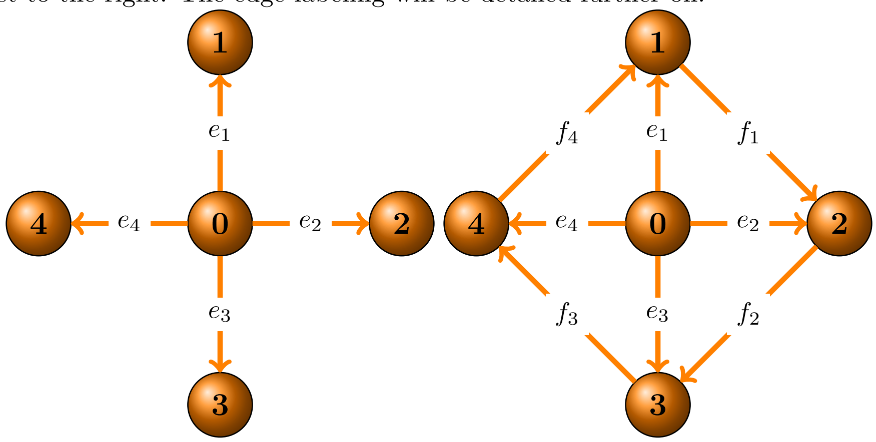
HOMEWORK, DUE FEB 14, VALENTINE'S DAY: CYCLE AND CUT SPACES; HOW THEY WORK ON WHEEL GRAPHS

Below is the star graph with 4 edges, $K_{1,4}$ with its associated wheel graph W_4 just to the right. The edge labeling will be detailed further on.



1. Look up the definition of 'star graph" and "wheel graph".

A) Draw the star graph $K_{1,5}$ with 5 edges and the wheel graph W_5.

B) The star graph with n edges is also referred to as the bipartite gaph $K_{1,n}$. Why?

Labelling of edges , vertices. We label the 'hub vertex' of the wheel as 0 and the other vertices $1, 2, \ldots, n$ in cyclic order. Label the edges coming out of the hub as $e_1, e_2, \ldots .e_n$, ordered pointing out so that $e_i = (0, i)$ points from 0 to i. Label the edges going around the big cycle as $f_1, f_2, \ldots, f_n$ with $f_i = (i, i+1)$ going from i to $i+1$. See figures above.

2. The star graph $K_{1,4}$ depicted above is a spanning tree for the wheel graph W_4 to its right.. Like any spanning tree, the star graph induces a basis for both the cycle space and the cut space of the wheel graph.

A i) Write out the corresponding basis for the cycle space.

Use the following convention. The symbol e_i will also mean the function that is 1 on edge e_i and 0 on all other edges. So $-e_i$ is the function that is -1 on e_i and 0 on all other edges. Similarly for f_i. *Example:* $e_1 + f_1 - e_2$ is the cycle vector corresponding to the triangle with vertices 012.

A ii) Write out the corresponding basis for the cut space.

B) Now draw another different spanning graph for W_4. List the edges that you used.

B i) Write out the corresponding basis for the cycle space defined by this choice of spanning graph.

B ii) Write out the corresponding basis for the cut space.

C i) Write out the change of basis matrix taking you from the cycle basis of A i) to that of B i)

D) i) How many cycles are there in W_4?

ii) Compare the number from (D) (i) to the total number of elements in the cycle space *when our underlying field is the field with 2 elements.*

3) A) Proceed to the general wheel graph W_n, and its spanning tree, the star graph $K_{1,n}$. Use the labeling system as above for vertices and edges.

Do the general analogue of problem 2A (i): write down the basis for the cycle space of W_n which is induced by the spanning tree $K_{1,n} \subset W_n$.

4) i) How many spanning trees are there in W_4?

ii) Verify that the number you found 'by hand" in part (i) agrees with the number given by the tree number formula (any cofactor of the Laplacian of W_4).

5) i) Describe the automorphism group of the star graph $K_{1,4}$. How many elements are in it? Is it Abelian?

ii) Describe the automorphism group of the wheel graph W_4. How many elements are in in it? Is it Abelian?

6) Repeat problem (5) for the general the star graph $K_{1,n}$ and wheel graph W_n.

7). Challenge problem. Repeat problem 4 for W_n: how many spanning trees are there in W_n?