7	≥ 8
$or(s)$	1	2	5	13	34	93	∞
$Author:$	$Engel$	$Giaro$ $Kumpera$ $Ruiz$	$Kumpera$ $Ruiz$	$Gaspar$	$Mormul$	$Mormul$	$Mormul$
$reference:$	1889	1978	1982	1985	1997	1998	1998

Although the entries of this table were obtained originally just for rank two Goursat distributions on R^{2+s} they hold for Goursat distributions of arbitrary rank k and corank

Me:

I’m working in distributions…

Facts. The Puiseux χic of every $\gamma \in \Pi_{Lag}(Leg(p))$ is the same.

Call the list of integers the "LPC" of p.

Fact: The LPC of p only depends on the RVT class of p!

So we have a map:

$$\text{RVT words} \longrightarrow \text{LPC's}.$$

Fact: Write αR^q for $\alpha \underbrace{RR\cdots R}_{q}$ where α is any RVT class.

$$LPC(\alpha R^q) = LPC(\alpha)$$ "invariant under stabilization"

N.B. Our book, Colley-Kennedy use the Puiseux χic of the FRONT proj of curves in Leg(P). Call this the 'PC'.

Curve must be properly alligned!

rel'n: $LPC = [m, \lambda_1, \lambda_2, \ldots, \lambda_k]$

$PC = [m, \lambda_1 + m, \lambda_2 + m, \ldots, \lambda_k + m]$

At a general level k there are F_{2k-3} RVT classes.

Some (eg R^k) form a single contact orbit.
Others break up into a finite number of orbits.
Others still break up into many “components”some of which are consist of a continua of distinct orbits: moduli are present.

Which RVT classes are single orbits and which break up?
How precisely do these latter break up so as to be decomposed into orbits?