The direct method of the calculus of variations. – by R. Montgomery
an application of weak convergence

1. The setting: c is a curve in $\mathbb{R}^n$ (resp. a Riemannian manifold M^n). The *action* of a path is given by: $A(c) = \int_c L(c, \dot{c})dt$. Thus A is a function of paths. Here the function L is a function on $\mathbb{R}^n \times \mathbb{R}^n$ (resp. on the tangent bundle TM of M) which we will take to be of the form $L = \frac{1}{2}\|\dot{c}\|^2 + U(c(t))$ where $\|\dot{c}\|^2 = \langle \dot{c}, \dot{c} \rangle$ is the usual squared length of the vector $\dot{c} = \frac{dc}{dt}$ in $\mathbb{R}^n$ (resp. the squared inner product of this tangent vector w.r.t. the Riemannian inner product) and $U : \mathbb{R}^n \to \mathbb{R}$ is a function, assumed "sufficiently smooth", and its negative is called the "potential energy". We say that L is the difference of the kinetic energy ($\frac{1}{2}\|\dot{c}\|^2$) and the potential energy.

REASON FOR INTEREST: Euler-Lagrange Principle. If we restrict A to the set of paths joining two given points in some given time, and if c minimizes A among all such paths, AND if c is "sufficiently smooth" then c satisfies Newton's equations:

$$\frac{d^2c}{dt^2} = \nabla U(c(t))$$

We will return to the proof of this principle later. The point is minimizing A provides us with a means of constructing solutions to Newton's equations.

2. Function spaces. $H^1([a, b], \mathbb{R}^n)$ ($[0, T]$) will be defined as the closure of the space of all absolutely continuous paths in $\mathbb{R}^n$ whose derivative is square integrable, and which are parameterized by the interval $[0, T]$. The standard H^1 norm is :

implies $c \in H^1$ dos cont ?

$$\|c\|^2_{H^1} := \int_0^T \{\|\dot{c}(t)\|^2 + \|c(t)\|^2\}dt.$$

+ not important, but prevents constants from having norm zero

Thus, in the particular case that $U(c) = \frac{1}{2}\|c\|^2$ we have $\|c\|_{H^1} = 2A(c)$. In this case L is the lagrangian for the "harmonic oscillator".

3. Fix a subset $X \subset H^1$ of curves, corresponding to "boundary conditions" or endpoint conditions on X. Two typical choices. Fix points $q_0, q_1 \in \mathbb{R}^n$ (resp. in M^n). Consider the set of all curves $c : [0, T] \to \mathbb{R}^n$ in H^1 which satisfy $c(0) = q_0$ and $c(T) = q_1$. The choice used by Gordon is to take X to be the set of all curves which are periodic of period T i.e. $c(0) = c(T)$ and which wind once around the origin. (on a manifold– take all curves which are periodic of period T and realize a given free homotopy class).

4. Set:

$$a(X) = inf_{c \in X} A(c).$$

By definition of "inf" this means there exists a sequence $c_n \in X$ of curves with the property that $inf A(c_n) = a(X)$. The *direct method of the calculus of variations* proceeds by completing the following steps.

STEP 1. Show that the c_n converge to some curve c_*. The sense of convergence will be weak convergence.

STEP 2. Show that this limit c_* is in X.

STEP 3. Show that c_* realizes the infimum: $A(c_*) = a(X) := lim\ inf_{c \in X} A(c)$. (This is "weak lower semicontinuity of A".)

STEP 4. Show that the differential of the action , $dA(c)$, at c, is zero. This would be "obvious" if X were finite-dimensional and open, for if the derivative were not zero, then we could move away from c in a direction which would further decrease the action.

STEP 5. Conclude from step 3 that c satisfies Newton's equations, which are the "Euler-Lagrange equations for our L.

5. SOBOLEV EMBEDDING.

A key to proceding is the Sobolev embedding theorem. Take $c \in H^1$ which is absolutely continuous. Then it is the integral of its derivative, and this derivative exists a.e. Thus:

$$c(t) - c(s) = \int_s^t \dot{c}dt$$

$|\int fg|^2 \leq \int |f|^2 \int |g|^2$ $\quad |\int \|\dot c\||^2 \leq \int \|\dot c\|^2 \int 1$

Now, using Cauchy-Schwartz, with $f = \|\dot{c}\|$ and $g = 1$, we find that:

$$\|c(t) - c(s)\| \leq \text{length } c = \int_s^t \|\dot{c}\| dt \leq \sqrt{|t-s|}\sqrt{\int_s^t \|\dot{c}\|^2 dt}.$$

From this we conclude that if $\|\dot{c}\|_{L^2} \leq M$ then $\quad \|\dot c\|_{L^2}$ ← correct? $\int_0^T$?

$$\|c(t) - c(s)\| \leq \sqrt{|t-s|}\cancel{\|\dot{c}\|_{L^2}}M$$

. Now,

$$\|c\|_{H^1}^2 = \|c\|_{L^2}^2 + \|\dot{c}\|_{L^2}^2 \geq \|\dot c\|^2_{L^2}$$

so, that we also have

$$\|c(t) - c(s)\| \leq \sqrt{|t-s|}\|c\|_{H^1}$$

This shows that those curves c lying in the H^1 ball form an equicontinous family, and hence the Arzela-Ascoli theorem. This shows that EVERY curve in H^1 is absolutely continuous, since this estimate shows that they are in fact in the Holder space $C^{\frac{1}{2}}$. The Arzela -Ascoli theorem implies that if we have a sequence c_n of H^1-curves which are bounded: $\|c_n\|_{H^1} \leq M$, and if $c_n(0)$ themselves are bounded, then the c_n admit a C^0-convergent subsequence. For they are equicontinuous (being Holder) and bounded.

SHOW THAT THE ALTERNATIVE NORM $\|\cdot\|_A$, with $\|c\|_A^2 := \|c(0)\|^2 + \int \|\dot{c}(t)\|^2 dt$ is equivalent to the H^1-norm which we have defined, namely the one whose square is $\int_0^T \|c(t)\|^2 + \int_0^T \|\dot{c}(t)\|^2 dt$

CONCLUDE: any H^1-bounded sequence forms a bounded equicts family, and therefore has a convergent subsequence.

CONCLUDE: every H^1 curve is C^0.

CONCLUDE: the inclusion: $H^1 \to C^0$ is cts (i.e. bdd). What is the bd?

Returning to our problem: We have this minimizing sequence c_n for the action A. Let us suppose for simplicity that we are interested in the fixed endpoint conditions:

BOUNDARY CONDITIONS: $c(0) = 0, c(T) = q_1$, fixed

with curves starting at 0, so that all of our sequence c_n passes through 0 at time 0. Let us also suppose

POTENTIAL CONDITION: U is non-negative: $U \geq 0$.

STEP 1.

Then: $\|c_n(0)\|^2 + \int_0^T \|\dot{c}_n(t)\|^2 dt \leq 2A(c_n)$ so that our sequence c_n is H^1 bounded. By the Banach-Alaoglu theorem, we may extract a weakly convergent subsequence c_{n_k} Following standard notational procedure, we rename this subsequence c_n. Thus:

$$c_n \rightharpoonup c_*.$$

This completes STEP 1.

This c_* is our potential future solution to NEWTON. We have produced it out of (hot?) thin air.

STEP 2. Our space X consists of those H^1 curves satisfing BOUNDARY CONDITION above. Our c_* is in X. Why?

Answer: the map $H^1 \to C^0$ is continuous in the weak topology.

STEP 3. c_* minimizes. Since $c_n \overset{H^1}{\rightharpoonup} c_*$, we have that $\|c_*\|_{H^1} \leq \|c_n\|_{H^1}$. The same is true using the norm $\|c(0)\|^2 + \int \|\dot{c}\|^2 dt$. – recall proof; second pf: lebesque dominated convg.

Thus $\frac{1}{2}\int \|\dot{c}_*\|^2 \leq \frac{1}{2}\int \|\dot{c}_n\|^2$. Also, $c_n \to c_*$ uniformly, i.e. in the C^0 topology. By potential assumption, this means that $U(c_n(t)) \to U(c_*(t))$ uniformly as well, and hence $\int U(c_n(t)) \to \int U(c_*(t))$. Adding these two observations we see that

$$A(c_*) \leq lim\, inf A(c_n) := a(X).$$

use $\int \|\dot c_* - \dot c_n\|^2$ & $\int \langle \dot c_*, \dot c_n \rangle \to \int |\dot c_*|^2$.

But $c_* \in X$. Therefore $A(c_*) = a(X)$, which is to say that c_* realizes the infimum.

STEP 5. Differentiating the action.

How do we differentiate functions from a Banach space? Just like we do for a regular vector space. Suppose $f : E \to \mathbb{R}$ is a function on the Banach space E. We form the difference quotients $\frac{1}{h}(f(x+he)-f(x))$. If this converges as $h \to 0$, then we define the limit to be the directional derivative of f at $x \in E$, written $df(x)(e)$, or sometimes $Df(x)(e)$, or $\nabla f(x)(e)$. If this derivative is linear in the direction e, it defines a linear functional. IF THIS LINEAR FUNCTIONAL IS BOUNDED then we say that f is differentiable at x, with derivative $df(x)$.

$df(x): E \to$

COMPUTATION:

$$dA(c)(e) = \int \langle \dot{c}(t), \dot{e}(t) \rangle + \langle \nabla U(c(t)), e(t) \rangle dt$$?

assuming that U is differentiable on $\mathbb{R}^n$.

?

Thus $dA(c)$ is the linear functional which is defined by the L_2 pairing of $\dot{e}$ with $\dot{c}$ plus the L_2-pairing of e with ∇U. This $dU(c)$ is a continuous linear functional on H^1.

A!

Suppose that $c \in X$, so that $c(0) = 0, c(T) = q_1$. If we also have $c + he \in X$, then we must have $e(0) = 0 = e(T)$. We thus set

$S = \{e \in H^1 : e(0) = 0 = e(T)\}$, which is the tangent space $T_c X$ to the space X at the curve c.

In the calculus of variations, such an e is said to be a "variation vanishing at the endpoints". (REMARK: Our X is an affine subspace of H^1, so that S is a linear subspace – the vector space on which this affine space is modelled.

Lemma: The differential $dA(c_*)$, which is a linear function $H^1 \to \mathbb{R}$, must annihilate this linear subspace $S := \{e \in H^1 : e(0) = 0 = e(T)\}$.

Proof. suppose not. Then there is an e in this subspace with $dA(c_*)(e) \neq 0$. By replacing e with $-e$ if neccessary, we may assume that $dA(c_*)(e) < 0$. But $dA(c_*)(e)$ is the derivative of the real-valued function $f(h) = A(c_* + he)$. If this derivative is negative, then for h sufficiently small, positive, we have $f(h) < f(0)$ which is to say that $A(c_* + he) < A(c_*)$. However, $c_* + he \in X$, and c_* is the minimum of A over all of X. CONTRADICTION.

Definition. *A curve (function) which satisfies $dA(c_*)|_S = 0$, where S is the subspace of variations vanishing at the endpoints is called a WEAK SOLUTION to Newton's equations.*

This word "weak solution" comes from PDE ~~and~~ is more used there.

INTEGRATION BY PARTS: If we ASSUME that c_* and e are sufficiently differentiable, here C^2 is good enough, then we can integrate by parts: $\frac{d}{dt}\langle \dot{c}_*, e \rangle = \langle (\frac{d}{dt})^2 c_*(t), e(t) \rangle + \langle \dot{c}_*(t), \dot{e}(t) \rangle$ From which it follows that

$$\int \langle \dot{c}_*(t), \dot{e}(t) \rangle = \int \langle \frac{-d^2}{dt^2} c_*(t), e(t) \rangle dt + \langle \dot{c}_*, e \rangle |_0^T = \int \langle \frac{-d^2}{dt^2} c_*(t), e(t) \rangle dt,$$

where we used in the last equality the endpoint conditions $e(0) = 0 = e(T)$. Consequently, under this assumption on c_* and e, we have

$$dA(c_*)(e) = \int \langle \underbrace{(\frac{-d^2}{dt^2} c_*(t)) + \nabla U(c_*(t))}_{=0}, e(t) \rangle dt$$

Now, if a real function g integrates with all functions h to be zero: if $\int gh = 0$ for all functions h with $h(0) = h(T) = 0$, then it stands to reason (and is easy to prove) that $g = 0$. Thus we have that c_* satisfies NEWTON: $\frac{-d^2}{dt^2} c(t) + \nabla U(c(t)) = 0$, PROVIDED we know, a priori , that c_* is twice differentiable. The big theoretical problem is, of course, that there is no reason this need be true!

FUNDAMENTAL LEMMA OF THE CALCULUS OF VARIATIONS. Suppose that $\int \langle \dot{c}, \dot{e} \rangle + \langle f, e \rangle dt = 0$ for all e

Then $f = \dot{c}$ (so ? is ...?)

Above, take $\dot{c} = 0$
and $f = -\frac{d^2}{dt^2} c_* + \nabla U c_*$