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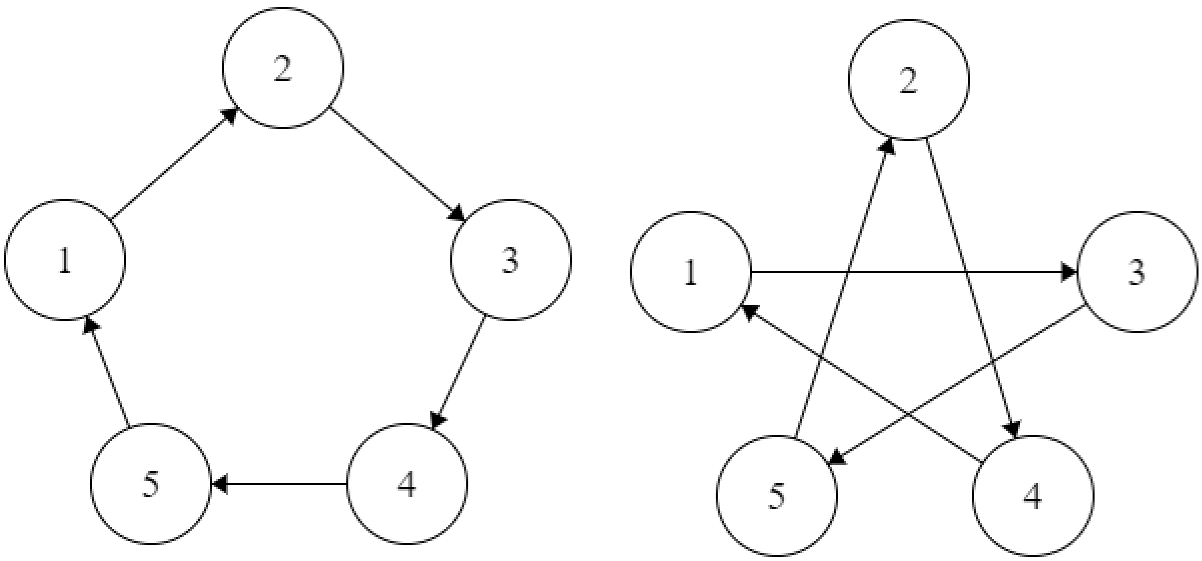
MATH 115

Homework 1

Note: The following graphs, despite the arrows on each edge, are not directional. All graphs depicted are undirected.

1.

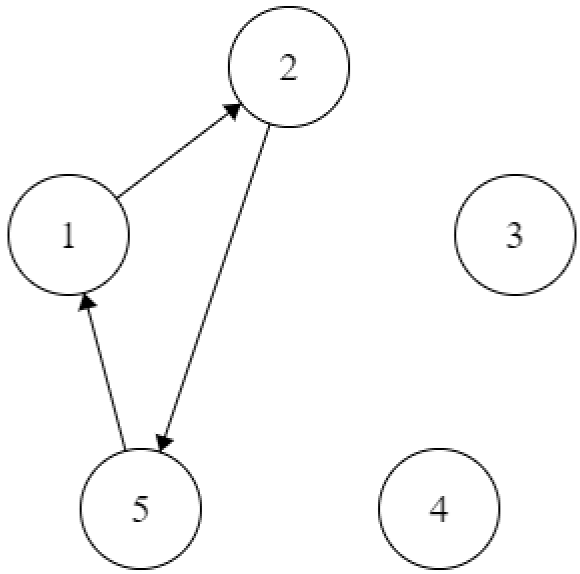


The above graphs are not directional—rather, the arrows dictate a path taken to produce an isomorphism Φ as follows:

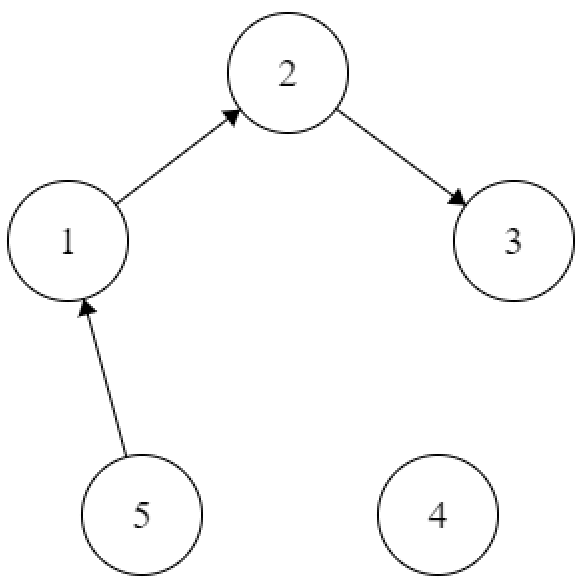
$$\boxed{\Phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}}$$

2. Up to isomorphism, there is only one 2-regular graph on 5 vertices.

Proof: Consider an undirected graph G with a set of vertices $V = \{1, 2, 3, 4, 5\}$, with vertices 1, 2, and 5 forming a cycle as shown:



The remaining ways in which vertices 3 and 4 can be connected to each other or the rest of G are not sufficient to create a 2-regular graph, as vertices 3 and 4 would have a degree of 1, or vertices 1, 2, or 5 would have a degree greater than 2.



Consider the graph above. Since connecting 3 and 5 would leave 4 disconnected, we are only left with connecting 3 to 4 and 4 to 5 in order to create a 2-regular graph. Therefore, there is only one 2-regular graph on 5 vertices.

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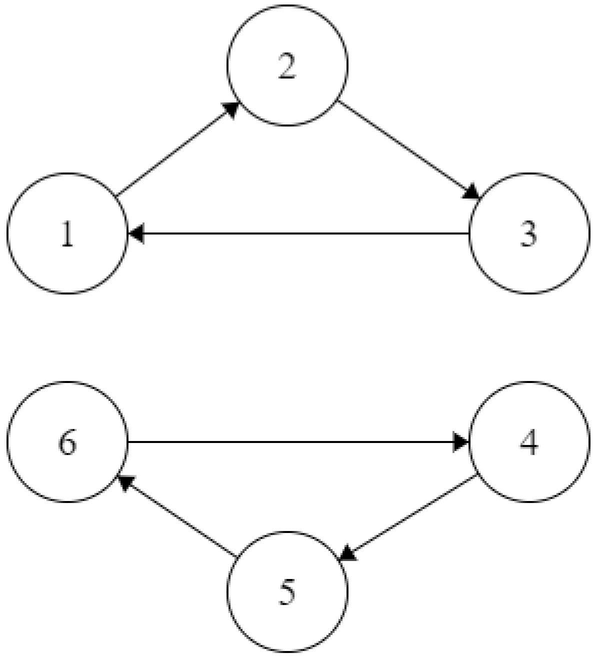
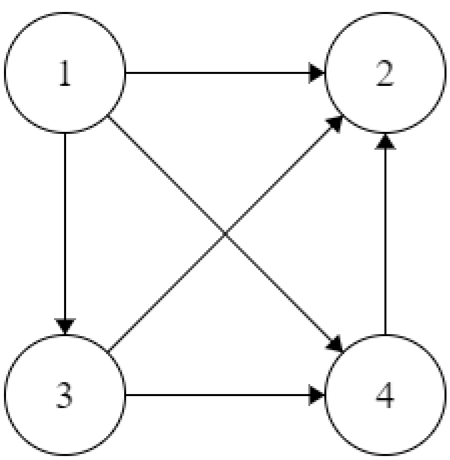
3. Up to isomorphism, there is only one 2-regular graph on 6 vertices.

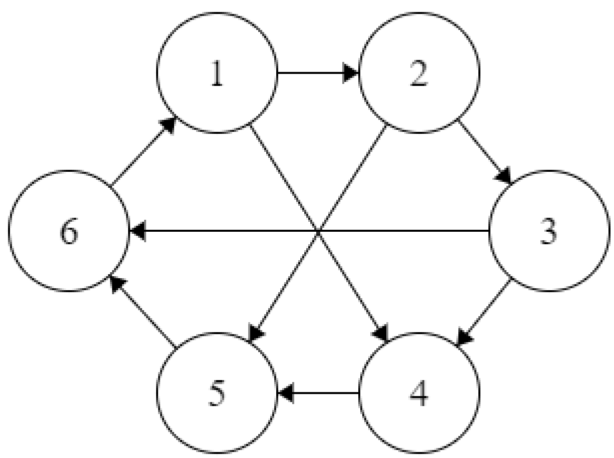
Counterexample:



4. The additional "connected" qualifier doesn't affect problem 2. However, my counterexample in problem 3 would be invalid, as it is a disconnected graph. The original statement would then be true, as all vertices would have to be connected to form the shape of a circle in order to keep each vertex at a degree of 2.

6.





7. There is a 3-regular graph on 5 vertices—false. Proof: Consider the following formula, which we will accept as true:

$$2m = \sum_{V=1}^{n} \deg(V)$$

Where m is the number of edges and n is the number of vertices. In other words, the sum of the degrees of all vertices in any graph is equal to twice the number of edges. For a 3-regular graph with 5 vertices, the sum of the degrees of all vertices is 15.

$$\sum_{V=1}^{5} \deg(V) = 15$$

$$\Rightarrow 15 = 2m$$

$$\Rightarrow 7.5 = m$$

So by the formula, we can see that a 3-regular graph on 5 vertices would require 7.5 edges, which is impossible. Therefore, there is no 3-regular graph on 5 vertices.

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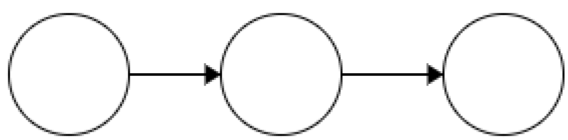
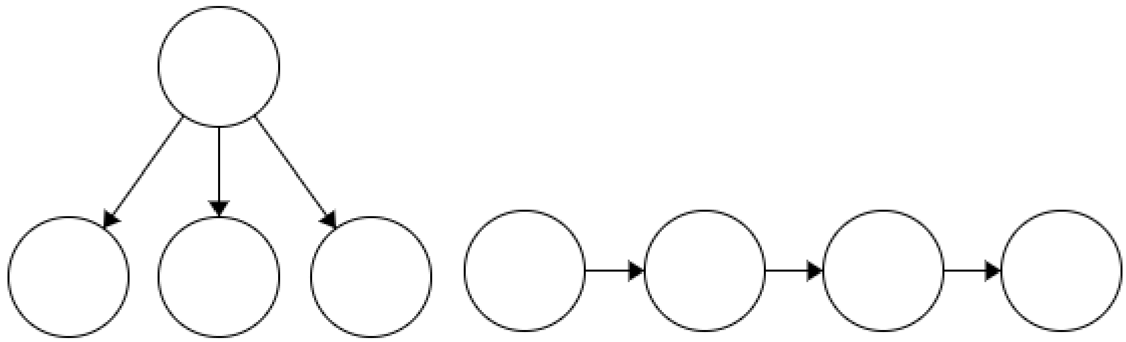
8. For brevity and sanity, I will only depict unlabeled isomorphisms of trees on 3, 4, and 5 vertices.

3 vertices.



There are only 3 ways to arrange labels here. 1-2-3, 1-3-2, and 2-1-3.

4 vertices.

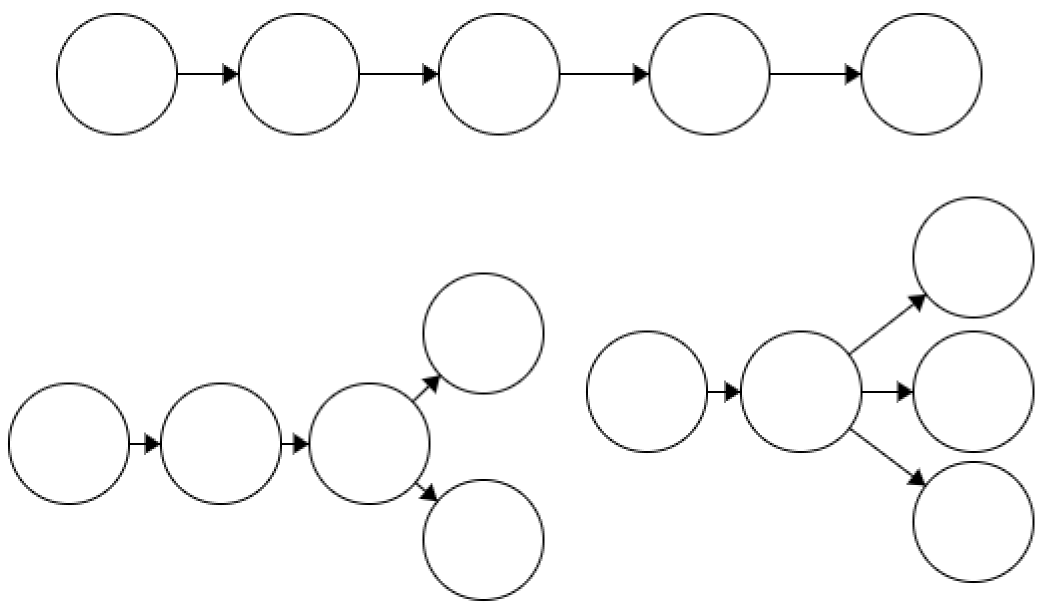


For the left isomorphism, there are only 4 ways to arrange the labels, as the only unique vertex is the one with degree 3. For the right isomorphism, there are

$$\frac{4!}{2} = 12$$

ways to arrange labels, as half of the possible arrangements are reflections of the other half. So there are 16 ways total.

5 vertices.



As with the previous case, there are only $\frac{5!}{2} = 60$ ways of arranging labels for the top tree. For the left tree, there are $5 * 4 * 3 = 60$ ways of arranging labels as well. For the right tree, there are only 5 ways of arranging labels, since the middle vertex is the only unique one, with a degree of 4. There are 125 ways total, which falls in line with the formula n^{n-2}, where n is the number of vertices.