1. Graph Theory, Lecture 1.

Lecture 1.

Lets start with a few pictures of graphs. Here is a linear graph.

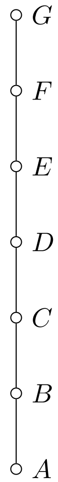
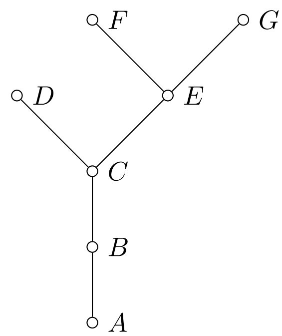
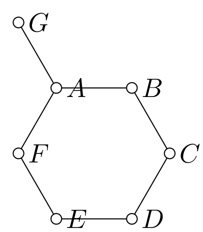


Figure 1. linear graph



A graph like this is called a tree:

Figure 2. a tree



Graphs may admit closed paths:

Figure 3. a hexagon loop

Graphs need not be connected:

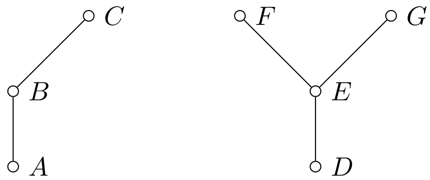


FIGURE 4. a forest consisting of two trees

Now, for the formal definition. A graph Γ consists of a vertex set V and an edge set E. Each edge $e \in E$ connect some pair of vertices, (v_1, v_2). The edges do not have directions so the same edge connects (v_2, v_1). There are never two edges connecting the same pair of vertices.

It follows that we can identify E with a subset of the set of unordered pairs of points from V. We can then unambiguously write $e = v_1 v_2 = v_2 v_1$ to specify an edge, $v_1 \neq v_2$.

Exercise 1 : If V consists of a finite number n of points, what is the largest size that E can be?

The above discussion yields two alternative equivalent definitions of a graph.

Def A. A graph Γ is a set of vertices V and a distinguished set of unordered pairs (= two element subsets) E.

Def B. A graph is a set of vertices V together with a symmetric "anti-equal" relation (*) $\sim$ on V, written $v \sim w$. "Symmetric' means that $v \sim w$ iff $w \sim v$. 'Anit-equal' means that v is NOT related to itself. (no edge connecting v to itself.) In this definition the set of edges E consists of those pairs $\{v, w\} \subset V$ such that $v \sim w$.

(*) Recall from set theory or math 100: a 'relation' on a set V is simply a subset of $V \times V$.

We are primarily interested in the case where V is finite.

Example. V is a finite set. E is empty.

Voila!

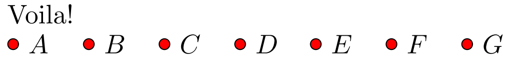


FIGURE 5. a completely disconnected graph

Example. V is a finite set of size n. E consists of all pairs of points from V. We call this graph "the complete graph on n vertices'. It is denoted by K_n.

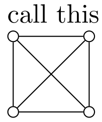


FIGURE 6. the complete graph on 4 vertices, K_4

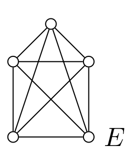


FIGURE 7. the complete graph on 5 vertices, K_5

Example. V is the set of vertices of the n-dimensional cube $0 \leq x_i \leq 1, i = 1, 2, \ldots, n$. Two vertices are connected if and only if we can get from one to the other by traveling along the boundary of the cube in one of the n coordinate directions. This is the same as saying that V consists of all vectors in n-space whose components are all 0's or 1's, and that two vertices are joined by an edge if and only if all but one of their components are equal. This graph is called the "1-skeleton" of the n-cube.

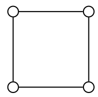


FIGURE 8. the 2-cube

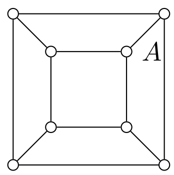


FIGURE 9. the 3-cube

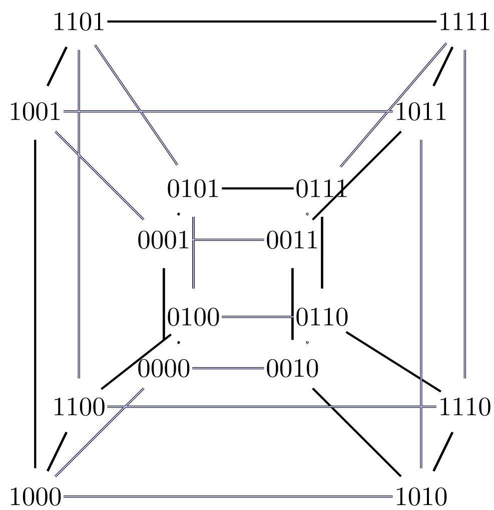


FIGURE 10. 4- cube, aka tesseract:

Compare to binary codes.

: Exercise 2: How many vertices for the n-cube? How many edges?

Example. V is a collection of cities. Two cities are joined by an edge if and only if there is a "direct road" going from one to the other.

Example. V is a collection of web pages. Two web pages are connected by an edge if and only if one web page has a link to the other.

The later example is more naturally a "directed graph", since just because page A links to page B does not mean that page B links to page A. In directed graphs, the edges have arrows. Alternatively, they consist of *ordered pairs.* Directed graphs are also called "digraphs".

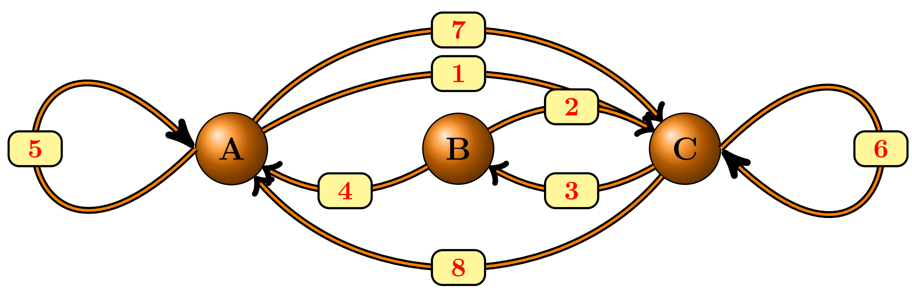
I am rather partial to infinite graphs.

Example. Graph paper. $V = \mathbb{Z}^2$, all pairs (m, n) of integers. Two vertices (m, n) and (p, q) are connected by an edge if and only if $|m - p| + |n - q| = 1$.

EXERCISE 3: Define what it means for two graphs Γ, Γ' to be isomorphic.

Besides directed graphs there are other kinds of graphs, ones with loops, or multiple edges connecting the same pair of vertices. When we need to distinguish the type

of graph we originally defined from these more general types graphs we will refer to the original type of graphs defined above as "simple graphs". For the most part though, we will be only talking about simple graphs.
Here is a directed graph with loops and multiple edges allowed :



EXERCISE 4: Erase the arrows on the edges here. Can I turn this graph into a simple graph by deleting 5 edges? 4 ? 3? Which edges?
NOTATION. We write Γ for a graph. Then $V(\Gamma)$ is its vertex set and $E(\Gamma)$ is its edge set.

2. Metrics on a graph. Emeddings of graphs. Subgraphs.

To do. In class. Lecture 2.
Define a 'path' on a graph.
Imagine all edges as intervals of length 1. Thus the distance between two adjacent vertices is 1.
Define the distance between two vertices.
Define what it means for a graph to be connected.
Recall the definition of a metric space. Show that with this notion of distance, a connected graph is a metric space.
Define the diameter of a graph.
For the 10 examples, what are the diameters of the graph?
What is the diameter of K_n? Of the n-cube?
What is the traveling salesman problem?
HARD: what does it mean to say that the traveling salesman problem is NP complete?
COMBINATORIAL PROBLEMS.
Picture [Gromov] shrinking the edges for $\mathbb{Z}^2$ to converge to $\mathbb{R}^2$.
Exercise [Hard]: For 'graph paper' $\mathbb{Z}^2$, make all the lengths ϵ instead of 1 and call the resulting metric space $\epsilon\mathbb{Z}^2$. Define what I mean when I say that as "' as $\epsilon \to 0$ $\epsilon\mathbb{Z}_2 \to (\mathbb{R}^2, d_1)$". Here d_1 is the "L_1-metric": $d_1(x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$.

3. Incidence Matrix

A graph Γ is completely encoded by its "adjacency matrix". This is a square matrix, $A = A(\Gamma)$ whose columns and rows are labelled by the vertices $v \in V$. We put

$$A_{vw} = \begin{cases} 1, & \text{if } vw \in E \\ 0, & \text{else.} \end{cases} \quad . \tag{1}$$

EXERCISE. Write out the adjacency matrices for all of the examples above, and in particular figures 1-10.
Define the degree d_A of a vertex.
Exercise. Let $\vec{1}$ be the column vector each of whose entries is 1. Show that $A_\Gamma \vec{1}$ is the vector whose Ath entry is the degree d_A of vertex A.
Define what it means for a graph Γ to be k-regular.
Answer: the degree d_A of every vertex A is the same and equal to k.
Cor: Γ is k-regular if and only if $A_\Gamma \vec{1} = k\vec{1}$.
Draw some 3 regular graphs on 6, 7, and 9 vertices.
Graph invariants from the adjacency matrix.
A graph invariant is a numerical invariant independent of the isomorphism class of the graph. Examples are number of vertices, number of edges, number of cycles, number of spanning trees, diameter, degree sequence, ...
If we permute the rows of columns of A, we induce an isomorphic graph. Thus, to get invariants out of $A(\Gamma)$ we need permutation invariants. The most immediate such invariants are the spectral invariants.
Recall the characteristic polynomial.

$$det(A - \lambda I) = \lambda^n + c_2 \lambda^{n-2} + \ldots + c_n.$$

Show that , indeed , $c_1 = 0$.
Show that c_2 is the number of edges in Γ.
Show that c_3 is the number of triangles.
Show that that the number of walks of length r from vertex v to vertex w is A^r_{vw}.

4. Combinatorics

Draw all graphs on 3 vertices. On 4.
How many (labelled) graphs are there on n vertices?
What does it mean for two graphs to be isomorphic?
Question: can a 2-regular graph be disconnected?
Question: How many distinct 2 -regular connected graphs are there on n vertices?
EXERCISE: How many 3 regular graphs on 7 vertices are there? Draw them.
Draw all NON-ISOMORPHIC graphs on 3 vertices. On 4.
Draw all NON-ISOMORPHIC graphs with 3 edges on 5 vertices. On 6 vertices. On 7 vertices.
Polya derived a formula for, and algorithms for computing generating function which counts the number of distinct graphs on n vertices with k edges. (See web link. And final ch. of Bollabas.) This polynomial is written $g_n(z) = 1 + z + 2z^2 + \ldots$ Its kth coefficient, g_{nk}, is the number of distinct graphs have n vertices and k edges.

$$g_3(z) = 1 + z + z^2 + z^3$$

$$g_4(z) = 1 + z + 2z^2 + 3z^3 + 2z^4 + z^5 + z^6$$

$$g_5(z) = 1 + z^2 + 2z^2 + 4z^3 + 6z^5 + 6z^6 + 6z^7 + 4z^7 + 2z^8 + z^9 + z^{10}$$

$$g_6(z) = 1+z+2z^2+5z^3+9z^4+15z^5+21z^6+24z^7+24z^8+21z^9+15z^{10}+9z^{11}+5z^{12}+2z^{13}+z^{14}+z^{15}$$

$$g_7(z) = 1+z+2z^2+5z^3+10z^4+21z^5+41z^6+65z^7+97z^8+131z^9+148z^{10}+148z^{11}+131z^{12}+\ldots+z^{21}$$

Explain why g_{n2} is always 2, once $n > 3$.

Explain why g_{n3} eventually stabilizes: there is an N_0 such that for $n > N_0$ we have $g_{n3} = 5$.

Explain why g_{nk} eventually stabilizes: there is an N_0 such that for $n > N_0$ we have $g_{nk} = g_{N_0,k}$.

Explain why the g_n are symmetric: If d is the degree of g_n then $g_{nk} = g_{n,d-k}$.

Explain why the degree d of g_n is $d = \binom{n}{2}$.