REVIEWS

Edited by **Gerald B. Folland**
Mathematics Department, University of Washington, Seattle, WA 98195-4350

Symmetry in Mechanics: A Gentle, Modern Introduction. By Stephanie Frank Singer. Birkhäuser, Boston, 2001, xii + 193 pp., ISBN 0-8176-4145-9, $29.95.

Reviewed by Richard Montgomery

The problem of explaining planetary motion has been central to a great deal of mathematics and physics over the past three centuries. From the Newtonian perspective, the fundamental problem is to analyze the motion of two bodies subject to an inverse-square-law attractive force:

$$\begin{aligned}\frac{d^2\mathbf{x}_1}{dt^2} &= Gm_2\frac{\mathbf{x}_2-\mathbf{x}_1}{r^3},\\ \frac{d^2\mathbf{x}_2}{dt^2} &= Gm_1\frac{\mathbf{x}_1-\mathbf{x}_2}{r^3},\end{aligned} \tag{1}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the positions of the two bodies as a function of time t, $r = \|\mathbf{x}_1 - \mathbf{x}_2\|$ is the distance between them, m_1 and m_2 are their masses, and G is Newton's gravitational constant. Translations, rotations, and time-translations are *symmetries* of (1): if $\mathbf{x}_i(t)$ is a solution, then so are $\mathbf{x}_i(t) + \mathbf{c}$ for any $\mathbf{c} \in \mathbb{R}^3$, $R\mathbf{x}_i(t)$ for any rotation R, and $\mathbf{x}_i(t + c)$ for any $c \in \mathbb{R}$. Corresponding to each of these symmetries is a quantity that is *conserved* in the sense that it remains constant along solutions: namely, the total linear momentum

$$\mathbf{P} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2, \tag{2}$$

the total angular momentum

$$\mathbf{L} = m_1\mathbf{x}_1 \times \mathbf{v}_1 + m_2\mathbf{x}_2 \times \mathbf{v}_2, \tag{3}$$

and the total energy

$$H = \frac{1}{2}\left(m_1\|\mathbf{v}_1\|^2 + m_2\|\mathbf{v}_2\|^2\right) - \frac{Gm_1m_2}{r} \tag{4}$$

(where $\mathbf{v}_i = d\mathbf{x}_j/dt$).

These conserved quantities together with their associated symmetries can be used to reduce the number of variables in (1). First, instead of taking the position vectors $\mathbf{x}_i$ as the dependent variables, we take $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ and $\mathbf{x}_{CM}$, the position of the center of mass of the system:

$$\mathbf{x}_{CM} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2}.$$

By using a Galilean transformation $\mathbf{x}_i(t) \to \mathbf{x}_i(t) + t\mathbf{v}_0$, also a symmetry of (1), we can arrange that $\mathbf{P} = \mathbf{0}$, so that $\mathbf{x}_{CM}$ is constant. It remains to solve for $\mathbf{r}$. Subtracting the second equation of (1) from the first, we obtain

$$\frac{d^2\mathbf{r}}{dt^2} = -G(m_1 + m_2)\frac{\mathbf{r}}{r^3}, \tag{5}$$

an equation known today as the *Kepler problem*. Next, for a fixed value of the angular momentum $\mathbf{L}$, the motion of $\mathbf{r}$ takes place in a plane perpendicular to $\mathbf{L}$. Introducing polar coordinates (r, θ) in this plane (so r is still $\|\mathbf{r}\|$), one finds that the energy H is a function of r alone:

$$H = \frac{1}{2}\left[\mu\left(\frac{dr}{dt}\right)^2 + \frac{\|\mathbf{L}\|^2}{\mu r^2}\right] - \frac{m_1 m_2}{r}, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{6}$$

Consequently, the dynamics becomes encoded in a differential equation for r alone. Using the differential relation $\|\mathbf{L}\|\, dt = \mu r^2\, d\theta$, one obtains a differential equation for r as a function of θ rather than of t—an equation that can be solved explicitly. The solution curves $r = r(\theta)$ are the conic sections of Kepler's law.

Perhaps the biggest step forward in classical mechanics after Newton was Hamilton's reformulation of the subject. For the two-body problem, it begins by considering the space of possible positions $(\mathbf{x}_1, \mathbf{x}_2)$ of the two bodies together with the space of corresponding momenta $(m_1\mathbf{v}_1, m_2\mathbf{v}_2)$. We denote the position space $\mathbb{R}^3 \times \mathbb{R}^3$ by $\mathbb{E}$ and identify the momentum space with the dual space $\mathbb{E}^*$; elements of $\mathbb{E}$ and $\mathbb{E}^*$ are conventionally denoted by $q = (q_1, \dots, q_6)$ and $p = (p_1, \dots, p_6)$. If one considers the energy H given by (4) as a function of q and p, then Newton's equations (1) are equivalent to Hamilton's equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}. \tag{7}$$

That is, *in the "phase space" $\mathbb{V} = \mathbb{E} \times \mathbb{E}^*$, the solutions are the integral curves of the vector field $X_H = (\nabla_p H, -\nabla_q H)$.*

The power of Hamilton's reformulation (7) of Newton's equations stems from an underlying geometric structure, *symplectic geometry*. To wit, the vector space $\mathbb{V} = \mathbb{E} \times \mathbb{E}^*$ carries a natural nondegenerate skew-symmetric bilinear form, or *symplectic form*,

$$\omega((q, p), (q', p')) = p(q') - q(p').$$

(Nondegeneracy means that if $\omega((p, q), (p', q')) = 0$ for all (p', q'), then $(p, q) = (0, 0)$.) In the language of differential geometry, ω is a two-form:

$$\omega = \sum dp_i \wedge dq_i.$$

Now, given any smooth function f on $\mathbb{V}$, its differential df is a one-form, and the symplectic form ω can be used to convert it into a vector field. Namely, the *Hamiltonian vector field X_f* associated to f is defined by

$$\omega(X_f(x), \xi) = df(x)(\xi),$$

where $x = (q, p) \in \mathbb{V}$ and ξ is a tangent vector at x; more concretely, $X_f = (\nabla_p f, -\nabla_q f)$. It is not hard to show that *the flow generated by any Hamiltonian* *vector field preserves the symplectic form* ω. When $f = H$, this flow describes the time evolution of the two-body system.

The Hamiltonian method applies to very general problems in classical mechanics. The "phase space" for a general problem is a *symplectic manifold* P, i.e., a manifold equipped with a closed nondegenerate two-form ω. Time evolution is by a one-parameter group of transformations of P that preserve ω. This group is generated by the Hamiltonian vector field X_H, where H is the total energy function.

The idea of using symmetries and conserved quantities to reduce the number of variables in the two-body problem also generalizes. In the general setting, a symmetry group is an action of a Lie group G on a symplectic manifold P by transformations that preserve the symplectic form. Given such an action, each element ξ of the Lie algebra $Lie(G)$ defines a vector field ξ_P (the "infinitesimal generator") whose flow preserves the symplectic form. It is legitimate to ask that there be a linear and G-equivariant map $\xi \to J^\xi$ from $Lie(G)$ to the space of smooth functions on P such that ξ_P is the Hamiltonian vector field X_{J^ξ}. If such a map exists, as it does in almost all examples of physical interest, the group action is called "Hamiltonian." In this case, for each $z \in P$ the map $\xi \to J^\xi(z)$ is linear, so we can regard J as a map from P to the dual space $Lie(G)^*$. As such, it is called the *momentum mapping* associated with the group action.

Suppose now that P is equipped with a Hamiltonian function H that is preserved by the group action, so that $dH(\xi_P) = 0$ for all $\xi \in Lie(G)$. Since $dH(\xi_P) = \omega(X_H, \xi_P) = -dJ^\xi(X_H)$, J is constant on trajectories of H and so is a "conserved quantity." (This is the Hamiltonian version of Noether's theorem.) Now fix a value $\nu \in Lie(G)^*$ of the momentum. The group G acts on $Lie(G)^*$ by the coadjoint action; let G_ν be the subgroup of G that preserves ν. A theorem of Meyer and Marsden-Weinstein states that the quotient space $P_\nu = \{z \in P : J(z) = \nu\}/G_\nu$ is itself a symplectic manifold, provided that G acts freely on P; it is called the *symplectic reduction* of P at ν. Moreover, the Hamiltonian function H descends to a function H_ν on P_ν, and one is reduced to studying the dynamics of this lower-dimensional system (P_ν, H_ν).

In the case of the two-body problem (1), G is the group of translations and rotations of $\mathbb{R}^3$, acting on the phase space $(\mathbb{R}^3 \times \mathbb{R}^3) \times (\mathbb{R}^3 \times \mathbb{R}^3)^*$ in the obvious way. The Lie algebra of G can be identified with $\mathbb{R}^3 \times \mathbb{R}^3$, representing the generators of translations and rotations, and the translation and rotation components of the momentum map are just the (physical!) linear and angular momenta given by (2) and (3). Moreover, the reduced space P_ν (with $\nu = (\mathbf{P}, \mathbf{L}), \mathbf{L} \neq \mathbf{0}$) can be identified with $(0, \infty) \times \mathbb{R}$ (with coordinates r and $p = \mu(dr/dt)$), and the reduced Hamiltonian H_ν is (6).

The mathematics outlined in the preceding paragraphs is the subject of Stephanie Frank Singer's book, which uses the Kepler problem to introduce modern symplectic geometry and the theory of symplectic reduction. To quote from its preface:

> Chapter 1 presents the derivation of Kepler's laws of planetary motion from Newton's laws of gravitation in the style of a typical American undergraduate physics text. Chapter 8 presents the same argument in the language of modern symplectic geometry. The chapters in between develop the concepts and terminology necessary for the final chapter, providing a detailed translation between the quite different languages of mathematics and physics.

Included in the intermediate chapters are introductions to (and motivations for) differentiable manifolds and Lie groups. Singer does beautifully what she sets out to do, getting to the core of her subjects with a minimum of fuss.

I know of no other books filling the niche that this one fills. There are a number of mathematical lecture notes and textbooks that cover symplectic geometry and reduction and their connections with mechanics, but all at a significantly more advanced level. (Notable among these are Bryant's lecture notes [3].) The only prerequisites for Singer's book are a familiarity with vector calculus and a passing knowledge of college-level physics.

Beyond providing a working introduction to modern differential geometry and a dictionary between nineteenth century physics and twentieth century mathematics, the main achievement of this book is to provide readers with a ready entrance to the many applications of symplectic geometry and to the exciting and beautiful recent developments in the area. We now describe a few of these applications and developments.

Symplectic manifolds arise in many contexts. The cotangent bundle of any manifold carries a natural symplectic structure. Analysis on cotangent bundles plays a central role not only in Hamiltonian mechanics (where the bundles arise as phase spaces), but in the analysis of partial differential operators (propagation of singularities, pseudodifferential operators, and Fourier integral operators; see Taylor [13]) and the theory of nonlinear optimal control as formulated by Pontrjagin et al. [12]. On the other hand, since the real and imaginary parts of a Hermitian inner product are a real inner product and a real symplectic form, Riemannian and symplectic geometry meet in the realm of complex manifolds to form the theory of Kähler manifolds (see Wells [16]).

Any Lie group G acts on the dual space of its Lie algebra by the coadjoint action. Each orbit under this action carries a G-invariant symplectic structure. (In fact, the orbit through $\nu \in Lie(G)^*$ is the symplectic reduction at ν, as described earlier, of the cotangent bundle of G.) Kirillov, Kostant, and others, inspired by the idea of "quantization," have developed a tight correspondence between the coadjoint orbits of G and its irreducible unitary representations. This "orbit picture" has been a guiding principle for modern developments in representation theory (see Wallach [15] and Vogan [14]).

Let X be a Hermitian $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$, and let $diag(X) \in \mathbb{R}^n$ be the vector of diagonal entries $(X_{11}, \dots, X_{nn})$. A theorem of Schur asserts that $diag(X)$ lies in the convex polyhedron whose vertices are the points $(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)})$ for arbitrary permutations σ. This classic result can be placed in the context of symplectic reduction. Indeed, let G be the unitary group $U(n)$. The space of Hermitian matrices can be identified with $Lie(G)^*$. The $U(n)$-orbit through X is the set of all matrices with the same eigenvalues as X (with the same multiplicities). The map $X \to diag(X)$ turns out to be the momentum map for the action of the diagonal subgroup of $U(n)$ on this orbit. Guillemin and Sternberg [6] and Atiyah [2] have proved a vast and beautiful generalization of this fact: Let P be a symplectic manifold on which a torus T^k (i.e., the k-fold power of the circle group) acts in a Hamiltonian fashion. Then the image $J(P)$ of P under the momentum map for the T^k action is a convex polytope in $\mathbb{R}^k = Lie(T^k)$, and the vertices of this polytope are the images of the fixed points of the action. The proofs of this convexity theorem use Morse theory in an essential way. The convexity theorem has led to a variety of surprising results in combinatorics and representation theory (see, for example, Knutson and Tao [9]).

I would feel criminally negligent if I were to discuss symplectic geometry without mentioning fantastic developments initiated by Arnol'd [1] and Gromov [5]. Arnol'd, based in part on his work on the motion of a charged particle traveling through a magnetic field, conjectured that every symplectic map of a certain type ("exact") on a compact symplectic manifold has at least as many fixed points as a certain topological invariant (the sum of the Betti numbers if all the fixed points are nondegenerate). Floer proved a special case of Arnol'd's conjecture by inventing what we now call Floer homology. A crucial ingredient for Floer's work was an earlier invention of

Gromov, pseudoholomorphic curves—a kind of partial complex analysis available on symplectic manifolds. Every symplectic manifold carries a natural volume form, and every transformation that preserves the symplectic form is volume-preserving. Consequently, two domains in phase space that are symplectically equivalent must have the same volume. To what extent is the converse also true? Gromov used his theory of pseudoholomorphic curves to construct symplectic invariants of domains finer than their volumes, thus yielding answers to this question. The works of Arnol'd, Floer, and Gromov have led to a new field of mathematics, "symplectic topology"; the reader can consult Hofer and Zehnder **[8]** for a survey of this field.

One glaring omission in Singer's bibliography is Milnor's lovely exposition **[10]** of the Kepler problem (5) that appeared in this MONTHLY almost twenty years ago. There Milnor shows how the solutions to the Kepler problem with energy h correspond to geodesics on a three-dimensional Riemannian manifold of constant curvature $-2h$. Thus all the classical geometries—spherical ($h < 0$), Euclidean ($h = 0$), and hyperbolic ($h > 0$)—are embedded within the Kepler problem! In particular, the negative energy solutions correspond to geodesics on the three-sphere, on which the four-dimensional rotation group $SO(4)$ acts. Thus the Kepler problem (5) for negative energies has an unexpected $SO(4)$-symmetry. (A priori, the symmetry group of the Kepler problem is the three-dimensional rotational group $SO(3)$.) The problem admits a corresponding momentum map with values in the dual of the Lie algebra of $SO(4)$, which is the sum of two copies of $Lie(SO(3))^* \cong \mathbb{R}^3$. The first $\mathbb{R}^3$ component is the angular momentum $\mathbf{L}$, and the second one is a vector quantity discovered by Laplace but more commonly known as the Runge-Lenz vector. This $SO(4)$-symmetry also exists in the quantum version of the Kepler problem; we recommend Guillemin and Sternberg [7] for details and references.

I was lucky enough to attend a few lectures of S. S. Chern just before he retired from Berkeley in which he said that the cotangent bundle (differential forms) is the feminine side of analysis on manifolds, while the tangent bundle (vector fields) is the masculine side. From this perspective, Hamiltonian mechanics is the feminine side of classical physics. Its masculine side is Lagrangian mechanics, which is formulated in terms of velocities (tangent vectors) rather than momenta (cotangent vectors) and focuses on the Lagrangian L (the difference of the kinetic and potential energies) rather than the Hamiltonian H (their sum). Trajectories $\mathbf{x}(t)$ that solve Newton's equations are those that are extrema of the action $A = \int L(\mathbf{x}(t), \mathbf{v}(t))\, dt$.

Singer does not touch upon this masculine side of mechanics, a good choice given the aims and scope of her book. However, Lagrangian mechanics is enjoying a resurgence within celestial mechanics, stemming in part from a 1970 paper of Gordon **[4]** on the planar Kepler problem. Consider the class $\mathcal{C}_k$ of all closed curves in the plane that wind k times around the origin in a time period T without passing through it, i.e., without colliding with the sun. Try to minimize the Kepler action over $\mathcal{C}_k$. Gordon showed that for $k = \pm 1$ any Keplerian ellipse of period T minimizes, but that if $|k| > 1$, then the infimum of the action over $\mathcal{C}_k$ is not realized. Rather, take a near-collision curve that comes within ϵ of the origin, winds around k times, and returns to its starting point. By letting $\epsilon \to 0$ one obtains a minimizing sequence that converges to a Keplerian collision-ejection solution, i.e., one that collides with the sun and elastically rebounds. The collision-ejection solutions have the same action as the $k = \pm 1$ Keplerian ellipses and lie on the boundary of every class $\mathcal{C}_k$. They allow one to pass from $\mathcal{C}_k$ for $k \neq \pm 1$ to $\mathcal{C}_{\pm 1}$ and thus decrease the action by violating the no-collision constraint. Gordon's result plays a central role in the recent rediscovery and existence proof by Chenciner and Montgomery [11] of figure-eight solutions for the planar three-body problem.

There are many more landscapes to explore from the base camp of the Kepler problem, but I am running out of space and starting to toot my own horn. In conclusion, Singer has done an excellent job of leading the reader from the Kepler problem to a view of the growing field of symplectic geometry.

REFERENCES

1. V. I. Arnol'd, First steps in symplectic topology, *Russian Math. Surveys* **41**:6 (1986) 1–21.
2. M. F. Atiyah, Convexity and commuting Hamiltonians, *Bull. London Math. Soc.* **14** (1982) 1–15.
3. R. L. Bryant, An introduction to Lie groups and symplectic geometry, in *Geometry and Quantum Field Theory*, D. S. Freed and K. K. Uhlenbeck, eds., American Mathematical Society, Providence, 1995, pp. 5–181.
4. W. B. Gordon, A minimizing property of Keplerian orbits, *Amer. J. Math.* **99** (1970) 961–971.
5. M. Gromov, Pseudoholomorphic curves in symplectic geometry, *Invent. Math.* **82** (1985) 307–347.
6. V. Guillemin and S. Sternberg, Convexity properties of the moment map, *Invent. Math.* **67** (1982) 512–538.
7. V. Guillemin and S. Sternberg, *Variations on a Theme of Kepler*, American Mathematical Society, Providence, 1991.
8. H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser, Basel, 1994.
9. A. Knutson and T. Tao, Honeycombs and sums of Hermitian matrices, *Notices Amer. Math. Soc.* **48** (2001) 175–186.
10. J. W. Milnor, On the geometry of the Kepler problem, this MONTHLY **90** (1983) 353–365.
11. R. Montgomery, A new solution to the three-body problem, *Notices Amer. Math. Soc.* **48** (2001), 471–481.
12. L. S. Pontrjagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*, Wiley-Interscience, New York, 1962.
13. M. E. Taylor, *Pseudodifferential Operators*, Princeton University Press, Princeton, 1981.
14. D. A. Vogan, Representations of reductive Lie groups, in *Proceedings of the International Congress of Mathematicians 1986*, American Mathematical Society, Providence, 1987, pp. 245–266.
15. N. R. Wallach, *Symplectic Geometry and Fourier Analysis*, Math Sci Press, Brookline, MA, 1977.
16. R. O. Wells, *Differential Analysis on Complex Manifolds*, Prentice-Hall, Englewood Cliffs, NJ, 1973.

University of California at Santa Cruz, Santa Cruz, CA 95064
rmont@math.ucsc.edu

Wave Motion. By J. Billingham and A. C. King. Cambridge University Press, Cambridge, U.K., 2000, 468 pp. Cloth: ISBN 0-521-63257-9, $110. Paper: ISBN 0-521-63450-4, $37.95.

Reviewed by **Jeffrey Rauch**

The mathematics of wave motion is a subject of interest to pure mathematicians, applied mathematicians, scientists, and engineers, and it has been the subject of books by people in all these disciplines. On the pure side, there are treatises on aspects of the rigorous theory of partial differential equations. Life is harder on the applied side, where one confidently uses ideas and computations that are not provably correct, and inductive reasoning from simple cases plays a large role. What, then, should be the distinction between an "applied mathematics" treatment of the subject and a "science/engineering" treatment?

In both types of books I expect to see mathematical modeling of physical phenomena involving some simplifications and approximations, followed by some analysis of