1. The incidence matrix, its kernel and cockerel.

The incidence matrix.

The entries of the incidence matrix are d_{ia} where i runs over an index set for the vertices (could be the vertices themselves!) and a runs over an index set for the edges. Thus the vertex set is

$$V(\Gamma) = \{v_i, i = 1, \dots, n\}$$

while the edge set is

$$E(\Gamma) = \{e_a : a = 1, \dots, m\}$$

To define the incidence matrix we choose an orientation for each edge, which means a 'positive' end and a 'negative' end for each edge.

Then

$$d_{ia} = \begin{cases} 1, & \text{if } v_i = \text{is the positive end of } e_a \\ -1, & \text{if if } v_i \text{ is the negative end of } e_a \\ 0, & \text{if } v_i \text{is not incident to } e_a \end{cases} .$$

Now recall that

$$C_1(\Gamma) \cong \mathbb{R}^m; \text{ coordinates } \phi \mapsto \phi(e_a)$$

and

$$C_0(\Gamma) \cong \mathbb{R}^n; \text{ coordinates } f \mapsto f(v_i).$$

For $\phi \in C_1(\Gamma)$ a function on edges we have

$$D\phi(v_i) = \Sigma d_{ia}\phi(e_a)$$

while for $f \in C_0(\Gamma)$ a function on vertices we have:

$$df(e_a) = \Sigma d_{ia} f(v_i)$$

Then

$$D = d^T$$

and they form a pair of linear operators: $C_0(\Gamma) \underset{D}{\overset{d}{\rightleftarrows}} C_1(\Gamma)$ In defining the transpose L^T of a linear operator L, we need an inner product on its domain and range. In our case of $D = d^T$, we need an inner product. For us, the transpose is relative to the following natural inner products on the two spaces:

$$\langle f, g \rangle = \Sigma_{i=1}^n f(v_i) g(v_i)$$
$$\langle \phi, \psi \rangle = \Sigma_{i=1}^m \phi(e_a)\psi(e_a)$$

Exercise 1. *Verify that these formulae for d, D are the same as the "intrinsic' formula: If $e_a = (v_i v_j)$ then*

$$df(e_a) = f(v_j) - f(v_i)$$

Notation: If the positive vertex is v_j and the negative vertex is v_i then we will also write $e_a = v_i v_j$. And

$$D\phi(v_i) = \Sigma\phi(v_j v_i) - \Sigma\phi(v_i v_k)$$

where the first sum is over edges with positive end v_i and the second sum is over those edges whose negative end is v_i.

Exercise 2. *Verify that the graph's Laplacian is given by $\Delta = Dd$.*

Theorem 1. *The kernel of d consists of the functions which are constant on each connected component of Γ.*

PROOF OF THEOREM. IN CLASS.

Corollary 1. *$dim(ker(d)) = c$ is the number of connected components of Γ. The dimension of the image of d is $n - c$. The dimension of kernel of D is $m - (n - c) = m - n + c$*

Proof of Cor. Use the rank plus nullity theorem: $dim(im(L)) + dim(ker(L)) = dim(dom(L)$ where $dom(L)$ is the domain of L.

For the second statement use the fact that $im(L)^{\perp} = ker(L^T)$ valid for any linear operator between inner product spaces. Then use the rank plus nullity theorem. QED

Corollary 2. *The kernel of the Laplacian consists of the functions which are constant on each connected component of Γ.*

Proof. The Laplacian is $d^T d$. It is a general fact that the kernel of $L^T L$ is equal to the kernel of L, for any linear operator L between inner product spaces. QED

CYCLE SPACE. Take a cycle in our graph. Orient it as per the standard cycle so the edges follow each other in a "circle". Now, for our given orientation of Γ and a given edge $e_a \in C$, that edge's orientation may or may not agree with the orientation we gave to C. We want to make it agree by using a choice of signs for a function supported on the edges of C. We call this the 'cycle function' of C and write it ξ_C :

$$\xi_C(e_a) = \begin{cases} 1, & \text{if } e_a \in C \text{ and its orientation agrees with } C \\ -1, & \text{if } e_a \in C \text{and its orientation is opposite of } C \\ 0, & \text{if } e_a \notin C \end{cases}.$$

Exercise. $D\xi_C = 0$ for any 'cycle function' C.

Later, we will see that $ker(D)$ is spanned by the cycle functions.

CUT SPACE. Fix a partition of $V(\Gamma)$ into two non-empty disjoint subsets, V_1, V_2. Thus $V = V_1 \cup V_2$ and consequently $V_2 = V \setminus V_1$. Let $H \subset E(\Gamma)$ be the set of all edges with one end in V_1 and the other in V_2. We call any such H a "cut set". We can choose an orientation for the edges of H by insisting that they 'go' from V_1 to V_2: so each $e \in E$ has positive end in V_2 and negative end in V_1. We will call this the "cut orientation" of the edges of H. We can now define the cut function associated to $H = H(V_1, V_2)$ by

$$\chi_H(e_a) = \begin{cases} 1, & \text{if } e_a \in H \text{ with cut-orientation agreeing with graph's orientation} \\ -1, & \text{if } e_a \in H \text{with cut-orientation opposite to graph's orientation} \\ 0, & \text{if } e_a \notin H \end{cases}.$$

Theorem 2. *The image of d is spanned by cut functions χ_H. This image equals the orthogonal complement of the kernel of D and is spanned by cut functions.*

Proof of theorem. For the first statement, let f_i denote the standard basis for $C_0(\Gamma)$, so that $f_i(v_j) = \delta_{ij}$. Compute that $df_i = -\chi_{H_i}$ where H_i is the partition with $V_1 = \{v_i\}$ and $V_2 = V \setminus \{v_i\}$.

For the second statement use that for any linear operator $L : X \to Y$ between inner product spaces we have $im(L)^{\perp} = ker(L^T)$ or $im(L) = ker(L^T)^{\perp}$. Here $L = d, L^T = D$.

From these theorems we have

$$dim(I'md) = dim(cutspace) = n - c$$

where , we recall c is the number of components of the graph.

Thus

$$dim(kerD) = dim(cyclespace) = m - (n - c)$$

Spanning trees to get a basis for the cut and cycle space

Suppose now that Γ is connected so that $c = 1$. Let $T \subset \Gamma$ be any spanning tree. To form T we had to take out a number of edges out of Γ.

Exercise 3. *If you add back to Γ any one of the edges you took away, the resulting graph has exactly one cycle*

Exercise 4. *If you take away from T a single edge the result is disconnected.*

Theorem 3. *Fix a spanning tree. Write $E(\Gamma) \setminus E(T)$ for the edges we had to take away from Γ to form T. For each edge $e_a \in E(\Gamma) \setminus E(T)$ let ϕ_a be the cycle function associated to the unique cycle we get by adding e_a back to T. Then the ϕ_a form a basis for the cycle space.*

Dimension count check. As we saw just a bit earlier $dim(cyclespace) = m-(n-1)$. The number of edges of T is $n - 1$. Hence the number of edges of $E(\Gamma) \setminus E(T)$ is $m - (n - 1)$, the correct number to form a basis for the cycle space.