GRAPH LAPLACIAN, DIFFERENTIAL. CYCLE BASIS.

RICHARD MONTGOMERY

Abstract. Working out the incidence matrix and Laplacian in terms of the differential on a graph.

1. Spectral Theory of Graphs.

References. The best source for this material, out of the texts we have listed, is chapter 4 of Biggs. Some of the material can also be found in ch. 12 of the online Murty and towards the end of ch 3 of Bollabas.

Write $C^0(\Gamma)$ for the vector space of all real-valued functions on the vertices of Γ. Several authors call this space "the vertex space". I will call it the space of functions on the graph. $C^0(\Gamma)$ is a real vector space of dimension $|V| = n$:

$$C^0(\Gamma) = \mathbb{R}^{V(\Gamma)} = \text{ real valued functions on } V$$

It has for a canonical basis, the "delta functions" (following Dirac), which are the functions $\delta_v : V \to \mathbb{R}$ which are one on v and 0 off of the vertex v: in symbols:

$$\delta_v(w) := \delta_{vw} = \begin{cases} 1, & \text{if } v = w \\ 0, & \text{if } v \neq w \end{cases} . \tag{1}$$

We sometimes just write v instead of δ_v.

Exercise. Show that any $f \in C^0(\Gamma)$ can be expressed uniquely as $f = \Sigma_{v \in V} f(v)\delta_v$.

1.1. Laplacian on a graph.

Define

$$\Delta : C^0(\Gamma) \to C^0(\Gamma)$$

by

$$(\Delta f)(v) = \Sigma_{e=vw \text{ an edge incident to } v}(f(v) - f(w)).$$

Exercise 1. *Show that relative to the canonical basis $\delta_v, v \in V$ we have that*

$$\Delta = DEG - A$$

where DEG is the diagonal matrix whose vv entry is $deg(v)$, and where $A = A(\Gamma)$ is the adjacency matrix of last week.

WARNING: Biggs writes Q for our Δ and Δ for our DEG.

Definition 1. *The 'tree number" of a graph Γ is the number of spanning trees of Γ.*

Date: January 19, 2016.

Example: If Γ is disconnected then its tree number is zero since it has no spanning trees.

Example. If Γ is the cyclic graph on n vertices then its tree number equals to n.

Example. If Γ is a tree then its tree number equals 1.

Recall the cofactor of an $n \times n$ matrix M. $C_{ij} = (-1)^{i+j} det(\hat{M})_{ij})$ where $\hat{M})_{ij}$ is the $n-1 \times n-1$ matrix we get by deleting the ith row and jth column of M.

Theorem 1. *Every cofactor of Δ is the same and equals to the tree number of Γ.*

APPLICATION. Cayley's formula. Since the complete graph on n vertices contains every labelled tree, its tree number is the total number of labelled trees on n vertices - the number Cayler counted and Prüfer proved is correct.

A) Write out the matrix Δ_n for Laplacian for the complete graph on n vertices (in terms of the standard basis!).

B) Let J_n be the n by n matrix all of whose entries are 1. Show that $\Delta_n = nI_n - J_n$ where J_n.

C) Show that the $ij = 1, 1$cofactor of Δ_n is $det(nI_{n-1} - J_{n-1})$.

D) Compute $det(nI_{n-1} - J_{n-1})$ by completing the following exercises which will allow us to compute $det(\lambda I_k - J_k)$ the characteristic polynomial of J_k, for k a positive integer.

D1) Show that the rank of J_k is 1 and its nullity is $k-1$.

D2) Show that the only nonzero eigenvalue of J_k is k.

D3) Conclude that $det(\lambda I - J_k) = \lambda^{k-1}(\lambda - 1)$.

D4) Now set $\lambda = n, k = n-1$ to finish off the computation of the cofactor.

1.2. Proof of tree number formula; more Laplacian facts.

Proposition 1. *(1) The constant functions are in the kernel of Δ.*

(2) The nullity of Δ equals the number of connected components of Γ.

Corollary 1. *If Γ is connected then the kernel of Δ is one-dimensional and consists of the constant functions.*

Recall: the cofactor of a matrix. Recall the cofactor formula for the inverse of a matrix. Recall that the product of a square matrix M and its cofactor matrix is equal to $det(M)Id$

....

Remark. It is really not so important that the functions are real-valued. They just need to take values in some field. It could be $\mathbb{C}, \mathbb{Q}$ or even the field with two elements. For definiteness, think of them as real valued.

We define a differential by

$$df(vw) = f(w) - f(v); vw \in E.$$

so that

$$d : C^0(\Gamma) \to C^1(\Gamma) := \text{functions on the edges}.$$

Oop! We have a problem here. Which comes first v or w? The edges, as we defined them, are unoriented: $vw = wv$. But for the differential to be defined we need to order them.

Definition 2. *An orientation of an edge is an answer to the question: does e go from v to w or from w to v? If the edge goes from v to w then the orientation* *is (v, w) and we call v the "positive end" and w the 'negative end. We write $e_+ = w, e_- = v$. If the edge goes from w to v then the orientation is (w, v). The two orientations are formal negatives: $-(v, w) = (w, v)$: reversing the pair, reverses the arrow , switches signs.*

An orientation of a simple graph is a choice of orientation for each edge. We write such a choice by putting a tilde over E: $\tilde{E}(\Gamma)$

Bollabas, p. 6, calls these "oriented graphs". So his "oriented graph" is a directed graph arising by choosing an orientation for a simple graph.

There are $2^{|E|}$ orientations of the graph. Fix one. By abuse of notation, continue to write E for $\tilde{E}$.

Definition 3. *$C^1(\Gamma)$ is the space of real-valued function on the oriented edges of the graph Γ and is called the "edge space".*

. Bollabas around p. 38. Biggs, p. 24.

Now we have a well defined differential

$$d : C^0(\Gamma) \to C^1(\Gamma)$$
$$df(e) = f(e_+) - f(e_-)$$

CF: Murty, p. 212. Eq (12.2). Murty write δ for d and calls the oriented edges "arcs".

Now $C^0(\Gamma)$ and $C^1(\Gamma)$ have a canonical basis, the vertices and edges respectively. Relative to this basis the matrix of the differential is an m by n matrix whose entries are all $0, 1$ or -1. Show that relative to this basis, the matrix of d is

$$\text{for } e \in \tilde{E}, v \in V : M_{e,v} = \begin{cases} 1, & \text{if } e_+ = v \\ -1, & \text{if } e_- = v \\ 0, & \text{else} \end{cases}.$$

TEXTS: this is the matrix B^T of p. 38 of Bollabas. The matrix B is called the "incidence matrix". This is the matrix D^T of Biggs, DEf. 4.2, again called the "incidence matrix" for the digraph. This matrix is NOT QUITE the matrix $M(G)^T$ from p. 7 of Murty. What is the difference between Murty's incidence matrix $M(G)$ and our M^T?

Exercise. Use the basis ... Write out the matrix for the differential on K_3 and K_4 as indicated.

Now, if V is finite, then $C^0(V)$ inherits a canonical inner product for which the δ_v's are orthonormal: Namely:

$$\langle f, g\rangle = \Sigma_{v \in V} f(v)g(v)$$

Similarly $C^1(E)$ has a canonical inner product:

$$\langle \alpha, \beta\rangle = \Sigma_{e \in \tilde{E}} \alpha(e)\beta(e)$$

We can then compute the dual of d by:

$$\langle df, \alpha\rangle = \langle f, d^* \alpha\rangle.$$

EXERCISE

Show that

$$(d^*\alpha)(v) = \Sigma_{e:e_+ = v}\alpha(e) - \Sigma_{e:e_- = v}\alpha(e).$$

Alternatively

$$(d^*\alpha)(v) = \Sigma_{e \in \tilde{E}} M_{e,v}\alpha(e).$$

with M as above.

Exercise. Verify that the matrix D of d^* and the matrix M of d are related by $M = D^T$.

Definition 4. *The Laplacian on the graph is the linear map*

$$\Delta = d^* d : C^0(\Gamma) \to C^0(\Gamma).$$

Exercise. Let DEG be the diagonal matrix whose entries are the degrees of the vertices: $DEG_{vv} = deg(v); DEG_{vw} = 0, v \neq w$. Show that

$$\Delta = DEG - A(\Gamma)$$

where $A(\Gamma)$ is the adjacency matrix described earlier.

TEXTS: prop 4.8, p. 27, Biggs. Bollabas, Theorem 6, p. 38. Biggs writes Δ for our DEG.

(Montgomery) Mathematics Department, University of California
E-mail address: rmont@ucsc.edu