

Gauge Theory: from Falling Cats to the Three-body problem.

... hello Illinois !

Unifying Mathematical theme: geometry of a principal **G**-bundle

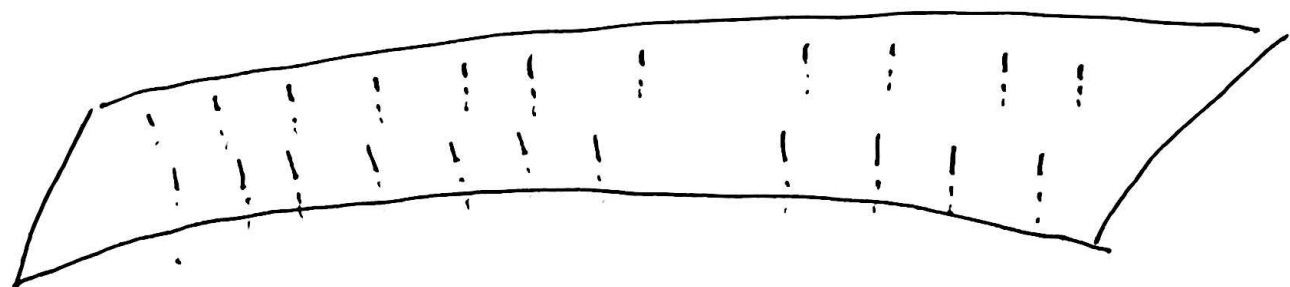
$$G \rightarrow Q \rightarrow S$$

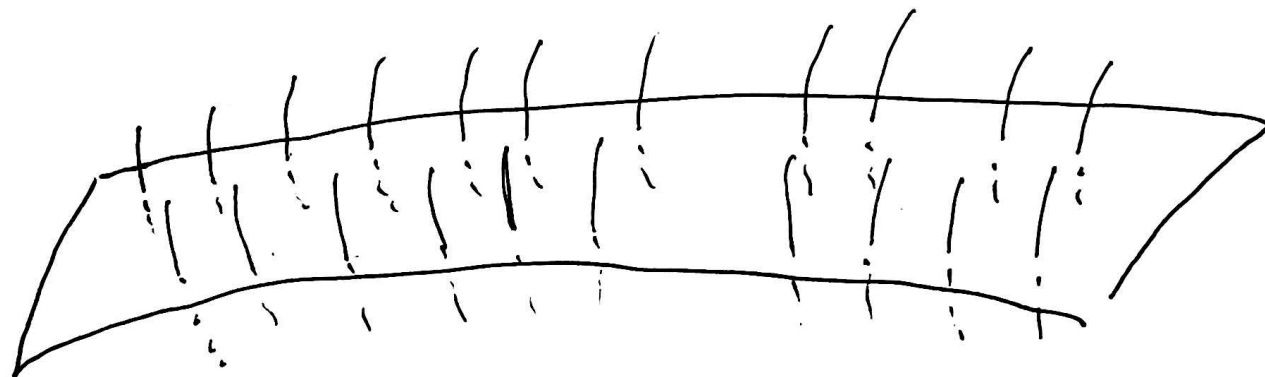
group

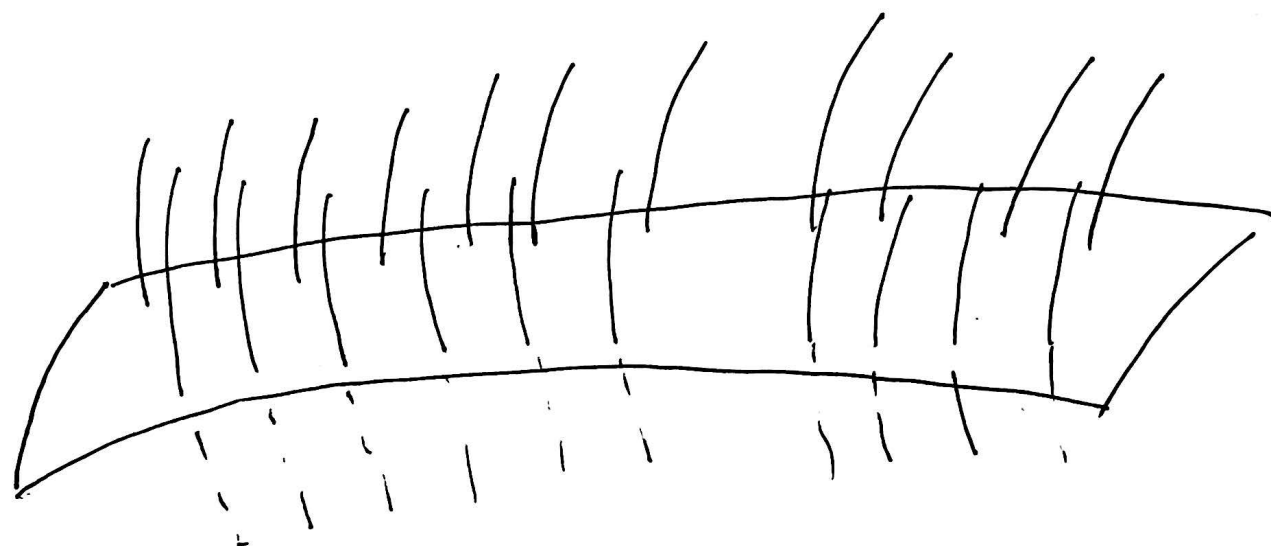
a space on
which G
acts

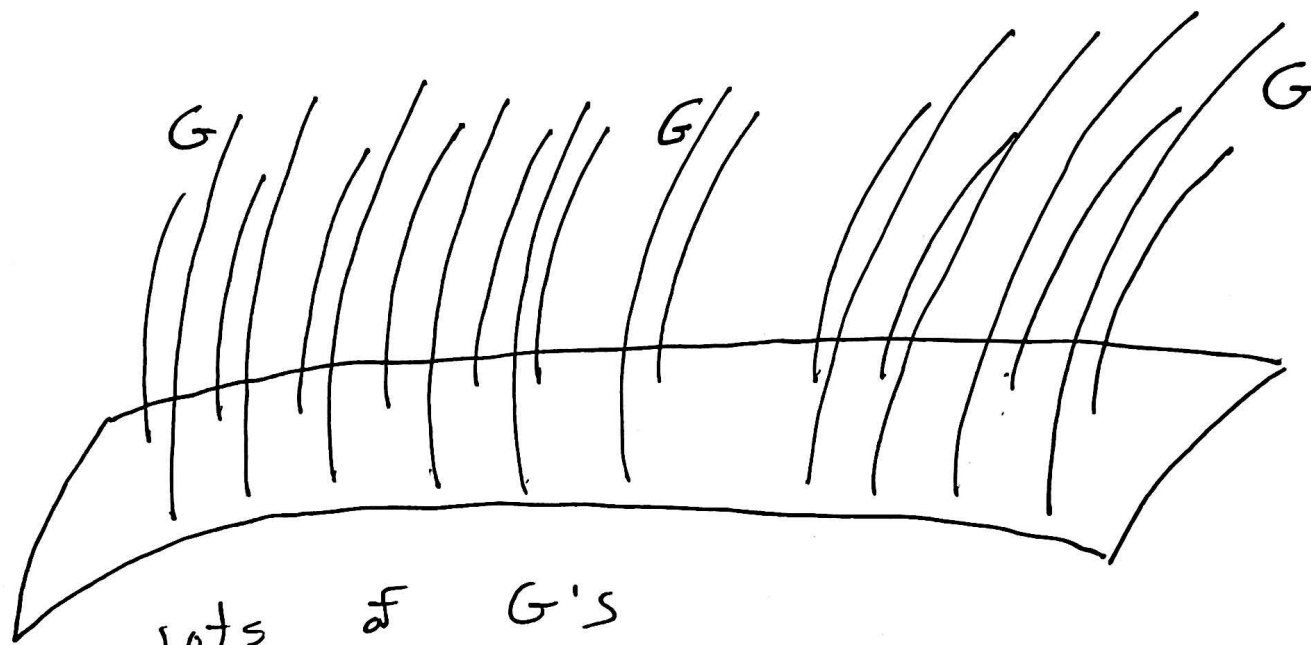
the quotient
space Q/G



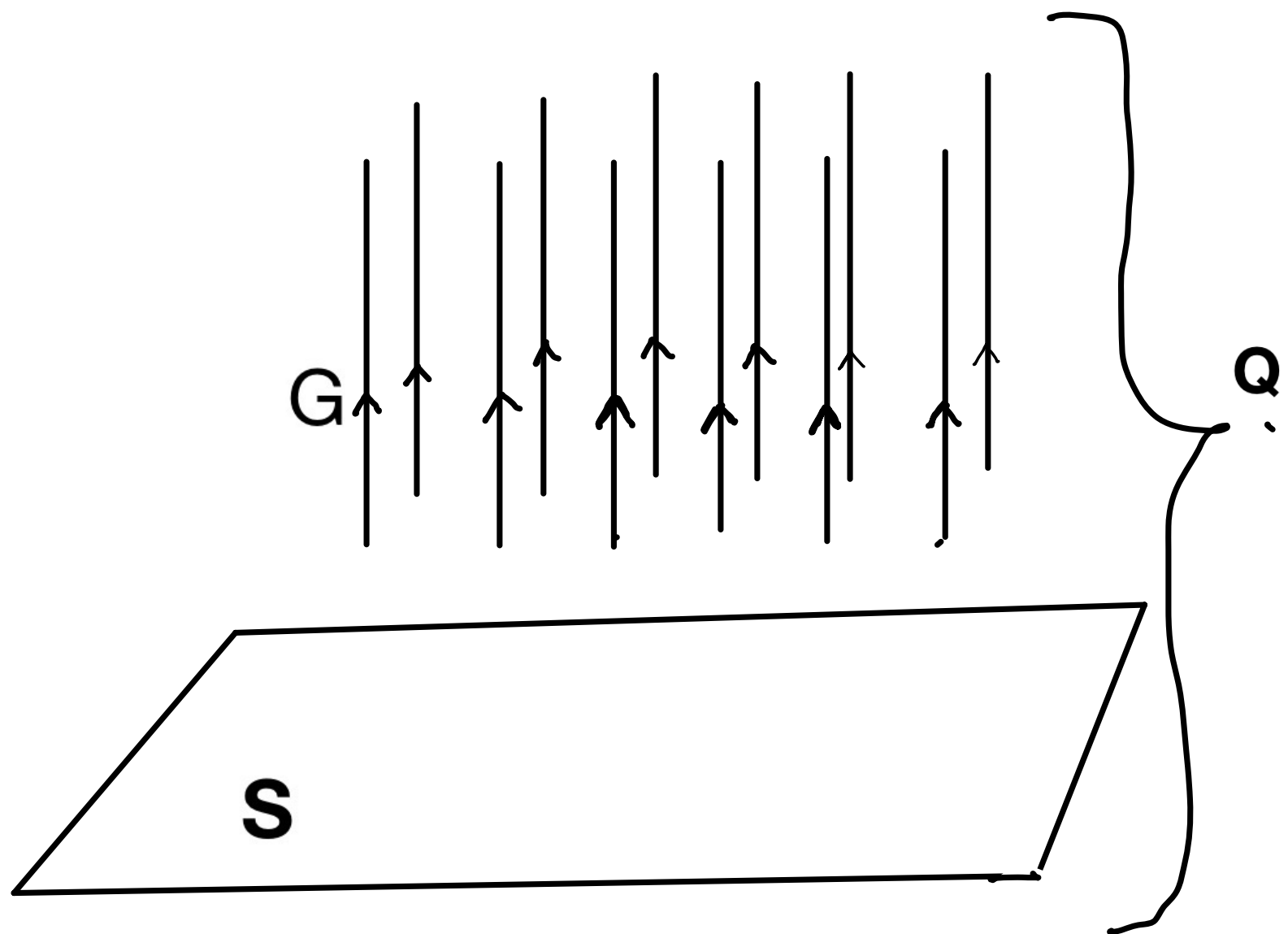




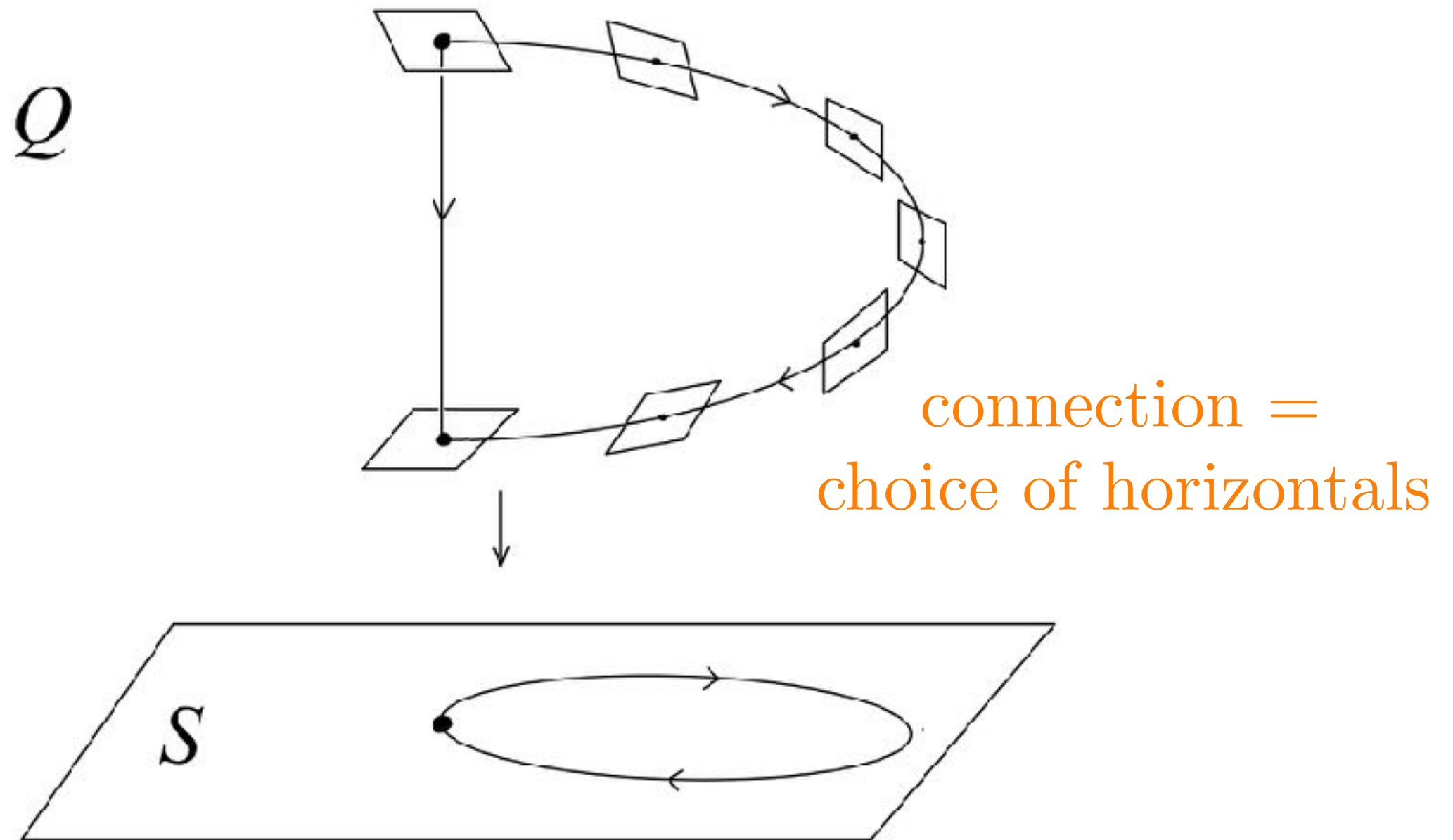


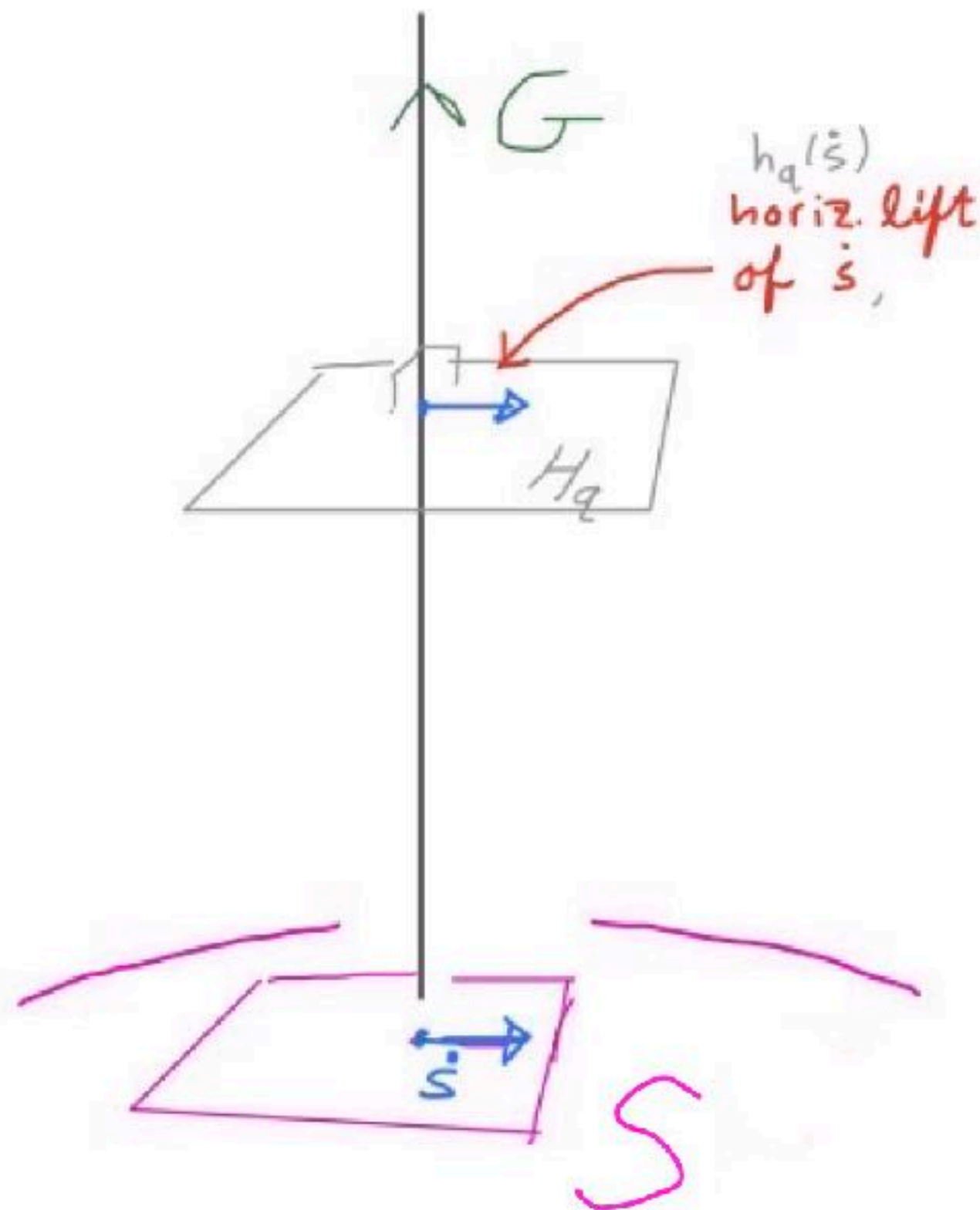


lots of G 's
one for each point of "space-time".



To lift paths from the base S
up to the total space Q use a connection :





gauge field =
 connection =
 choice of horizontals =
 choice of “PERPENDICULARS”
 to the fibers V



C. N. Yang and Robert Mills, 1999.

theory	group G fiber	total space Q	base space S	properties of gauge field A = connection
E and M	$U(1)$	$S \times U(1)$	space-time $\mathbb{R}^{3+1}$	vector potential (photon) variable
electroweak	$SU(2) \times U(1)$	$S \times G$	space-time	gauge potential (W and Z) variable
strong force QCD	$SU(3)$	$S \times G$	space-time	gauge potential (gluons) variable

Gauge theory: basic variables are
connections on a fixed principal bundle Berkeley. 1980-85

I witnessed a revolution in the understanding of
4-manifolds. It was driven by ideas from gauge
theory.

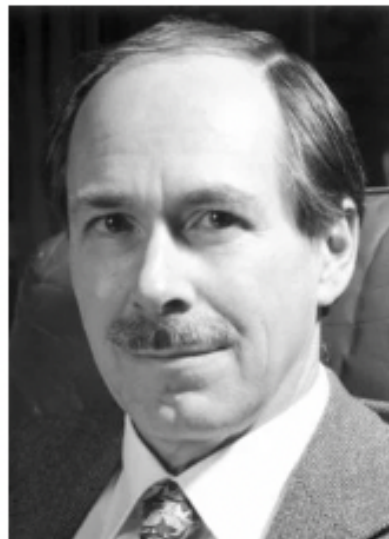
Method: attach the moduli space of solutions to
the “ASD Yang-Mills eqns”
to a 4-manifold.

results: exotic R^4 's ; which intersection
forms can be realized ..

Donaldson

Uhlenbeck

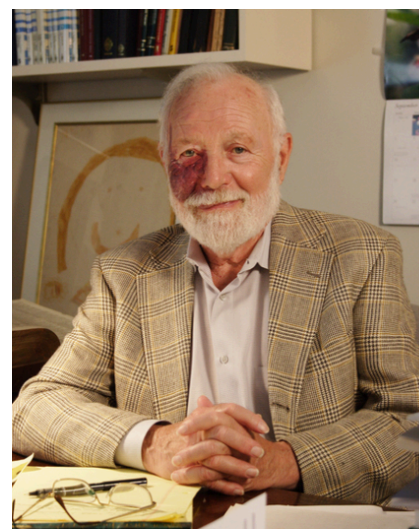
Taubes



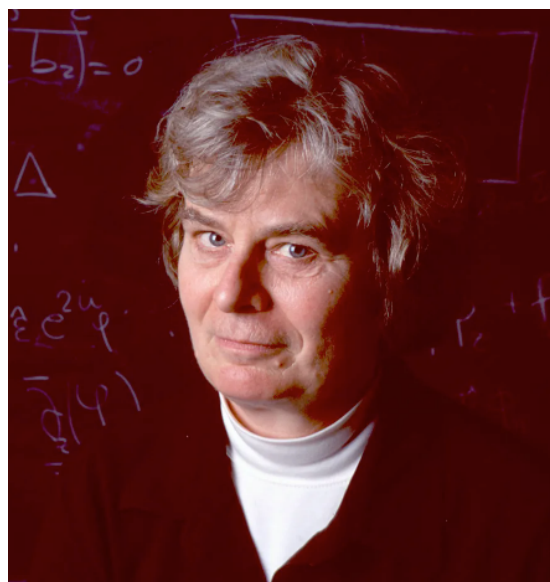
t'Hooft



Atiyah



Singer



Uhlenbek



Taubes



Donaldson



Geometry of self-propulsion at low Reynolds number

By **ALFRED SHAPER**[†] AND **FRANK WILCZEK**[‡]

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(Received 15 April 1987 and in revised form 12 July 1988)

The problem of swimming at low Reynolds number is formulated in terms of a gauge field on the space of shapes. Effective methods for computing this field, by solving a linear boundary-value problem, are described. We employ conformal-mapping techniques to calculate swimming motions for cylinders with a variety of cross-

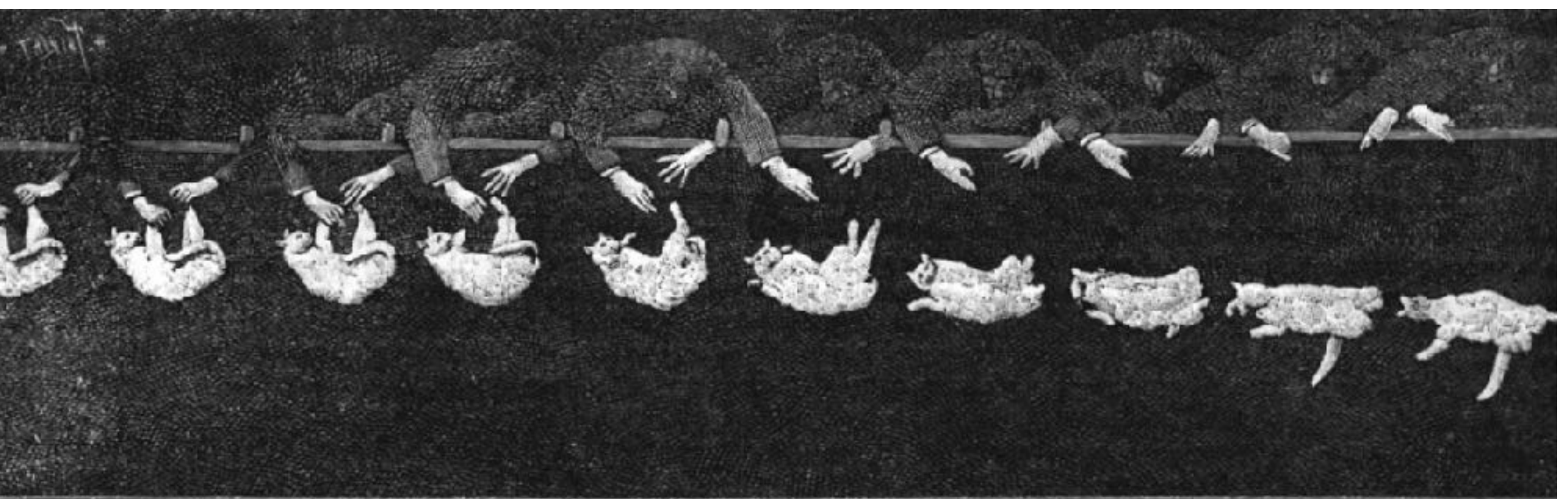


Nobel Prize
Frank Wilczek – Fact...

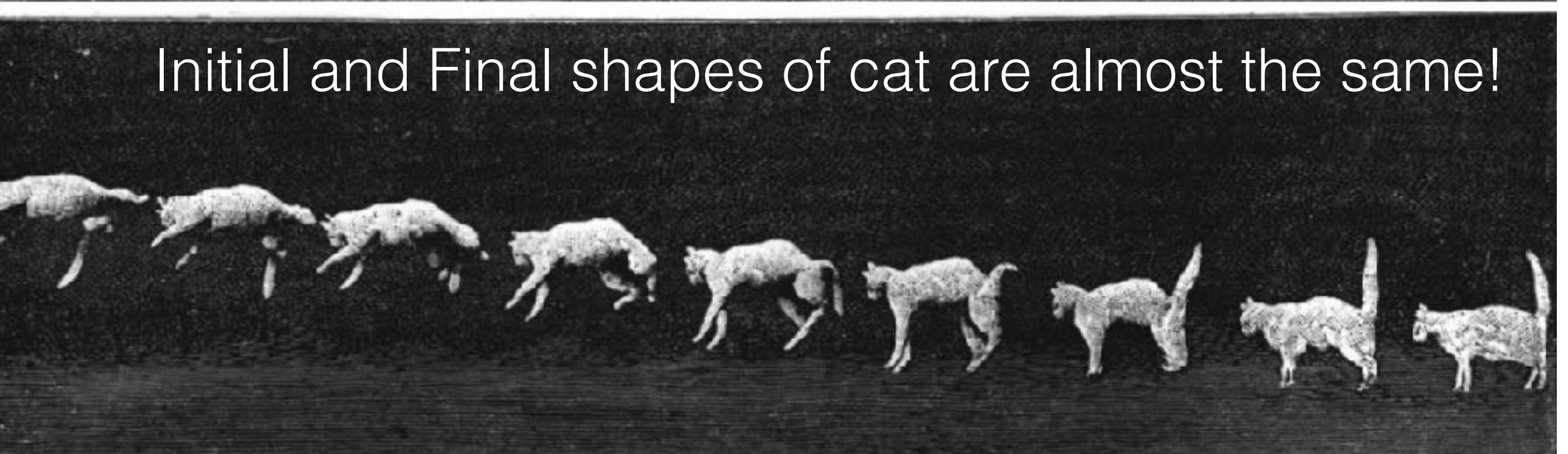


What is a shape?

.kinesthetics..
get up ..
freeze hands...

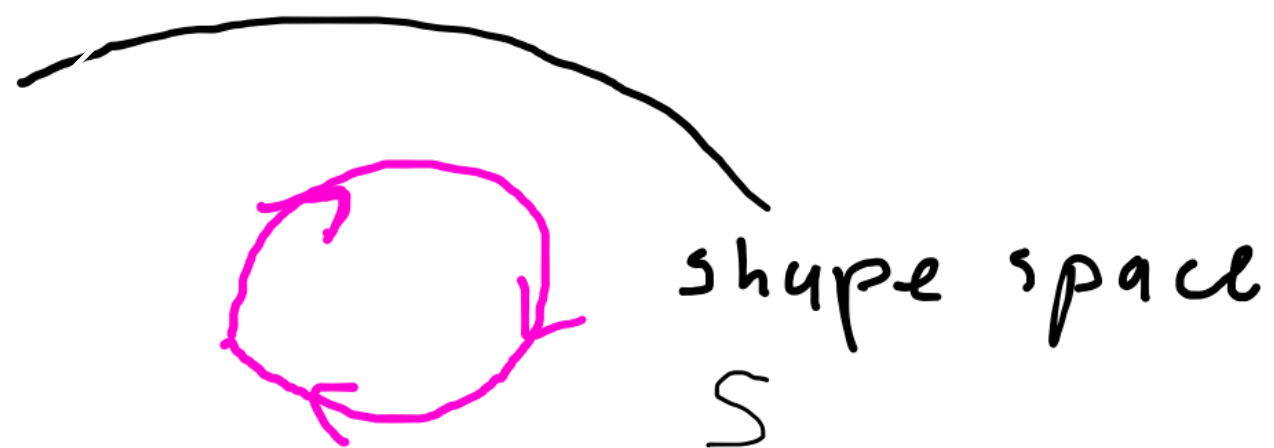
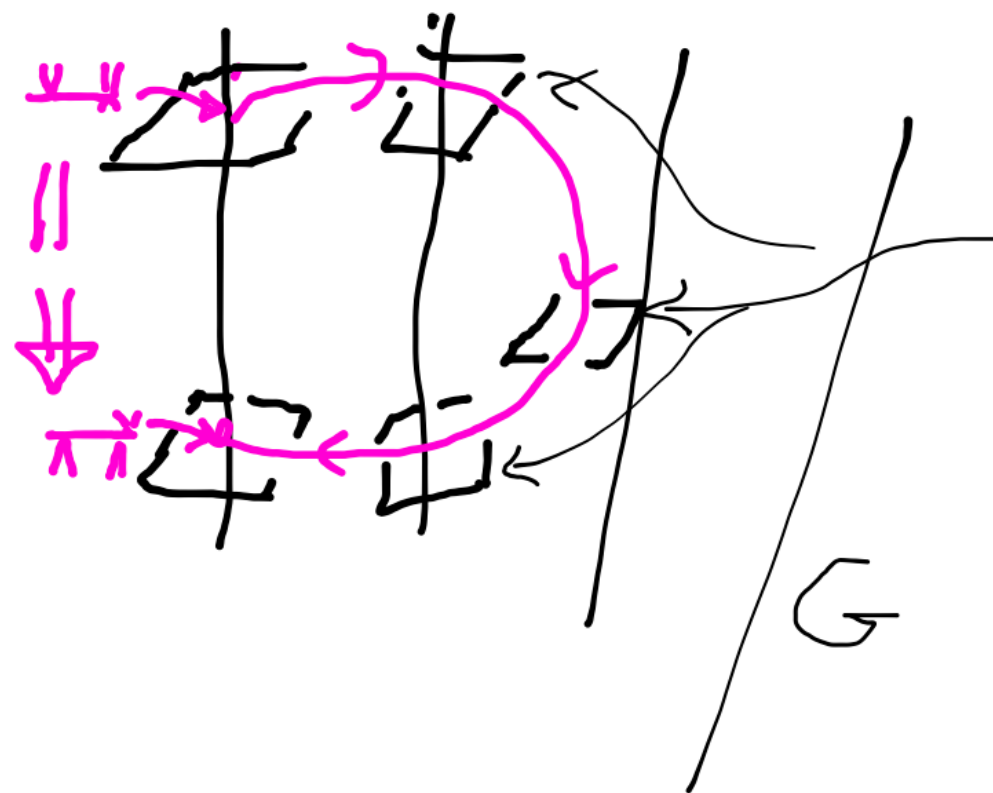


Initial and Final shapes of cat are almost the same!



(from 'Falling Cat' wiki page; a copy of a photo taken in 1894)

holonomy



loop in shape space

a principal **G**-bundle

$$G \rightarrow Q \rightarrow S$$

group
of rigid
motions

configuration
space of the cat;
points are
'located cats'

shape
space Q/G

Utility of principal bundle picture for understanding the strategies of the falling cat for righting herself

**A reorientation strategy
IS a loop in shape space**

**How does a loop in shape space
lift up to the total space Q ?**

i.e. what is the connection?

Connection defined by angular momentum = 0

so horizontal motions = motions with total
angular momentum zero

Why can we think of 'angular momentum' a gauge field?

FACT: a curve $q(t)$ in Q is perpendicular to the vertical space V if and only that curve has zero angular momentum.

$V =$ infinitesimal rotations

Perpendicular relative to what metric ?

KINETIC ENERGY METRIC.

Horizontal motions =
paths perpendicular to the group orbits,
or perpendicular to the 'vertical spaces'

where a 'vertical vector' is a vector tangent to a G-orbit

To show:

Horizontal motions =
motions with total angular momentum zero

Perpendicular **relative** to what metric on Q?

Model Q , the configuration space for the 'located cat' as a collection of point masses. So:

$$q = (q_1, q_2, \dots, q_N), q_a \in \mathbb{R}^3$$

represents a point of Q . Think of the q_a 's as 'marker points'. ('Foot', 'head', ...)
They have masses m_a . Define an inner product on Q for which the squared length of velocities v_a is twice their kinetic energy

K:

$$\begin{aligned} K(\dot{q}) &= \frac{1}{2} \sum m_a |v_a|^2, v_a = \dot{q}_a \\ &= \frac{1}{2} \langle v, v \rangle \end{aligned}$$

so that :

$$\langle q, q' \rangle = \sum m_a q_a \cdot q'_a$$

Mass metric

To work out

$$H_q = (V_q)^\perp$$

we need V_q , the tangent space to the group orbit through q :

$$G(q) = \{Rq := (Rq_1, \dots, Rq_N) : R \in SO(3) \text{ a rotation}\}$$

Infinitesimal rotations are given by cross products:

$$\frac{d}{d\epsilon} R(\epsilon) q_a \big|_{\epsilon=0} = \omega \times q_a, \quad \omega, q_a \in \mathbb{R}^3; R(0) = Id$$

So:

$$V_q = \{ \text{“}\omega \times q'' \text{”} : \omega \in \mathbb{R}^3 \}$$

where

$$\text{“}\omega \times q'' \text{”} = (\omega \times q_1, \omega \times q_2, \dots, \omega \times q_N)$$

Suppose that v in Q is mass-metric
perpendicular to all these vertical vectors:

$$\begin{aligned} 0 &= \langle v, \omega \times q \rangle \\ &= \sum m_a v_a \cdot (\omega \times q_a) \\ &= \sum m_a \omega \cdot (q_a \times v_a) \\ &= \omega \cdot (\sum m_a q_a \times v_a) \\ &= \omega \cdot (\sum q_a \times m_a v_a) \end{aligned}$$

for all $\omega \in \mathbb{R}^3$

This is true iff $\sum q_a \times m_a v_a = 0$

But this says the total angular momentum is zero!

$\sum q_a \times m_a v_a =$ the angular momentum associated to q, v

!!

Summary: A deformation, or 'motion' $q(t)$ of a located shape $q(0)$ is mass-metric perpendicular to the group orbits if and only if its total angular momentum is zero.

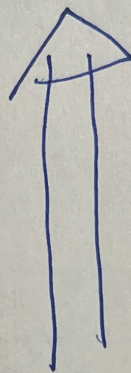
This connects geometry to physics !

The algebra and notation of last few slides works exactly as it stands for the N-body problem

Angular momentum IS the gauge-field.

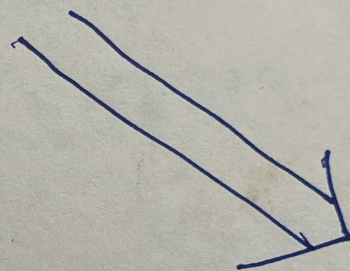
Ang. mom = 0 $\longleftrightarrow$ horizontal.

biology,
engineering



sub Riem
geom.

optimal
control,
geometry



N-body
problem

What is the optimal reorientation strategy for the falling cat?

What is the shortest loop (in S) with a given holonomy (in G)?

On solving the Isoholonomic problem.

Theorem. The optimal paths – the shortest loops with a given holonomy– are characterized by their projection $s(t)$ to S as follows.

Either

(I) $s(t)$ behaves like a ‘classical quark’ under the influence of the curvature $F = DA$,
or

(II) $\dot{s}(t)$ lies in a kernel of F :

$\exists$ a nonvanishing curve $\xi(t) \in \mathfrak{g}^*$ such that $\xi(t) \cdot F_{s(t)}(\dot{s}(t), \cdot) = 0$ holds along $s(t)$

$$\nabla_{\dot{s}} \dot{s} = \xi F(\dot{s}, \cdot)$$

$$(I) \quad \frac{D\xi}{dt} = 0$$

or

$$\ddot{s}^\mu + \Gamma_{\alpha\beta}^\mu \dot{s}^\alpha \dot{s}^\beta = \xi_a F_{\beta\gamma}^a \dot{s}^\beta g^{\gamma\mu}$$

$$\dot{\xi}_a + c_{ab}^d A_\mu^b \dot{s}^\mu \xi_d$$

If $\xi(t) = 0$ (“charge zero”) then eqns are $\nabla_{\dot{s}} \dot{s} = 0$,
i.e. the geod eqns on S . This fact is $\iff$ to the fact that geodesics on S
are the projections of geodesics in S perpendicular to the fibers of $\pi : Q \rightarrow S$.

$$(II) \quad 0 = \xi_a(t) F_{\beta\mu}^a \dot{s}^\beta$$

Proof. $U(1)$ case. The horizontal constraint rel. connection A reads

$$\dot{\theta} - \sum A_{\mu} \dot{x}^{\mu} = 0$$

Introduce a Lagrange multiplier λ to enforce horizontality of curves. Minimizing length is equivalent to minimizing kinetic energy at fixed time. THUS minimizers extremize

$$L = \frac{1}{2} \sum (\dot{x}^{\mu})^2 + \lambda (\dot{\theta} - \sum A_{\mu} \dot{x}^{\mu})$$

Find: $\dot{\lambda} = 0$. Think of $\lambda = e =$ ‘charge’.
Then $\lambda \dot{\theta} =$ exact time derivative. Ignore!

We have written the Lagrangian for a particle of mass one, charge e moving under the influence of the magnetic field $B = dA$.

QED for case (I)

Case (II) : almost all of us were taught a *wrong version* of the method of Lagrange multipliers. You must also account for the extremals of $\int(\dot{\theta} - \sum A_\mu \dot{x}^\mu)$

Problem: Min. F subject to constraint $G = 0$.

We are taught to form $F + \lambda G \dots$

This method can MISS minimizers that are critical points of G

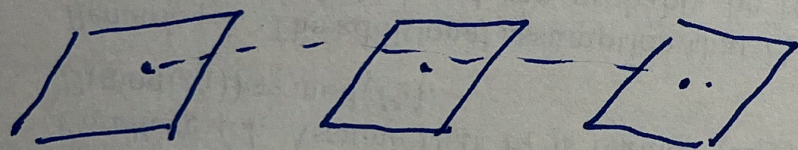
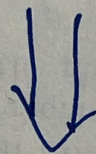
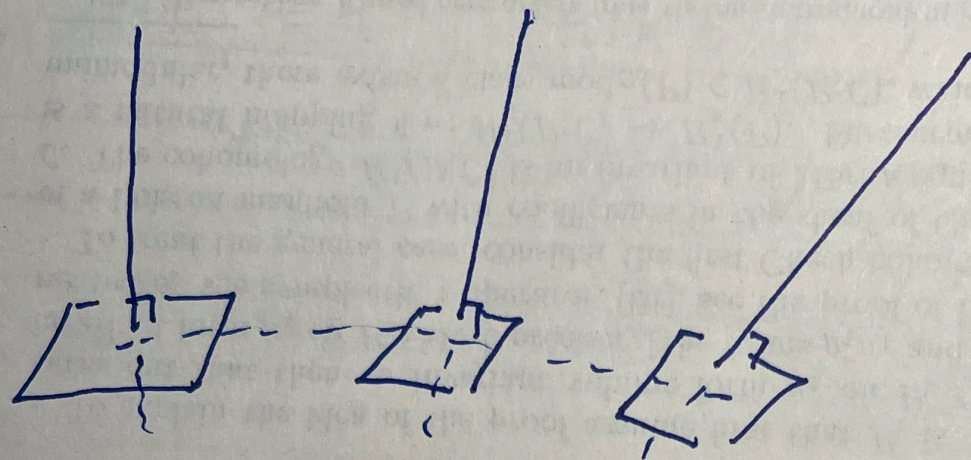
Correct ‘Lagrange multipliers’ approach:

$\lambda_0 F + \lambda G$ where $(\lambda_0, \lambda) \neq (0, 0)$.

Allows for $(\lambda_0, \lambda) = (0, 1)$.

In the planar case of $S = \mathbb{R}^2$, $G = \mathbb{S}^1$ these new extremals are the curves defined by $B = 0$ where B is the planar (scalar) magnetic field of $F = B(x, y)dx \wedge dy$.

[eventually Led to “Hearing the zero locus of a magnetic field”.]



Forget the
group!

Forget the V 's!

Keep the H 's
with their inner
products, $\Rightarrow$

"subRiemannian geometry"
aka, "Carnot-Carathéodory
geometry"

What is the optimal reorientation strategy for the falling cat?

What is the shortest loop (in S) with a given holonomy (in G)?

**What are the subRiemannian geodesics joining two given points of Q ?
(say the points q_0 and $q_1 = g q_0$, where g = desired holonomy)**

Case (I): ‘normal’

Case (II): ‘abnormal’ or ‘singular’

Theorem. [M-; 1989] There exist subRiemannian metrics which admit strictly abnormal geodesics:

“geodesics which do not satisfy the
(Hamiltonian) geodesic equations”.

These Abnormal geodesics are difficult to understand.
Their existence re-opened a basic question:

Are all subRiemannian geodesics smooth?

Open from 1989 to 2025.

Solved!! Posted in January 2025. !!

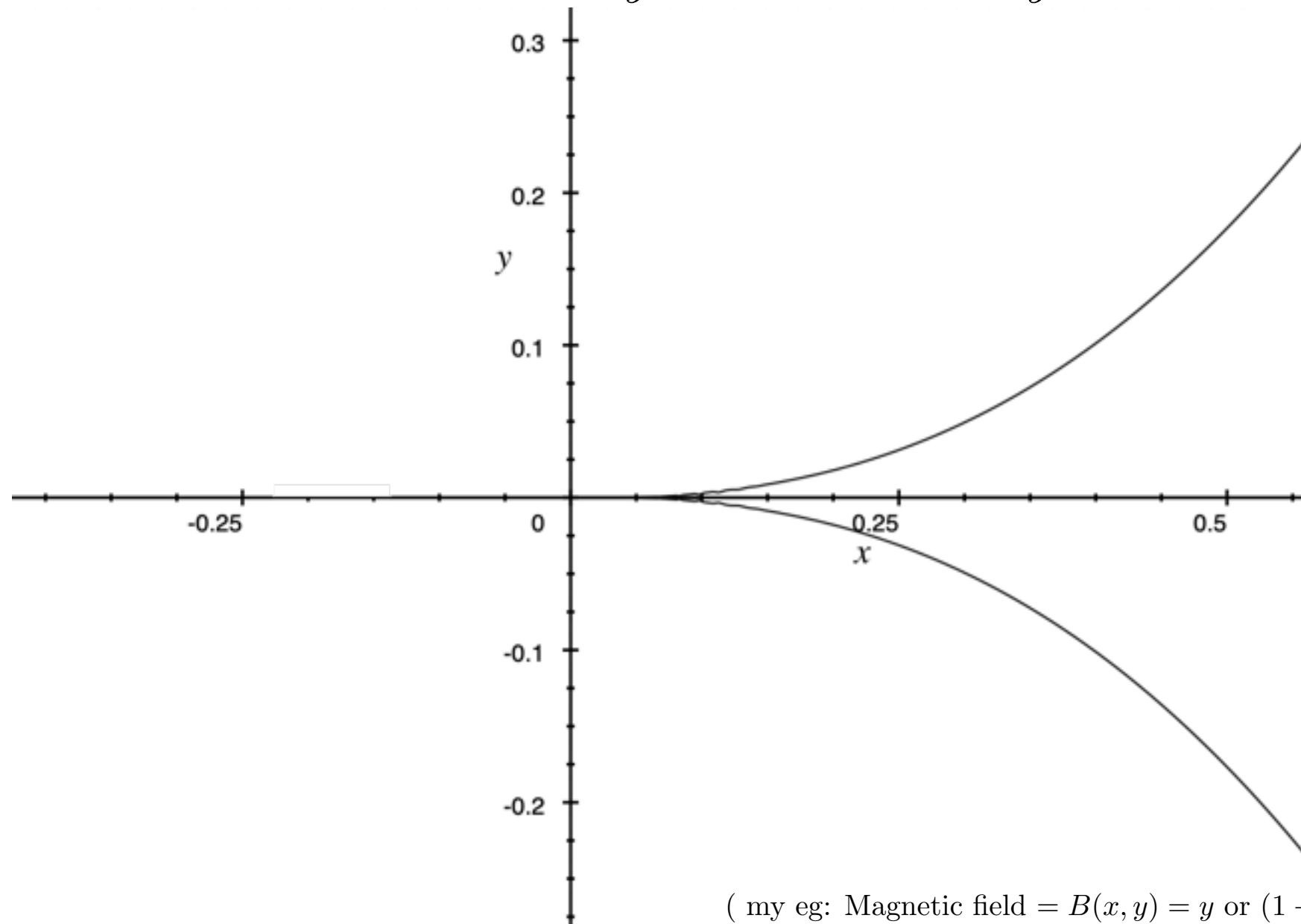
[arXiv:2501.18920](https://arxiv.org/abs/2501.18920) [[pdf](#), [other](#)] [math.DG](#)

[math.MG](#)

Not all sub-Riemannian minimizing geodesics are smooth

Authors: [Yacine Chitour](#), [Frédéric Jean](#), [Roberto Monti](#), [Ludovic Rifford](#), [Ludovic Sacchelli](#), [Mario Sigalotti](#), [Alessandro Socionovo](#)

Magnetic field = $B(x, y) = (y^5 - x^2)y$
 Distribution: $Span(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + (y^5 - x^2)^2 \frac{\partial}{\partial y})$



(my eg: Magnetic field = $B(x, y) = y$ or $(1 - (x^2 + y^2))$
 Distribution: $Span(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + y \frac{\partial}{\partial y})$

OPEN QUESTION

Can you have a non-smooth point of a sR geodesic as
and interior point of the geodesic arc?

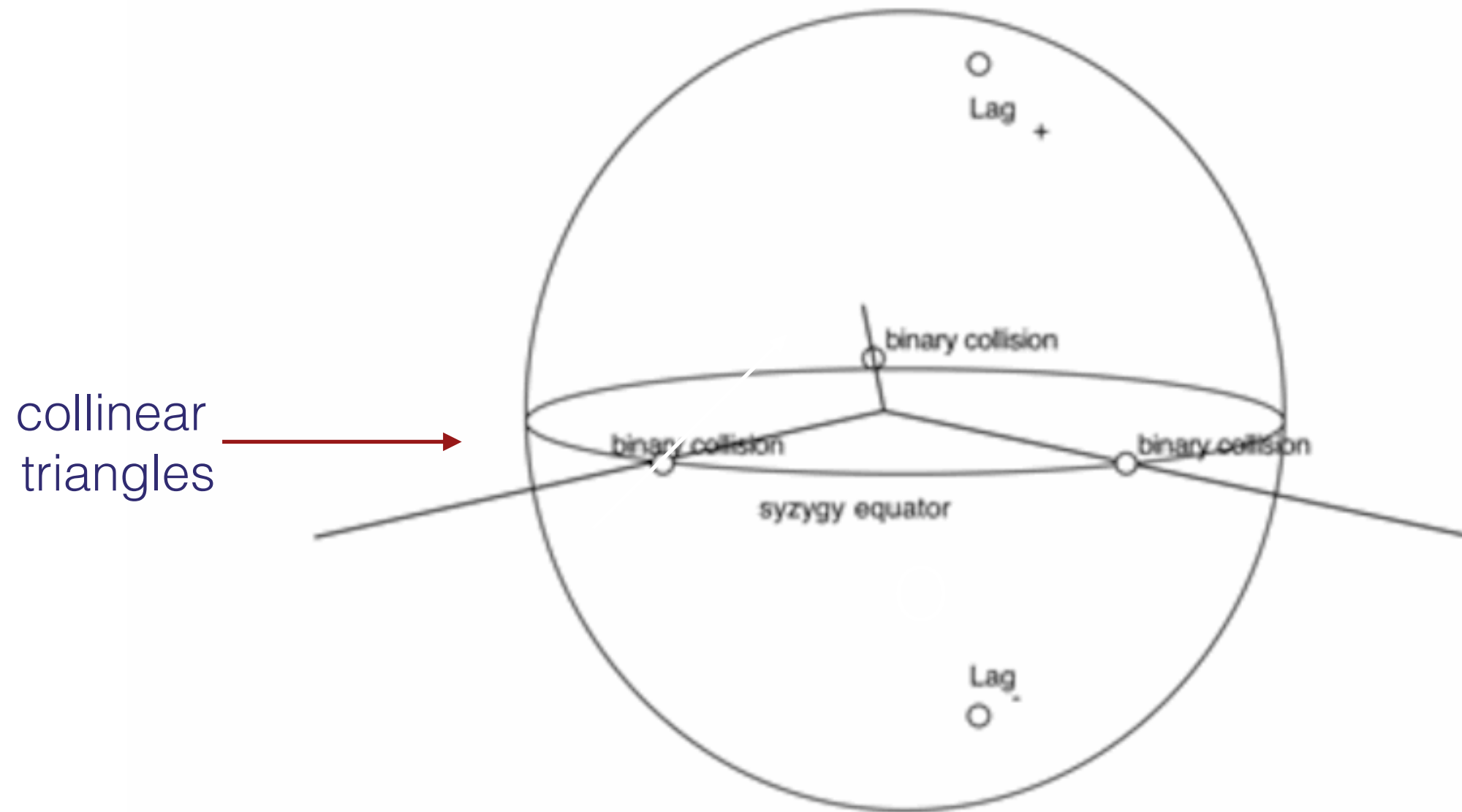
i.e.

Can you find a minimizing sR geodesic $c : [-\epsilon, \epsilon] \rightarrow Q$
parameterized by ARC-LENGTH and having 0 as a NON-Smooth point?

On to the three-body problem

...Hands... to form moving triangle

$d=2, N=3$:



SHAPE SPHERE

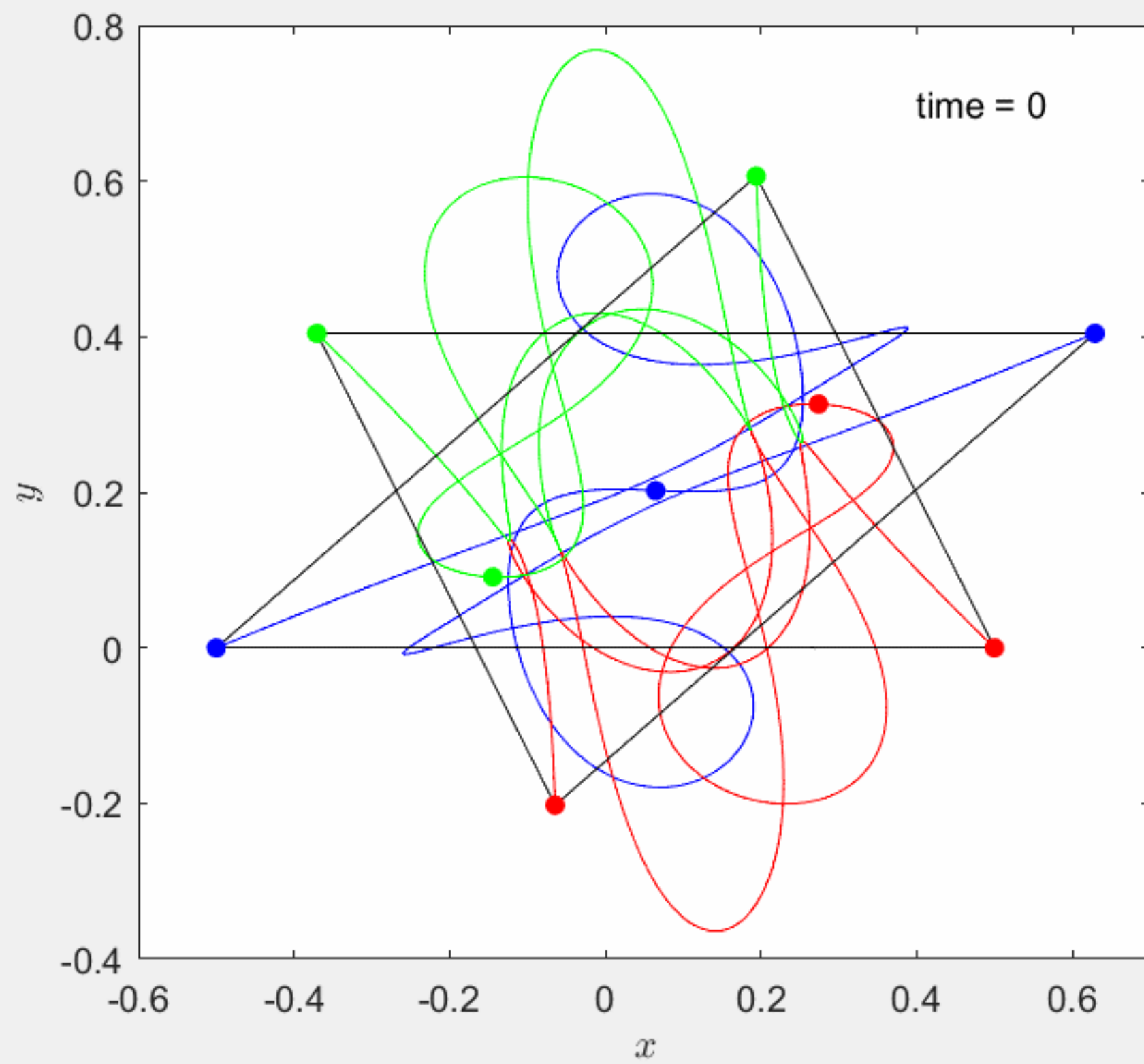
Oriented similarity classes
of triangles

$\subset$

SHAPE SPACE

Oriented congruence classes
of triangles

[100s](http://three-body.ipb.ac.rs/) by Suvakov & <http://three-body.ipb.ac.rs/>.

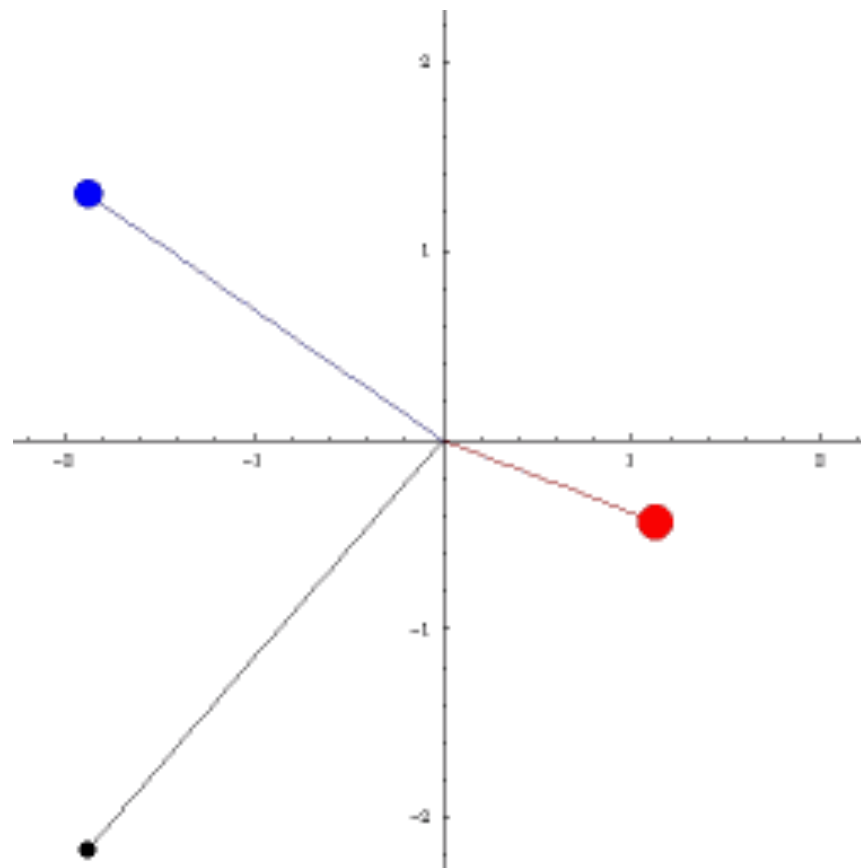


`Infinitely Many Syzygies'

Theorem. Every solution to the three-body problem with zero angular momentum and negative energy suffers collinearities EXCEPT for one family, the Lagrange equilateral 'dropped' family. If the time-domain of definition of the solution is infinite then it suffers infinitely many collinearities.

cone point = total collision

TCs = total collision solutions
= solutions which end in total collision



eg. The Lagrange homothety solution



Galileo 1632; "Dialogo .." :

The laws of physics are
invariant under my group.

My group contains your group G .

G = group of rigid motions of space
= translations, rotations, ~~reflections~~



Newton, Principia, 1687:

The laws of physics can be written as differential equations which are invariant under Galileo's group.

$$\ddot{q} = \nabla U(q)$$

$$U = G \sum_{a < b} \frac{m_a m_b}{r_{ab}} \text{ where } r_{ab} = |q_a - q_b|$$

$d=2, N=3$:

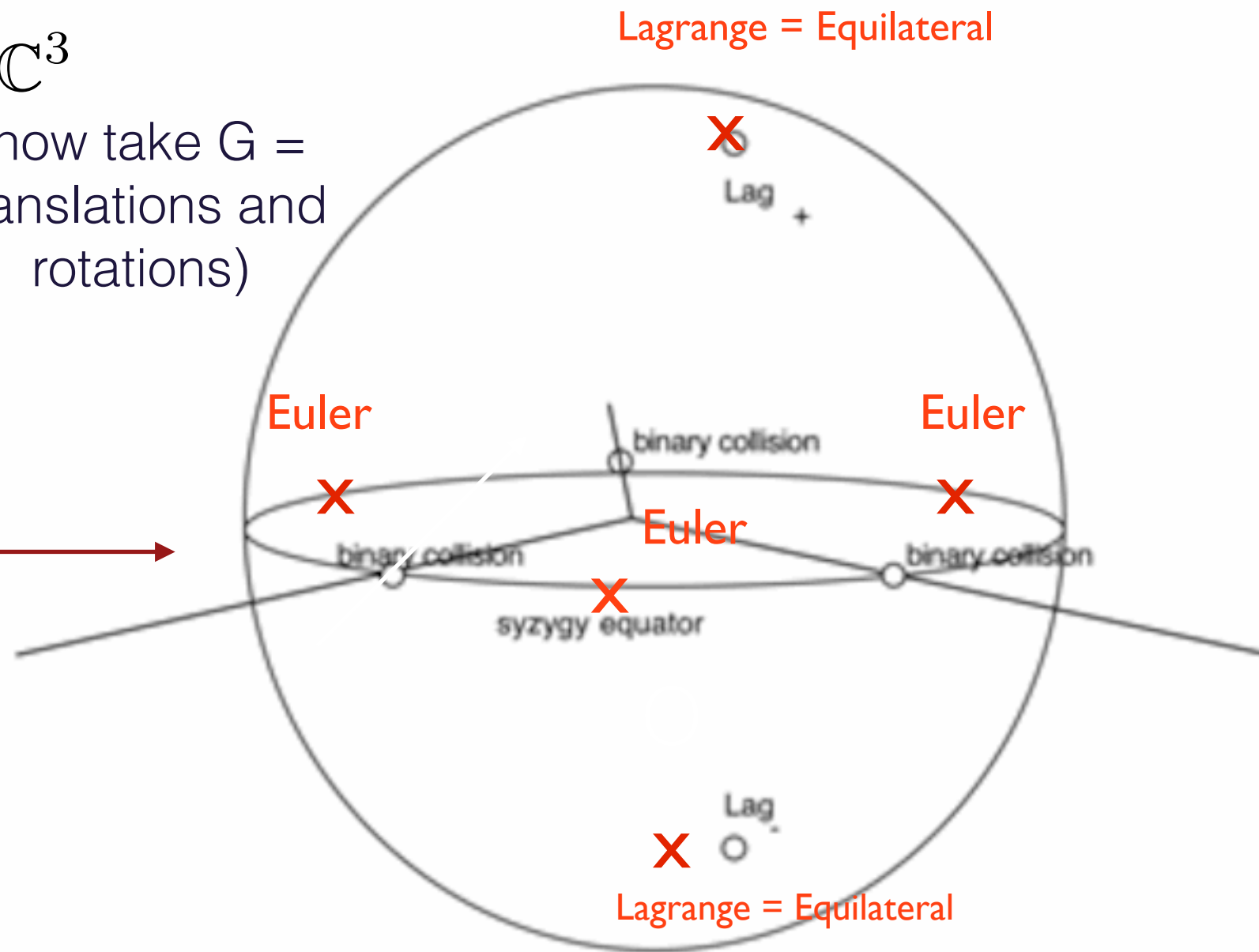
$$Q = (\mathbb{R}^2)^3 = \mathbb{C}^3$$

$\downarrow G$

(now take G =
translations and
rotations)

$$S = \mathbb{R}^3$$

collinear
triangles



SHAPE SPHERE

Oriented similarity classes
of triangles

$\subset$

SHAPE SPACE

Oriented congruence classes
of triangles

$$q = (q_1, q_2, \dots, q_N), q_a \in \mathbb{R}^3 \text{ or } \mathbb{R}^2$$

as before.

Positions q_a . Masses m_a .

Mass inner product: $\langle q, q' \rangle = \sum m_a q_a \cdot q'_a$

$$dU(q)(v) = \langle \nabla U(q), v \rangle$$

$$q = (q_1, q_2, \dots, q_N), q_a \in \mathbb{R}^3 \text{ or } \mathbb{R}^2$$

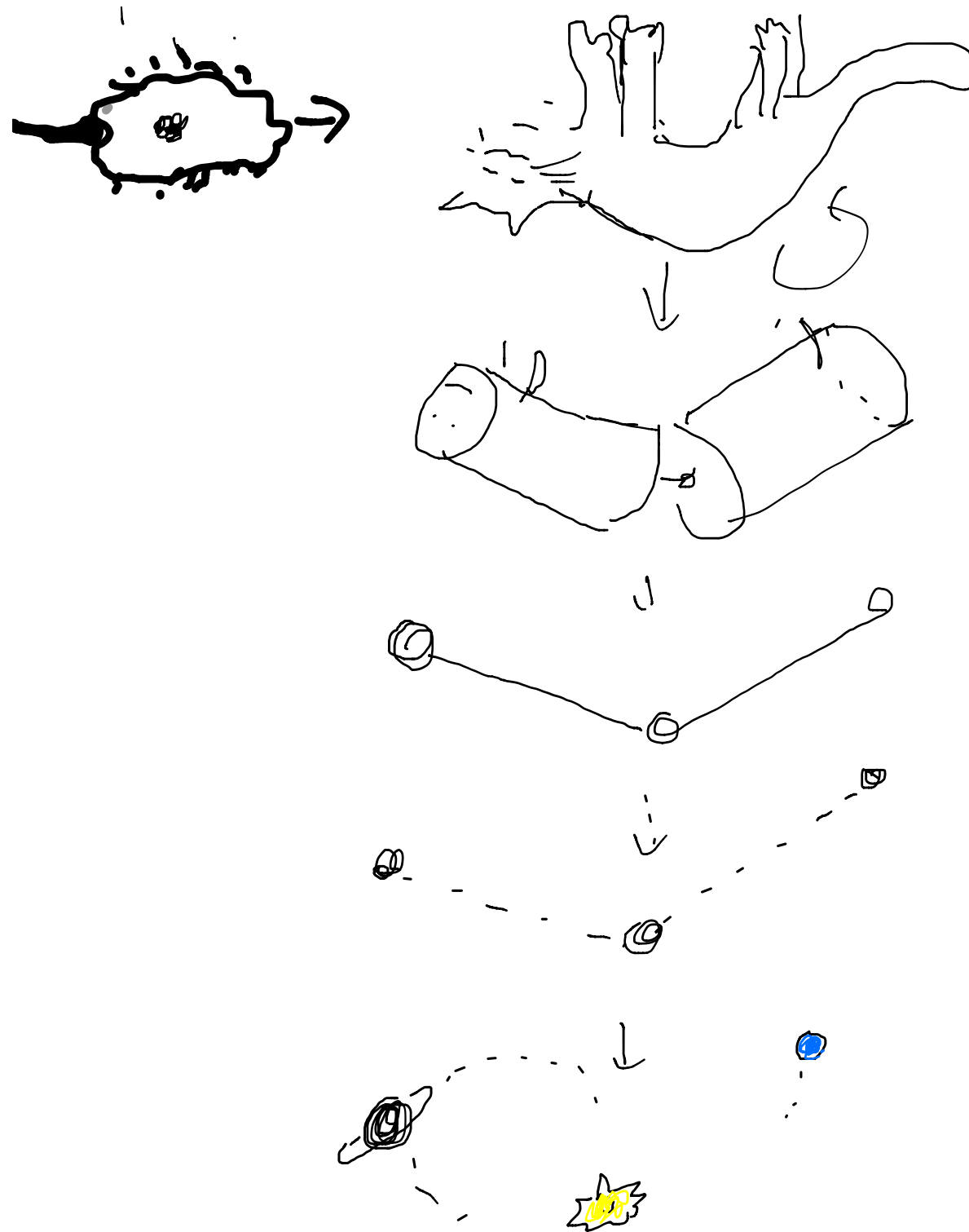
as before.

Positions q_a . Masses m_a .

Mass inner product: $\langle q, q' \rangle = \sum m_a q_a \cdot q'_a$

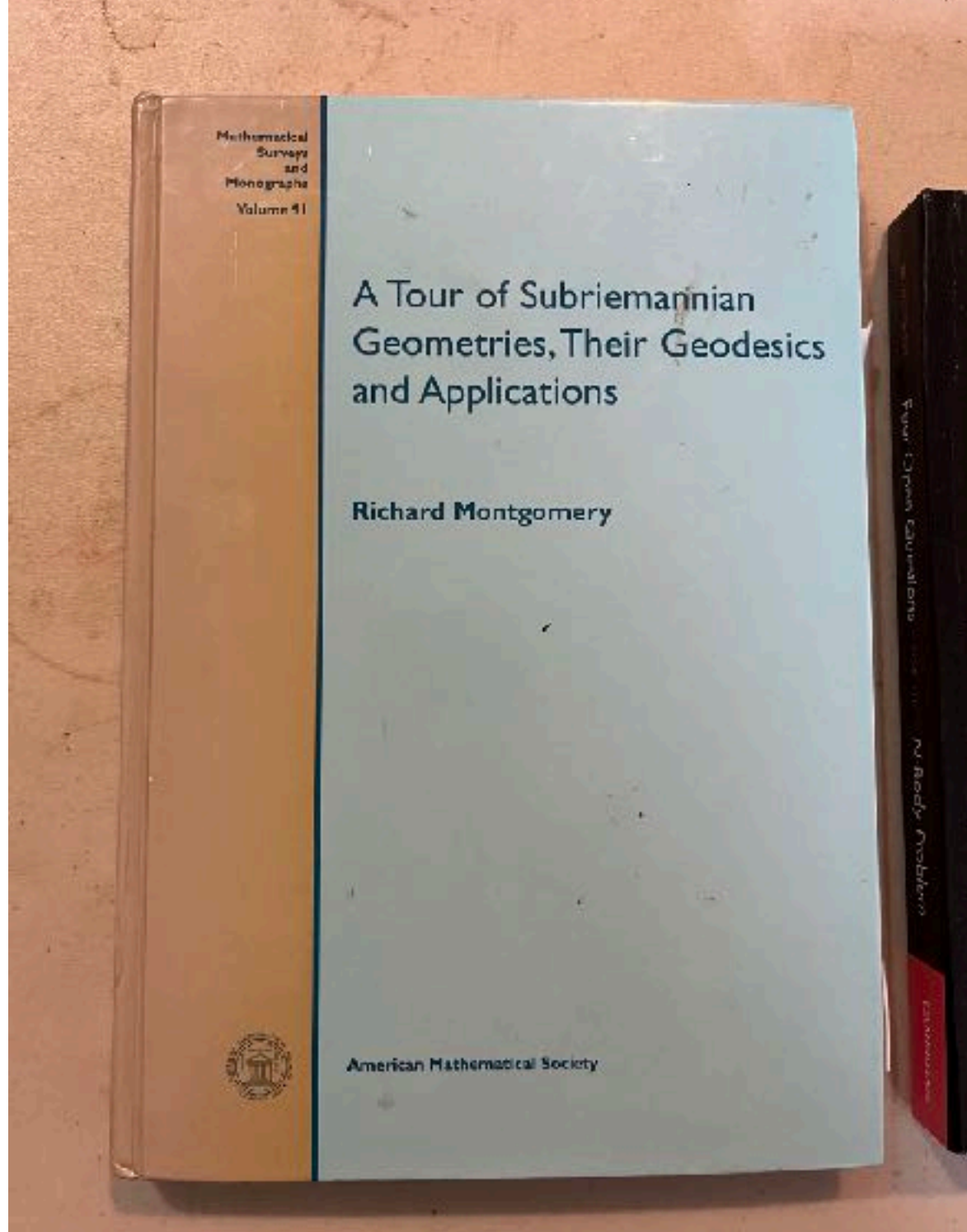
$$dU(q)(v) = \langle \nabla U(q), v \rangle$$

SUMMARY

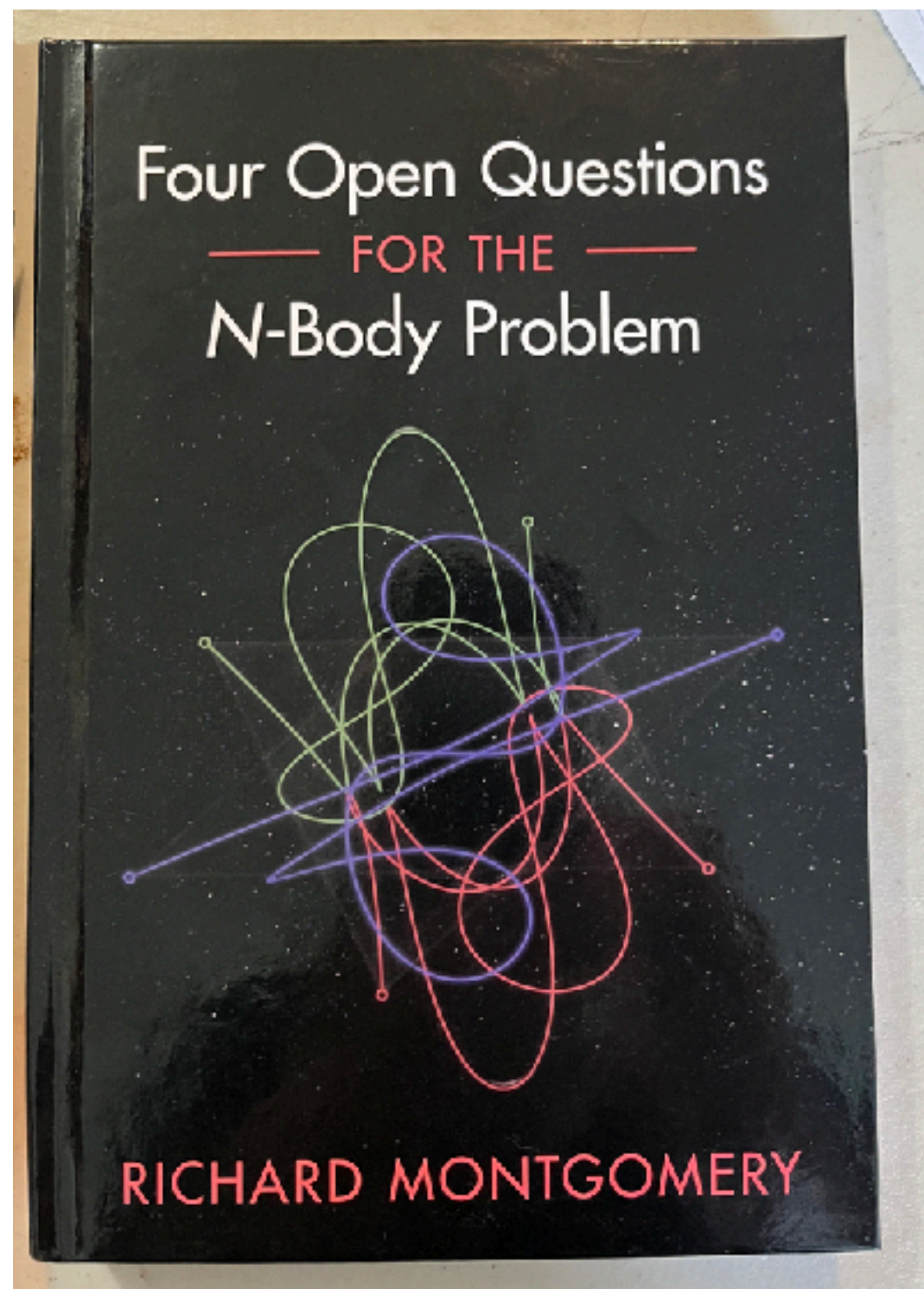


A summary of
my career.

theory	group G fiber	total space Q	base space S	properties of gauge field A = connection
E and M	$U(1)$	$S \times U(1)$	space-time $\mathbb{R}^{3+1}$	vector potential (photon) variable
electroweak	$SU(2) \times U(1)$	$S \times G$	space-time	gauge potential (W and Z) variable
strong force QCD	$SU(3)$	$S \times G$	space-time	gauge potential (gluons) variable
Falling Cat	$SO(3)$	config. space of located cats	shape space of cat	angular momentum zero defines horizontal FIXED!
Planar 3-body problem	$SE(2)$ $= SO(2) \ltimes \mathbb{R}^2$	config. space $(\mathbb{R}^2)^3 = \mathbb{C}^3$	shape space = = space of congruence classes of planar triangles $\mathbb{R}^3 = Cone(\mathbb{S}^2)$	angular momentum zero defines horizontal FIXED!
Planar N-body problem	$SE(2)$ $= SO(2) \ltimes \mathbb{R}^2$	config. space $(\mathbb{R}^2)^N = \mathbb{C}^N$	shape space = space of congruence classes of planar N-gons $= Cone(\mathbb{CP}^{N-2})$	angular momentum zero defines horizontal FIXED!
Berry phase in QM	$U(1) = \mathbb{S}^1$ group of phases	sphere in Hilbert space eg: $\mathbb{S}^{2N-1} \subset \mathbb{C}^N$	projective Hilbert space $\mathbb{CP}^{N-1}$	induced by Hilbert space inner product FIXED!
eg: $N = 2$		HOPF	FIBRATION!	



2000



2025

OR..

my web site; esp
the Monthly article

END !

OVERFLOW ...

Unifying theme: $G \rightarrow Q \rightarrow S$

$$G \curvearrowright_{isometric} (Q, ds^2) \rightarrow S = Q/G$$

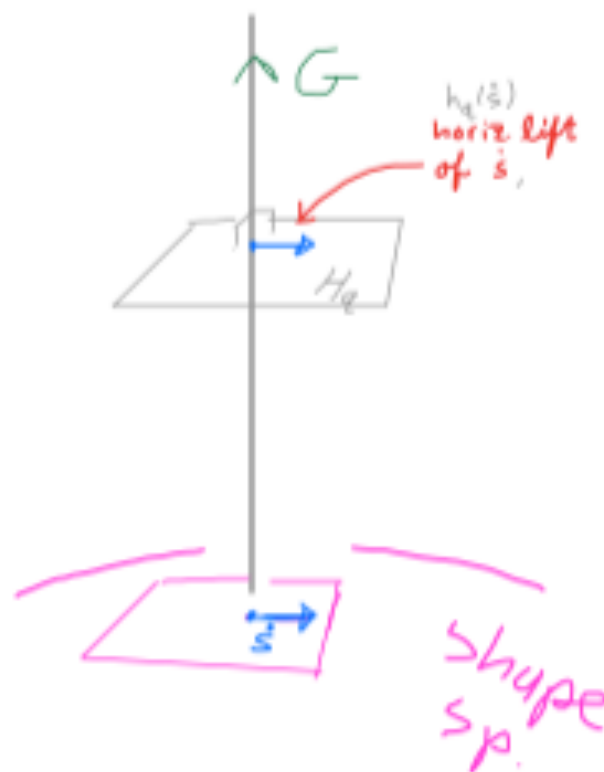
Lie group

acting freely
and isometrically

on a Riemannian
manifold

base space
= quotient space
= shape space,
inherits a Riem.
metric

and describe
the 'subRiemannian
geometry' here!

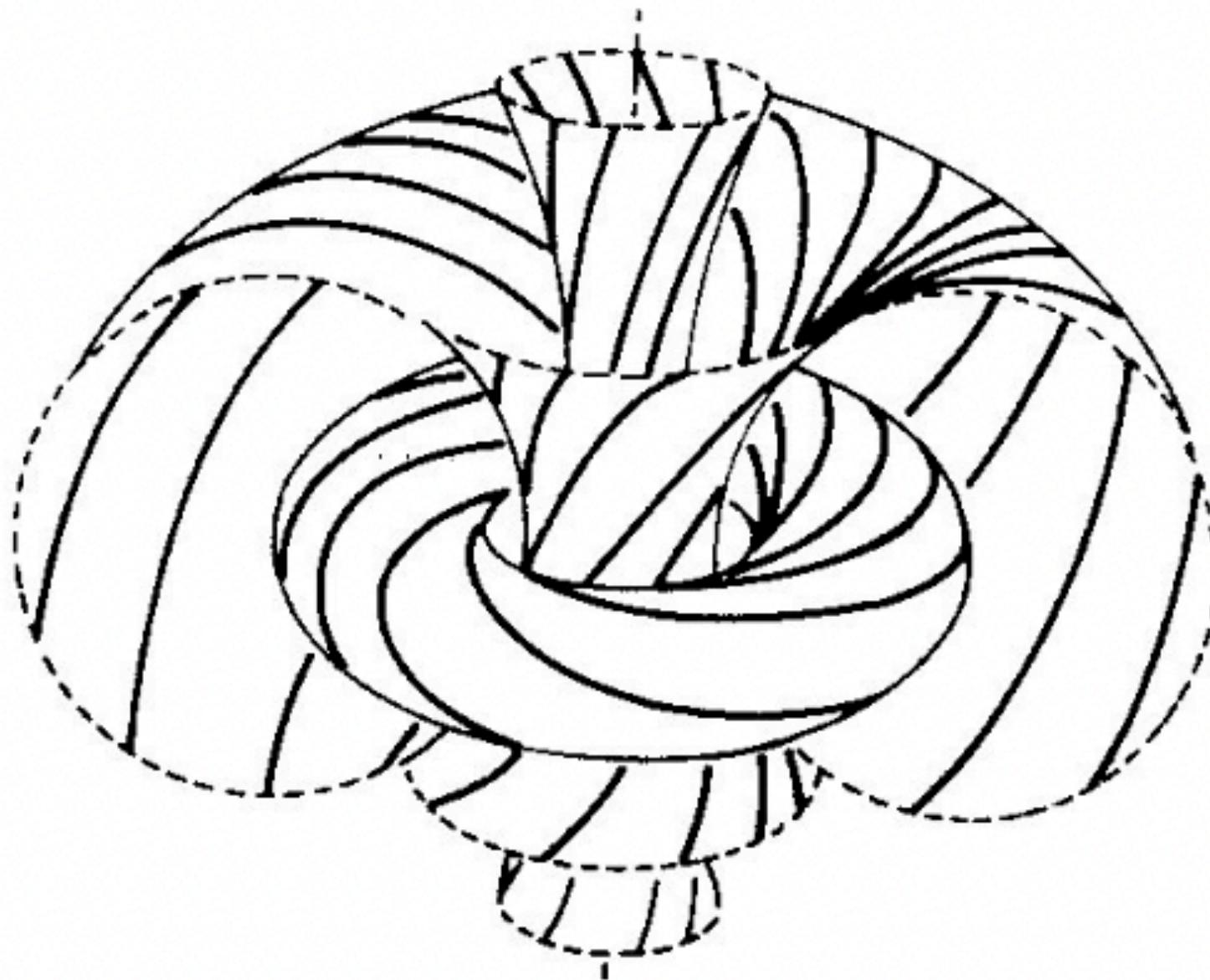


commercial break!
advertisement from our sponsor!

$$G \rightarrow Q \rightarrow S$$

$$S^1 \rightarrow S^3 \rightarrow S^2$$

Hopf!



*this talk is sponsored by
CIMAT and the
Hopf fibration*

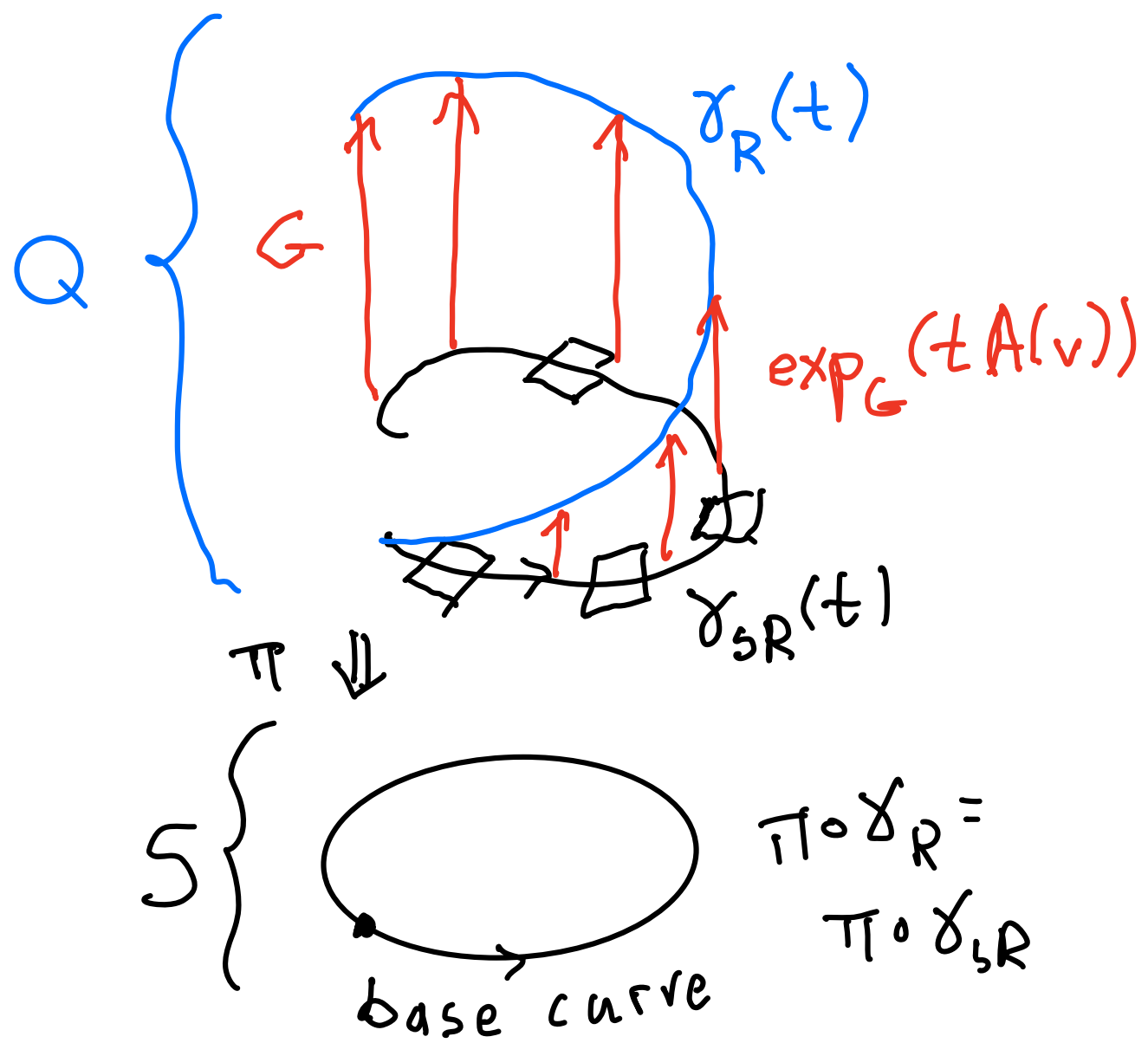
THEOREM 11.8 (on geodesics). Let $\pi : Q \rightarrow M$ be a principal G -bundle with a Riemannian metric of constant bi-invariant type. Let $\mathcal{H}$ be the induced connection, with $\mathfrak{g}$ -valued connection one-form A . Let $\exp_R$ be the Riemannian exponential map, so that $\gamma_R(t) = \exp_R(tv)$ is the Riemannian geodesic through q with initial velocity $v \in T_q Q$. Then any horizontal lift γ of the projection $\pi \circ \gamma_R$ is a normal subriemannian geodesic and is given by

$$\gamma(t) = \exp_R(tv) \exp_G(-tA(v))$$

where $\exp_G : \mathfrak{g} \rightarrow G$ is the exponential map of G . Moreover, all normal subriemannian geodesics can be obtained in this way.

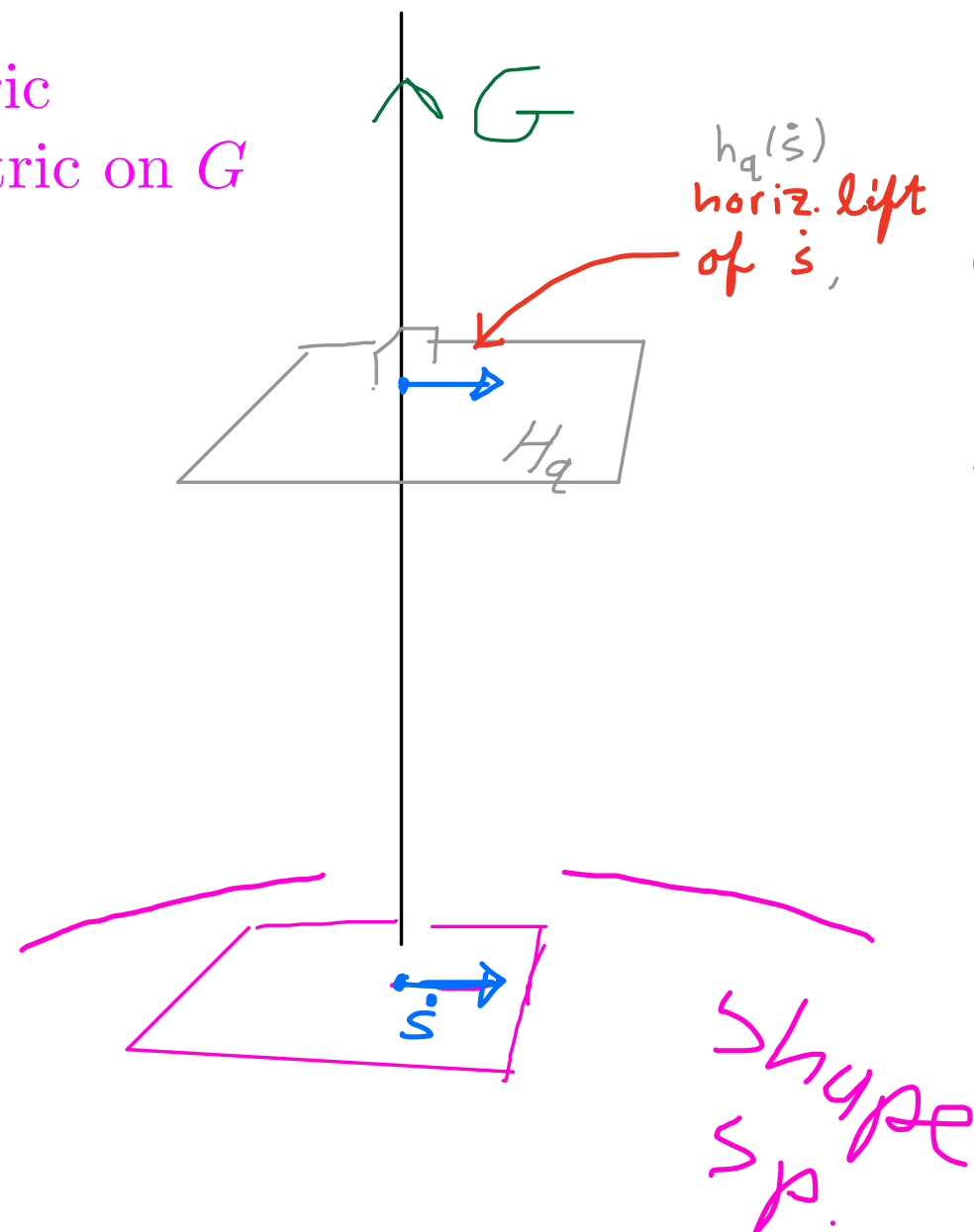
(*) constant bi-invariant type ?

(*) normal vs abnormal sR geodesics?



(*) constant bi-invariant type ?

metric on fibers is isometric
to a fixed bi-invariant metric on G



connection:
encodes horizontal
space's H

OPEN QUESTION (number 1 of book)

Are the CCs always isolated?

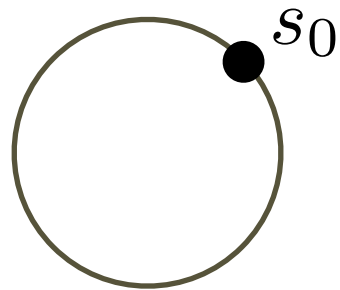
Regardless:

The only roads in to total collision are along central configurations:

Theorem [Chazy 1922] Sundman [1906] Suppose a solution $q(t)$ suffers a total collision as $t \rightarrow t_c$. Let $s(t) = q(t)/\|q(t)\|$ be its normalized configuration curve. Then the set of accumulation points of $s(t)$ as $t \rightarrow t_c$ is a closed connected subset of the set of normalized central configurations.

$$s(t) \in S = \text{sphere of normalized, centered configurations,} \\ = \{q \in Q : \sum m_a q_a = 0, \sum m_a |q_a|^2 = q\} \cong \mathbb{S}^{2N-3}$$

HOPF FIBRATION! $\pi : S \rightarrow \mathbb{CP}^{N-1} := \tilde{S} !$



Assume we are in the case of the PLANAR N-body problem.
Assume, for the given mass distribution, that the CCs are
isolated, so form a finite number of circles in the sphere S .
Then:

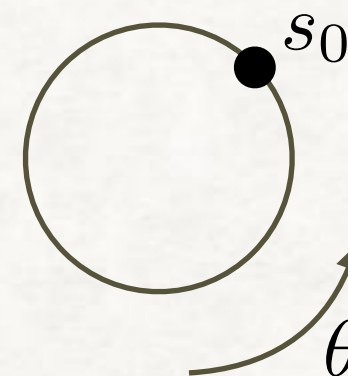
No Infinite Spin:

Theorem [Moeckel, Montgomery [2024]] Under these assumptions,
and letting $q(t)$, $s(t)$ be as above, then $\lim_{t \rightarrow t_c} s(t)$ exists:
the limit set is a point NOT a circle, but rather a single point.

said a bit
differently by the
other R.M.:

The Infinite Spin Problem

As a TC solution approaches collision, it's conceivable that the configuration spins or spirals so that its shape converges to a whole circle of CCs rather than to a specific CC on the circle. For nondegenerate CCs, Chazy showed that this does not happen, that is, infinite spin is impossible. Our main result is to extend this to degenerate CCs, provided they are isolated.



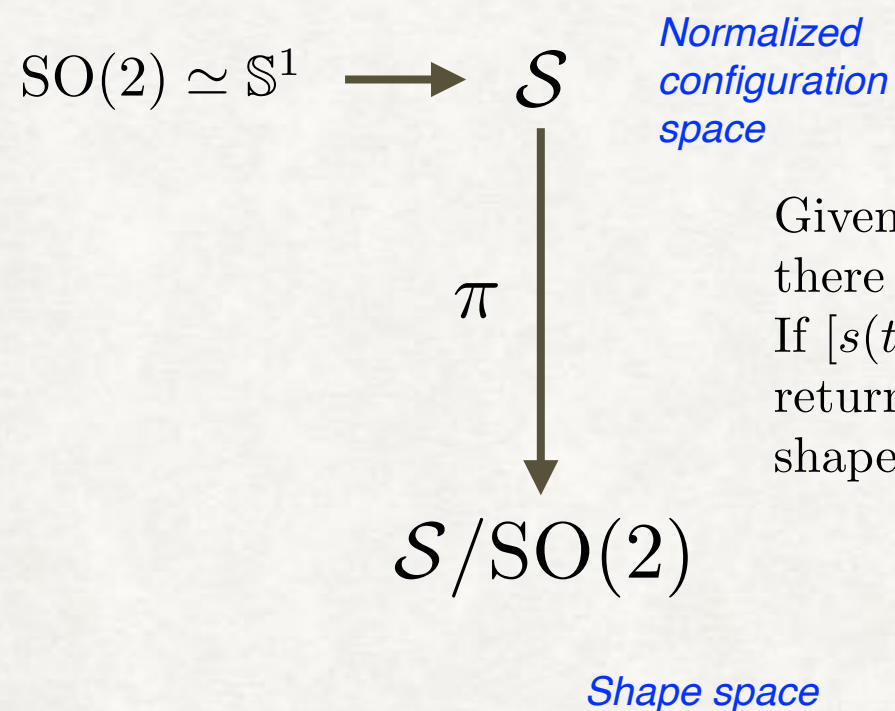
Theorem 1. *(with R. Montgomery) Suppose $q(t)$ is a TC solution of the planar n -body and suppose its shape $[s(t)] \in \tilde{S} = S/SO(2)$ converges to an isolated CC. If $\theta(t)$ is an angular variable on the corresponding circle of CCs in S , then $\theta(t)$ converges as $t \rightarrow T$ and so $s(t)$ converges to a particular CC in the circle.*

Nondegenerate critical points are isolated so this provides another proof of Chazy's result. We have seen that degenerate CCs exist, but all known examples are isolated. In fact it is a well-known conjecture that the reduced potential $U([s])$ always has a finite number of critical points which would certainly imply that they are all isolated, even if degenerate.

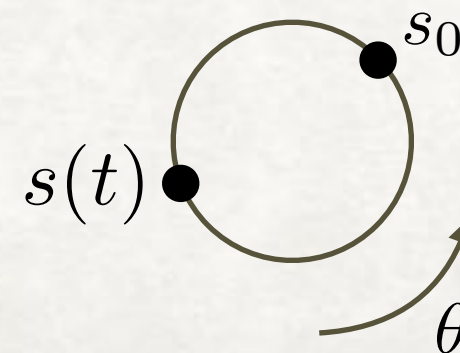
Theorem 1 -- Infinite Spin and Falling Cats

From a certain point of view, it seems intuitively clear that infinite spin can't occur simply because **the angular momentum of any TC solution must vanish**. How is it possible for the configuration of the bodies to spin around and around if there is no angular momentum ? But the same reasoning applies to the falling cat problem. How is it possible for a cat dropped with zero angular momentum is able to rotate to land on its feet ?

The explanation involves the curvature of the circle bundle $\pi : \mathcal{S} \rightarrow \mathcal{S}/\text{SO}(2)$.



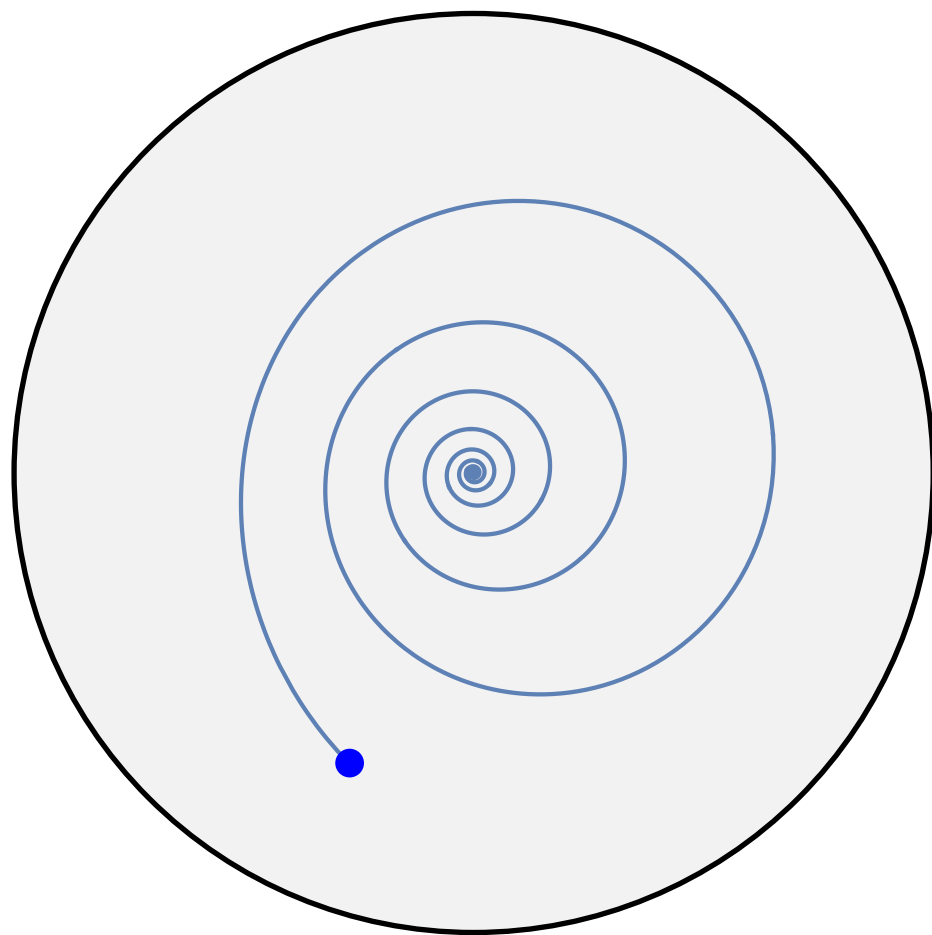
Given a curve $[s(t)]$ in shape space and an initial point s_0 with $\pi(s_0) = [s(0)]$ there is a unique *horizontal lift curve* $s(t)$ in $\mathcal{S}$ with zero-angular momentum. If $[s(t)]$ returns to its initial shape at some later time, the lifted curve may not return to the same point on the circle. Similarly, the cat can resume its original shape but with a different spatial orientation.



Example with Configurations of Three-Bodies

Consider the possible motions of three points in the plane with zero angular momentum. This will not be the gravitational three-body problem. Instead imaging three people on a slippery surface who are joined together by some imaginary stick. By pushing and pulling on the sticks, they can change the size and shape of the triangle, but they will always have zero angular momentum.

Curve in Shape Space



Shape converges to an equilateral triangle

Corresponding zero angular momentum motion

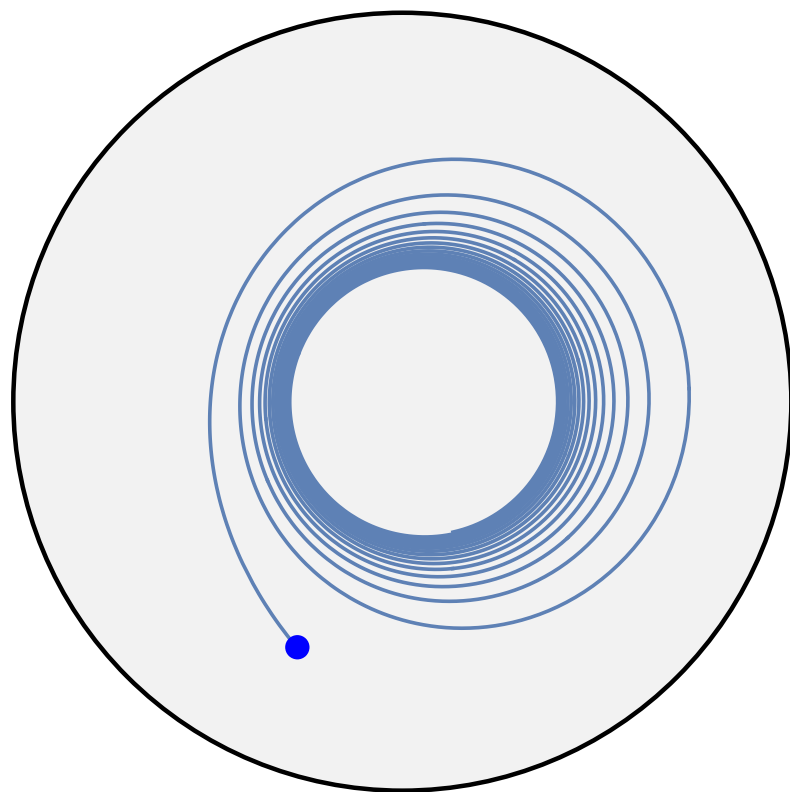


Finite amount of rotation

Example with Configurations of Three-Bodies

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Curve in Shape Space



Shape converges to an equilateral triangle

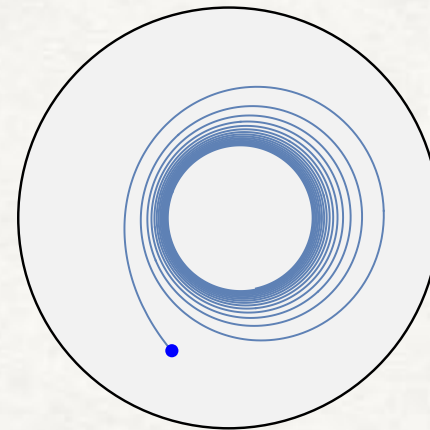
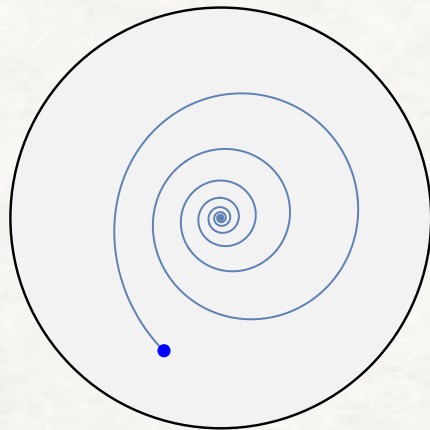
Corresponding zero angular momentum motion



Infinite Spin -- Can't happen for the nbp

Avoiding Infinite Spin -- Finite Arclength

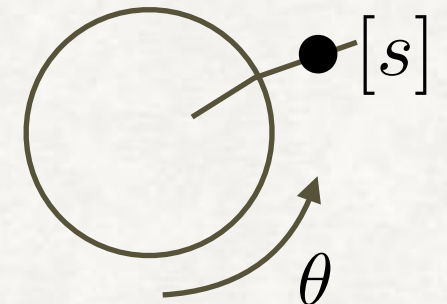
The difference between these two examples lies in the nature of the curve in the shape space. Both converge to the same shape but one is too slow -- the shape curve has infinite arclength.



For the planar n -body problem it's possible to introduce local shape-cross-angle coordinates near a give circle of CCs. Given a shape curve $\gamma(t) = [s(t)]$ from this product neighborhood, the change in angle can be reconstructed by integration of a one-form.

$$\Delta\theta = \int_{\gamma} \Omega([s]) d[s].$$

Then Theorem 1 follows from



Theorem 3. *If $\gamma(t) = [s(t)]$ is the shape curve of a TC orbit converging to an isolated CC then γ has finite arclength (with respect to the mass metric).*

Flow on the Center Manifold

If a TC solution converges to a nondegenerate restpoint on the collision manifold, the convergence will be exponential and the finiteness of the arclength follows easily (Chazy's result). For degenerate restpoints one needs to study the flow on the center manifold. The crucial steps are:

- The dynamics on the center manifold is well approximated by a gradient ODE: $x' = -\hat{\nabla}W(x)$
- An argument based on the Lojasiewicz gradient inequality proves finiteness of arclength.

The gradient $\hat{\nabla}$ is with respect to a Riemannian metric in local coordinates on the center manifold induced by the mass norm.

Lojasiewicz Gradient Theorem

Theorem (Lojasiewicz). *Consider a gradient ODE $x' = -\nabla F(x)$ where $F(x)$ is analytic in some open set in $\mathbb{R}^k$ and let $x(t)$ be a solution which has a limit point $x_0 \in U$. Then $x(t)$ has finite arclength and converges to x_0 as $t \rightarrow \infty$.*

The proof uses a gradient inequality

Theorem (Lojasiewicz). *If $F(x)$ is analytic near $x = x_0$ then there is some $0 \leq \alpha < 1$ such that*

$$|\nabla F(x)| \geq |F(x) - F(x_0)|^\alpha$$

in some neighborhood of x_0 .

We have $x' \simeq -k\hat{\nabla}W(x)$ but $W(x) = V(x, \phi(x))$ need not be analytic. But we can show that for some $c > 0$

$$c|\hat{\nabla}W(x)| \geq |\tilde{\nabla}V(x, \phi(x))| \geq |V(x, \phi(x)) - V(x_0, \phi(x_0))|^\alpha = |W(x) - W(x_0)|^\alpha$$

A lemma

V is analytic

END!

<i>Control</i>	<i>Gauge Theory</i>	<i>Deformable Body</i>
control law $\dot{q} = h(q)u$	equations of parallel transport	total angular momentum is zero
	$dq + A(x, g)dx = 0$	
state space	total space Q	configuration space
variables g transverse to contr.	fiber $\pi^{-1}(s)$	all rotations of a given shape s
	structure group G	group of rigid rotations
	Lie algebra of G	space of angular velocities
directly controlled variables x	base space S	shape space
	bundle projection π	to each config. q assigns its shape $x = \pi(q)$
	local trivialization or choice of gauge	robot coordinates are one example
controls u	tangent vectors to base space	shape deformations
$u(t)$ steers from q_0 to q_1	q_1 is the parallel translate of q_0 along $x(t)$ where $\dot{x} = u(t)$	
	the map $q_0 \mapsto q_1$ is called the holonomy when $x_0 = x_1$	$g_0 \mapsto g_1$ is the reorientation of the body
Chow's control. criterion	Ambrose-Singer Thm.	What reorientations are possible?
Lie bracket conditions on contr. vector field	curvature conditions	
optimal control for a quadratic cost function	motions of a charged particle in the master Yang-Mills field	most efficient shape deformat. yield. a desired reorient.



A summary of
my career.

mag. field= (ang. mom. strength)*Dirac monopole

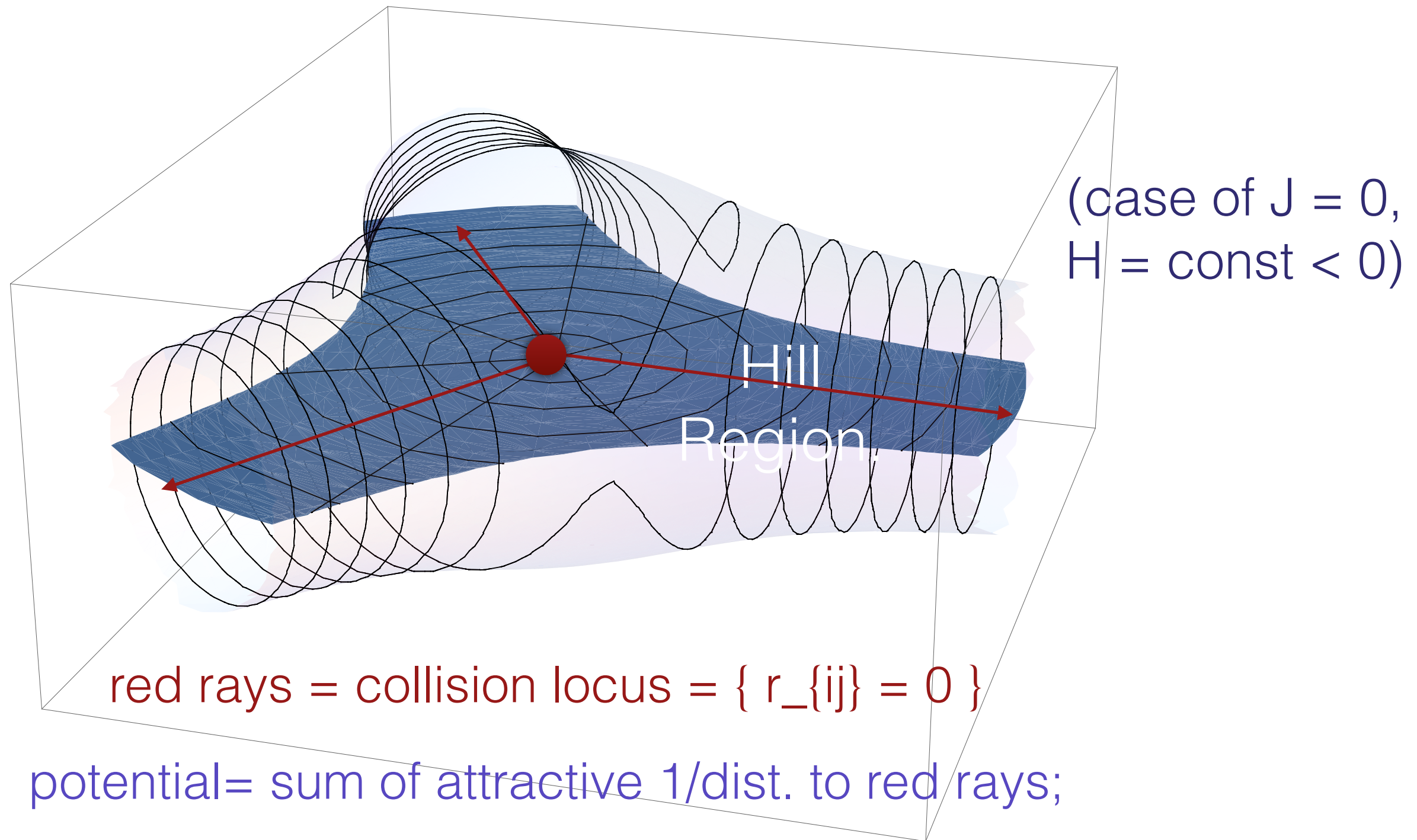
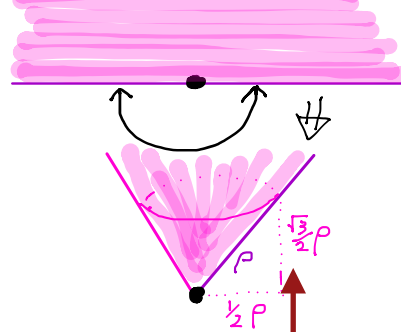
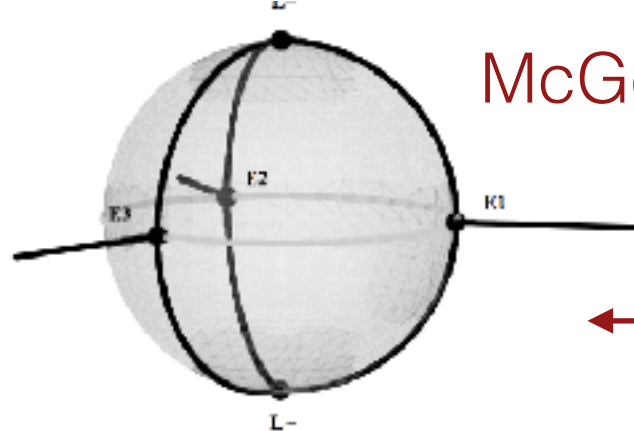


Figure: Hill region for negative energy $-h$. That is, the part of shape space S which is the image of $\{q: \text{there exists a } v \text{ such that } H(q,v) = -h\}$. Boundary: 'brake conditions'. Motions happen here.

Kepler Cone;
inextendibility of minimizers
past collision.



all syz. seq
realized



McGehee blow-up+
Moeckel

regularize . &
play grav. billiards



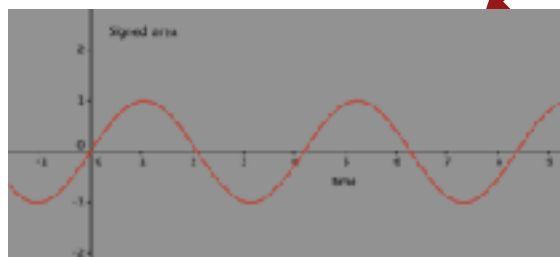
Bestiary of
Danya Rose

Mechanical intuition ,
conformal transf.
Albouy's 'dispositiones'

...

variational methods
plus discrete symmetries

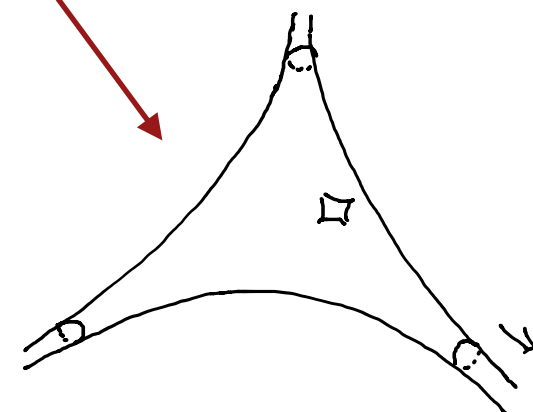
Jacobi-Maupertuis metric,
Riem geom: curvature



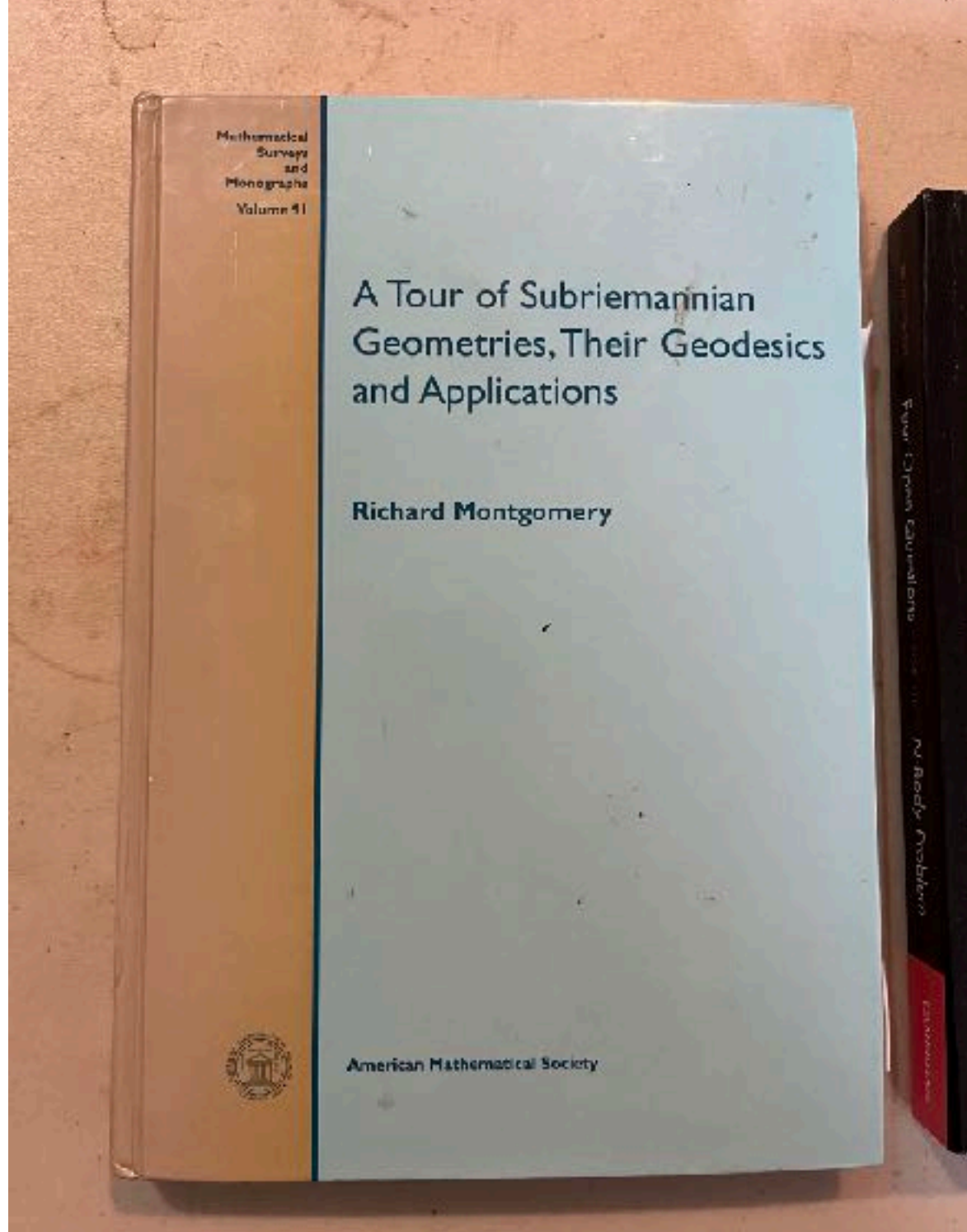
'Infinitely many...'



figure 8; choreos

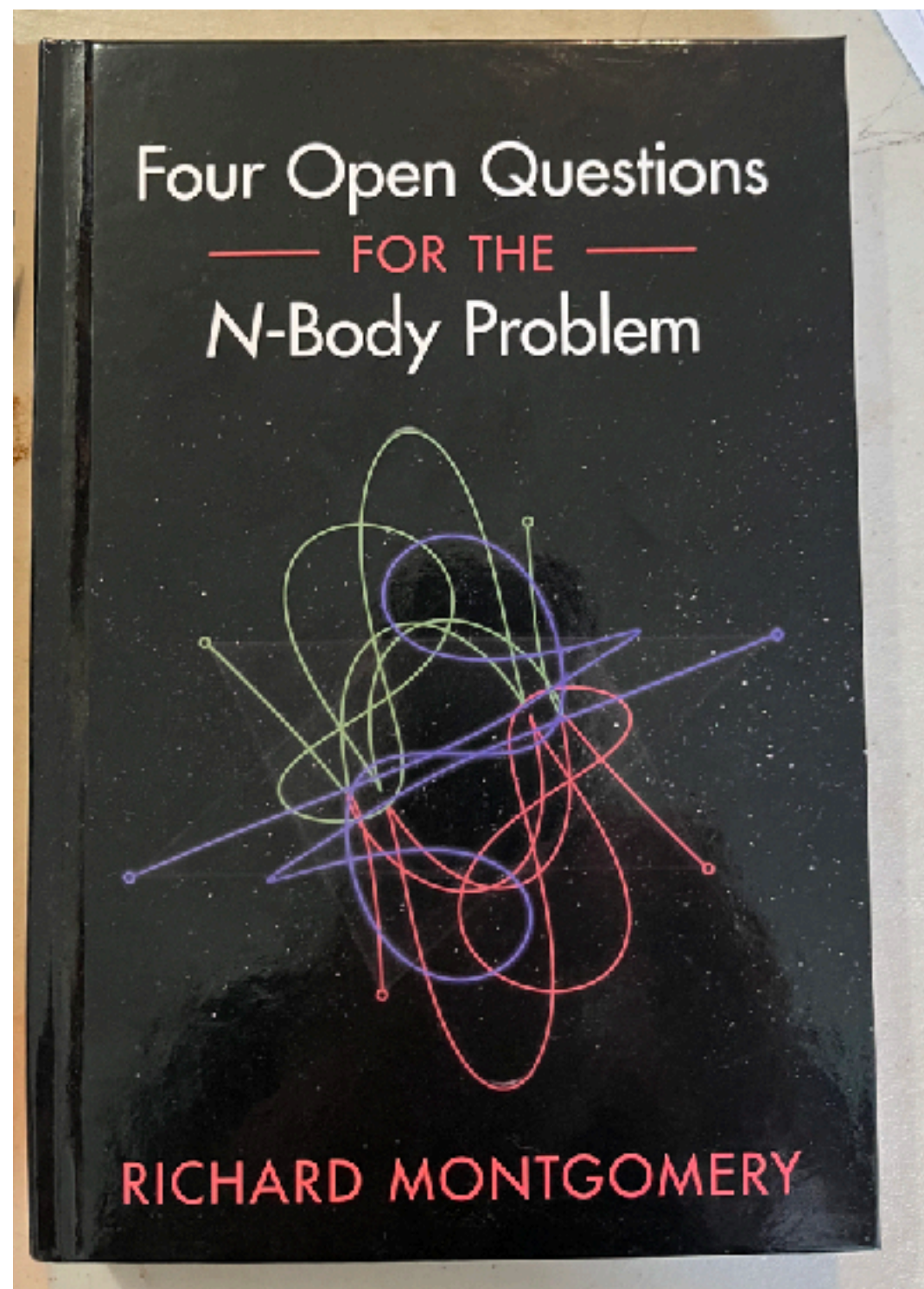


'Hyperbolic Pants ...'



2000

thank you CIMAT!



2025

END!