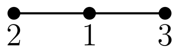
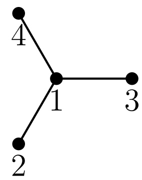
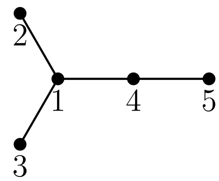
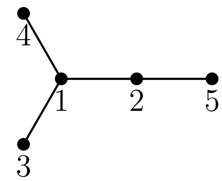
#0.5(a): Given a minimally connected graph G, let us assume that it contains a cycle. If we identify the cycle by vertices $(v_1, v_2, \ldots, v_n)$, then let us remove the edge (v_1, v_n) from G, creating a new graph G'. Let us now examine two arbitrarily-selected vertices of G'; call them a and b. Because G' is a subgraph of G, a and b are also vertices of G, and therefore have a path P connecting them. We will now show that a and b are path-connected in G' on a case-by-case basis. In the case where $(v_1, v_2) \notin P$, then every edge of P is in G', so P is a path on G' and we are done. If $(v_1, v_2) \in P$, then let us construct a new path as follows: The first part is the subpath of P from a to the fist occurance of a $v_j \in \{v_1, v_2, \ldots, v_n\}$. The last part is the subpath of P from the last occurance of a $v_k \in \{v_1, v_2, \ldots, v_n\}$ to b. The middle part is made up of the vertices $v_j, v_{j+1}, \ldots, v_k$ or $v_j, v_{j-1}, \ldots, v_k$, depending on whether $j \leq k$ or not. All the vertices in the first and last part are adjacent, because those regions of P come before and after (v_1, v_n) respectively, and all the vertices in the middle part are adjacent by their definition. Showing that no vertices are repeated follows from the definition as well. The combination of all three is therefore a path in G' connecting a and b. The existance of such a path in both cases, combined with the arbitrary selection of a and b means that G' is path-connected. But then G could not be minimally connected; a contradiction. Thus no minimally connected graph can contain a cycle.

#0.9: (a) 

(b) 

(c) 

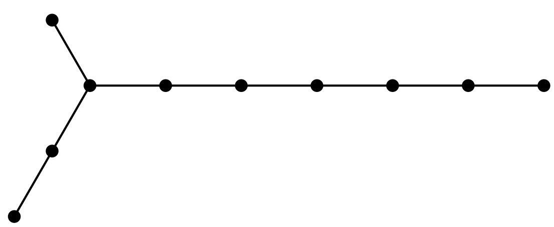
(d) 

In the case of (a) and (b), the only number in the symbol is 1, so everything else must be leaves connected to 1. In the case of (c), the first two numbers in the symbol are both 1, so 2 and 3 are the first two leaves, both connected to 1. After performing appropriate deletions, we are left with the symbol (4) over vertices 1, 4, and 5. The result is similar to (a); 1 and 5 are both leaves of 4. Adding back the 2 and 3 leaves to 1 gives us the finished graph. (d) is quite similar, except that 2 is present in the symbol, so the first two leaves are 3 and 4, and then the 3-vertex graph after deletions is 1-2-5.

#1.2(a): Let us find an upper bound on the number of labelled trees. For every labelled tree, there is an unlabelled counterpart (i.e. the tree derived by removing the labels). We can therefore be guaranteed to count every labelled tree by counting every possible labelling of every possible unlabelled tree. Since there are $n!$ possible ways to uniquely label n vertices, we get the following inequality:

$$n^{n-2} = \#\text{ of labelled trees on } n \text{ vertices} \leq n! \cdot T(n)$$

Since certain labellings of unlabelled trees produce isomorphic labelled trees (for instance 1-2-...-n and n-...-2-1), the inequality becomes strictly less than. Dividing both sides by $n!$, we get $\frac{n^{n-2}}{n!} < T(n)$



#1.2 supplement:
There are 10! possible labellings of the tree by assigning the numbers 1-10 each to a vertex. To show that they are distinct, let us assume that two labellings lead to isomorphic labelled trees. Then consider the function which takes each vertex on one tree to the correspondingly labelled vertex on the other. Because the two trees are isomorphic, this is an automorphism. Because the labellings were different, it is not the identity. But this is a contradiction to there being no non-identity automorphisms. Thus each of the 10! labellings is distinct.

I couldn't figure out the extra credit, but I did notice a couple interesting things. First, when describing a tree automorphism, it is sufficient to note where the leaves are sent. Second, the midpoint of a maximum path on a tree will be the midpoint of every maximal path on that tree. (Sketch of proof: Assume two maximum paths have different midpoints. Connect the two midpoints via a path and add on half of one maximal path and half of the other. The resulting path will be longer than maximal; a contradiction.)