18.312: Algebraic Combinatorics — Lionel Levine

Lecture 19

Lecture date: April 21, 2011 — *Notes by: David Witmer*

1 Matrix-Tree Theorem

1.1 Undirected Graphs

Let $G = (V, E)$ be a connected, undirected graph with n vertices, and let $\kappa(G)$ be the number of spanning trees of G.

Definition 1 (Laplacian matrix of undirected graph) *The Laplacian matrix L of G is equal to $D - A$, where*

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

such that d_i is the degree of vertex i, i.e. the number of edges incident to vertex i, and A is the adjacency matrix of G such that

$$A = (a_{ij}),$$

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{else.} \end{cases}$$

Theorem 2 (Matrix-Tree Theorem, Version 1)

$$\kappa(G) = \frac{1}{n}\lambda_1 \lambda_2 \ldots \lambda_{n-1},$$

where λ_1, λ_2, ..., λ_{n-1} are non-zero eigenvalues of the Laplacian matrix L of G.

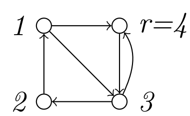
1.2 Directed Graphs

We can give another version of the Matrix-Tree Theorem for directed graphs. First, we need to define spanning trees and Laplacian matrices for directed graphs. Let $\Gamma = (V, E)$ be a directed graph.

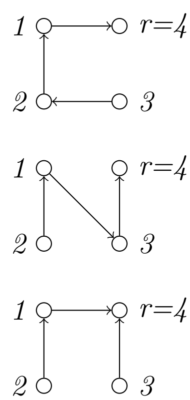
Definition 3 (Oriented spanning tree) *An oriented spanning tree of* Γ *rooted at* $r \in V$ *is a spanning subgraph* $T = (V, A)$ *such that*

1. *Every vertex* $v \neq r$ *has out degree 1.*
2. r *has out degree 0.*
3. T *has no oriented cycles.*

Example 4 *Consider the following directed graph:*



It has three oriented spanning trees:



Definition 5 (Laplacian matrix of directed graph) *The Laplacian matrix* L *of* Γ *is equal to* $D - A$*, where*

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

such that d_i *is the out degree of vertex* i*, i.e.* $\#\{j \in V | (i,j) \in E\}$*, and* A *is the adjacency matrix of* Γ*.*

Theorem 6 (Matrix-Tree Theorem, Version 2) *Let*

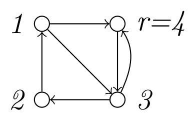
$$\kappa(\Gamma, r) = \#\{\text{oriented spanning trees of } \Gamma \text{ rooted at } r\}$$

and L_r be the Laplacian matrix of Γ with the row and column corresponding to vertex r crossed out. Then

$$\kappa(\Gamma, r) = \det L_r$$

where L_r is the Laplacian matrix L with row and column r removed.

Example 7 *Consider the directed graph from the previous example:*



Then we see that

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

so

$$L = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 1 \end{pmatrix}$$

and

$$L_r = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix}$$

Then

$$\det L_r = 2 \cdot 1 \cdot 2 + -1 \cdot -1 \cdot -1 = 3$$

which matches what we found in the previous example.

We will prove this version of the Matrix-Tree Theorem and then show that it implies the version for undirected graphs.

Proof: Reorder the vertices of Γ so that r is the nth vertex. Then $\det L_r = d_1 d_2 \dots d_{n-1} -$ (other terms), since L_r has the d_i's on the diagonal and either -1 or 0 for the off-diagonal entries. $d_1 d_2 \dots d_{n-1}$ counts the number of subgraphs H of Γ such that each vertex $v \neq r$ has out-degree 1. So we have that

$$H = T \cup C_1 \cup \dots \cup C_k,$$

where T is an oriented tree rooted at r and each C_i is an oriented cycle.

Then

$$\det L_r = \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma) L_{1,\sigma(1)} \dots L_{n-1,\sigma(n-1)}.$$

Let $\operatorname{fix}(\sigma) = \{i \mid \sigma(i) = i\}$. Then we have

$$\det L_r = \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma) \prod_{i \in \operatorname{fix}(\sigma)} d_i \prod_{i \notin \operatorname{fix}(\sigma)} L_{i,\sigma(i)}.$$

$\prod_{i \notin \operatorname{fix}(\sigma)} L_{i,\sigma(i)}$ is only non-zero when $(i, \sigma(i)) \in E$ for all $i \notin \operatorname{fix}(\sigma)$. In this case,

$$\prod_{i \notin \operatorname{fix}(\sigma)} L_{i,\sigma(i)} = (-1)^{n-1-|\operatorname{fix}(\sigma)|}.$$

We wish to write

$$\det L_r = \sum_{\text{subgraphs } H \subset \Gamma} C_H,$$

where C_H is 1 if H is an oriented spanning tree and 0 otherwise. Any permutation σ consists of fixed points and cycles. A subgraph $H = T \cup C_1 \cup \dots \cup C_k$ arises from σ if and only if the union of all cycles C_i of H contains all vertices not fixed by H, which, in turn, is true if and only if $T \subseteq \operatorname{fix}(\sigma)$.

We can then conclude that

$$C_H = \sum_{\{\sigma \in S_{n-1} \mid T \subseteq \operatorname{fix}(\sigma)\}} \operatorname{sgn}(\sigma)(-1)^{n-1-|\operatorname{fix}(\sigma)|}.$$

Our goal is then to show that C_H is 1 when H is a tree and 0 otherwise. When H is a tree, $H = T$ and there are no cycles. Then all vertices are in $|\operatorname{fix}(\sigma)|$ and σ is the identity permutation. The sign of the identity permutation is 1 and $n-1$ points are fixed, so $C_H = 1$.

Lastly, we need to show that $C_H = 0$ if $k \geq 1$, i.e. if H has a cycle. For each C_i, we can either choose $C_i \subset \operatorname{fix}(\sigma)$ or C_i to be a cycle of σ. Let $i_1, \dots, i_l$ be the indices of the C_i's that are formed from vertices in cycles of σ. All other points must be fixed by σ, so

$$\operatorname{sgn}(\sigma) = (-1)^{(|C_{i_1}|-1)+\dots+(|C_{i_l}|-1)}.$$

This means that

$$C_H = \sum_{\{i_1,\dots,i_l\} \in [k]} (-1)^{(|C_{i_1}|-1)+\dots+(|C_{i_l}|-1)} (-1)^{|C_{i_1}|+\dots+|C_{i_l}|}.$$

So,

$$\begin{aligned} C_H &= \sum_{S \subseteq [k]} (-1)^{|S|} \\ &= \sum_{l=0}^{k} \binom{k}{l} (1-1)^k \\ &= 0 \text{ if } k \geq 1. \end{aligned}$$

□

1.3 Proof of the Matrix Tree Theorem, Version 1

Now we will show that Version 2 of the Matrix Tree Theorem implies the version for undirected graphs.

Proof: Given undirected graph G, let Γ be the directed graph with edges (i, j) and (j, i) for every edge of G. We first observe that there is a bijection between the set of oriented spanning trees of Γ rooted at r and the set of spanning trees of G. We can take any oriented spanning tree of Γ rooted at r and get a spanning tree of G by disregarding the root and the orientation of the edges. For any spanning tree T of G, we can get an oriented spanning tree of Γ by orienting edges along the unique path from each vertex to r. Such a path exists because T is connected and is unique because T has no cycles. Then

$$n\kappa(G) = \sum_{r=1}^{n} \kappa(\Gamma, r).$$

Let L be the Laplacian matrix of Γ. Then the characteristic polynomial of L is

$$\chi(t) = \det(tI - L).$$

It is true that

$$\sum_{r=1}^{n} \det L_r = (-1)^{n-1}[t]\chi(t),$$

where $[t]\chi(t)$ is the coefficient of t in $\chi(t)$.

So, we have that

$$
\begin{aligned}
n\kappa(G) = \sum_{r-1}^{n} \det L_r &= (-1)^{n-1}[t]\chi(t) \\
&= (-1)^{n-1}[t]\prod_{i=1}^{n}(t-\lambda_i), \text{ where the } \lambda_i\text{'s are eigenvalues of } L \text{ and } \lambda_n = 0 \\
&= (-1)^{n-1}(-1)^{n-1}\lambda_1 \dots \lambda_{n-1} \\
&= \lambda_1 \dots \lambda_{n-1}.
\end{aligned}
$$

Therefore,

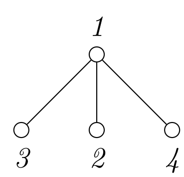
$$\kappa(G) = \frac{1}{n}\lambda_1 \dots \lambda_{n-1}.$$

□

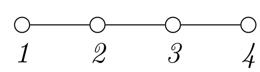
2 Cayley's Theorem

Theorem 8 (Cayley's Theorem) *The number of trees on n labeled vertices is n^{n-2}.*

Example 9 *Consider trees containing 4 vertices. There are $16 = 4^{4-2}$ total, 4 of the form*



and 12 of the form



Proof: Any tree on n vertices is a spanning tree of the complete graph K_n, so we can apply Version 2 of the Matrix-Tree Theorem. So,

$$\kappa(K_n) = \frac{1}{n}\lambda_1 \dots \lambda_{n-1},$$

where

$$\lambda_1, \dots, \lambda_{n-1}$$

are the non-zero eigenvalues of the Laplacian matrix

$$L = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix} = nI - J,$$

where J is the $n \times n$ matrix of ones.

J has the ones vector as one of its eigenvectors. The remaining $n-1$ eigenvectors are of the form

$$\begin{pmatrix} \vdots \\ 1 \\ -1 \\ \vdots \end{pmatrix},$$

so J has eigenvalues $n, 0, \ldots, 0$, with 0 having multiplicity $n-1$. This implies that L has eigenvalues $0, n, \ldots, n$, with n having multiplicity $n-1$.

So,

$$\kappa(K_n) = \frac{n^{n-1}}{n} = n^{n-2}.$$

□

3 Eigenvalues of the Adjacency Matrix

Let G be an undirected, connected graph with n vertices. Let P_l be the number of closed paths in G of length l:

$$P_l = \#\{(v_0, v_1, \ldots, v_{l-1}, v_l = v_0) \mid (v_i, v_{i+1}) \in E \text{ for } i = 0, 1, \ldots, l-1\}$$

Theorem 10

$$P_l = \phi_1^l + \cdots + \phi_n^l,$$

where $\phi_1, \ldots, \phi_n$ are the eigenvalues of the adjacency matrix A of G.

Proof: We observe that

$$(A^l)_{ij} = \#\{\text{paths of length } l \text{ from } i \text{ to } j\}.$$

So,

$$P_l = (A^l)_{11} + (A^l)_{22} + \cdots + (A^l)_{nn} = \mathrm{Tr}(A^l).$$

Note that this holds for both directed and undirected graphs.

Since G is undirected, A is symmetric, which means that A is diagonalizable so there exists some S such that

$$SAS^{-1} = \begin{pmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_n \end{pmatrix}.$$

So,

$$\begin{aligned} P_l &= \mathrm{Tr}(A^l) \\ &= \mathrm{Tr}(SA^lS^{-1}) \\ &= \mathrm{Tr}((SAS^{-1})^l) \\ &= \mathrm{Tr}\begin{pmatrix} \phi_1^l & 0 & \cdots & 0 \\ 0 & \phi_2^l & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_n^l \end{pmatrix} \\ &= \phi_1^l + \cdots + \phi_n^l. \end{aligned}$$

□