HW, DUE FEB 27, ON THE HYPERCUBE GRAPH.

Look up 'hypercube graph" on wiki. Focus on the right hand top panel of the wiki page for the hypercube graph which lists its characteristics.

Also look up "Hamming distance".

1. Draw Q_2 as a square in the plane. According to wiki it's automorphism group has $2!2^2$ elements. Every one of these elements can be represented as a rotation or reflection. Draw pictures representing each automorphism.

2. Draw Q_3 as the edges and vertices of a unit cube in space. According to wiki the automorphism group of Q_3 has.... how many elements? All of these again are realized by isometries; rotations and reflections . Sketch a few

3. *Background:*The unit n-cube in $\mathbb{R}^n$ is the closed set of all vectors $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ such that $0 \leq x_i \leq 1$ for all $i = 1, 2, \ldots, n$. Its vertices are obtained by maxing out these inequalities: all the x_i are either 0 or 1. Its edges are realixed by maxing out all but one of these inequalities and letting the remaining one "slide": thus the edge joining $(0, 0, \ldots, 0)$ to $(1, 0, \ldots, 0)$ is given by $0 \leq x_1 \leq 1$ while $x_j = 0, j > 1$. Together, these vertices and edges form the hypercube graph Q_n. (In topology they would be called the "1-skeleton" of the n-cube.)

Problem: Verify [= prove] that the following is an equivalent description of the hypercube Q_n. Its vertices are the vectors in the n-dimensional vector space

$$V = \mathbb{Z}_2^n$$

over the field $\mathbb{Z}_2$ of two elements. Two vertices are adjacent if and only if the Hamming distance between them is 1.

4. Any fixed $v \in V$ defines a "translation action" of V on itself by sending $s \mapsto s + v$. Verify that each such translation action is an automorphism of the hypercube graph.

5. Any fixed permutation $\sigma \in S_n$ of the n coordinate labels defines an action on V by

$$\sigma(x_1, x_2 \ldots, x_n) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)})$$

Verify that this mapping defines an automorphism of Q_n.

6. A) What is the order of the group of all translations as in problem (4)?

B) What is the order of the group of all permutation actions as in problem (5) ?

C) Compare the two numbers with the number wiki given for the order of $Aut(Q_n)$. Discuss.

7. Look up "Cayley graph". One speaks of the 'Cayley graph of a group G upon choosing a finite set of generators S". V above forms a group under (vector) addition. Take as generating set S for V the standard basis. Show that the Cayley graph of the group V with the generating set S is the hypercube graph. [Use the undirected version of the Cayley graph].

8. [EXTRA CREDIT] Repeat problem 7, but with another cyclic group $\mathbb{Z}_k$ in place of $\mathbb{Z}_2$. Thus, in place of V take the Abelian group $\mathbb{Z}_k^n$ so that our the numbers s_k in our sequences $(s_1, s_2, \ldots, s_k)$ now have k possibilities instead of just two possibilitied. How many vertices? How many edges? Draw out what happens for $\mathbb{Z}_3^3$.