HW 1.

1.

1. Background. A "Cantor set" is a compact set which is totally disconnected and such that there are no isolated points. The standard "throw away the middle thirds"' Cantor set was the first example. Any countable product of finite sets $\Pi_{i\in I}F_i$ of finite sets F_i, is another class of examples. It is a theorem that any two Cantor sets are homeomorphic.

For our "standard Cantor set " we take all the $F_I = \mathbb{Z}_2 = \{0,1\}$.

$$\Sigma = \mathbb{Z}_2^{\mathbb{Z}}$$

which we can view as the space of bi-infinite sequences of 0's and 1's. So an element of Σ is a sequence $\omega = (\omega_i)_{i\in\mathbb{Z}}$ with each ω_i being 0 or 1.

Define the "shift map"' $F : \Sigma \to \Sigma$ by

$$F(\omega)_i = \omega_{i+1}.$$

Define a measure on Σ by declaring that, for each $i \in \mathbb{Z}$

$$\mu(\{\omega : \omega_i = 1\}) = \frac{1}{2} = \mu((\{\omega : \omega_i = 0\})$$

This is the "coin flip" measure of probability theory.

a) Show that F is a homeomorphism

b) Show that μ is a Borel measure

c) Show that F is measure preserving

d) Prove that F is ergodic.

e) Prove that F is mixing

Cantor set to be the product space $\mathbb{Z}_2^{\mathbb{N}_+}$, ($\mathbb{N}_+$ is the set of all positive integers) endowed with the product topology. An element of the Cantor set is then an infinite sequence $(\sigma_i)_{i\in\mathbb{N}}$ of 1's and 0's; $\sigma_i \in \{0,1\}$. the sequence labelled by the positive integers. Consider the map F whcih sends to

$$;_2 : C_2 \to [0,1]; F_2(\sigma) = \Sigma_{i\in\mathbb{N}_+}\sigma_i 2^{-i}$$

a) Show that F_2 is continous and onto.

b) If we give $\mathbb{Z}_2$ the "coin-flip" measure (each element has probability 1/2) then the Cantor set inherits a probability measure. (The product of probability spaces is a probability space, so that the Cantor set has a probability measure on it.) Show that F_2 is an isomorphism in the sense of measure theory: it is onto, and the map is measure preserving: $\mu(F_2^{-1}(I) = |I|$ for any interval I.

hint: consider dyadic intervals.

d)Show that F_2 is a measure preserving semi-conjugacy between the Bernoulli shift on the Cantor set and the doubling map (mod 1) on the interval.

e) Repeat (a)-(c) for $F_N : \mathbb{Z}_N^{\mathbb{N}_+} \to I$.

f) Use the fact that there is a bijective map from two disjoint copies of $\mathbb{N}_+$ to $\mathbb{N}_+$ to define an onto map $C \to I \times I$. Repeat, to establish the existence of an onto map from the Cantor set ONTO the n-cube. Onto any compact n-manifold.

1. A gradient flow. Let $\mathbb{T}$ be the flat torus with standard coordinates θ_1, θ_2 mod 2π. Let $V = -\cos(\theta_1)cos(\theta_2)$. Locate the equilibria. Describe each type (source, saddle, sink). Sketch the flow lines .

2. Hamiltonian flows. For $M = T^*\mathbb{R} = \mathbb{R} \times \mathbb{R}$ let $H = (1/2)p^2 + P(x)$ where P is a polynomial. Sketch the phase portraits in case
a) P is linear
b) P is homogeneous quadratic.
c) P is cubic. Do a few cases.
d) $P = (x-1)^2(x+1)^2$.

3. Again as in 2. Again P is polynomial. Is the flow complete? Find a proof or a counterexample.

4. N-dimensional oscillator. This has for its Hamiltonian $H(q,p) = (1/2)\langle p, p\rangle^2 + \langle q, Aq\rangle$ where $p, q \in \mathbb{R}^n$, where we use the standard inner product $\langle \cdot, \cdot \rangle$ to identify $\mathbb{R}^n$ with its dual, and where A is a positive definite symmetric matrix. Prove that the closure of the typical orbit is a k-torus, for some $k \leq N$. Describe the maximal k in terms of the eigenvalues of A.

5. Guckenheimer-Holmes. Exer. 5.1.2 and 5.1.3 of p. 234.
These exercises are on the Smale Horseshoe and are best solved using symbolic dynamics. Let $\Gamma \subset I^2$ be the subset of the square $S = I^2$ which never leaves the square in forward or backward time.
5.1.2. Show that all the periodic orbits are of saddle type. Locate the periodic orbits with period 4 or less and write out their symbol sequence. Show that Λ contains a countable infinity of heteroclinic and homoclinic orbits. Show that Λ contains an uncountable number of orbits which are not periodic.
5.1.3 Show that Λ contains a dense orbit.

6. A gradient system in $\mathbb{R}^n$ is given by $\dot{x} = -\nabla V(x), x \in \mathbb{R}^n$ where V is a smooth function. What is special about the linearization of a gradient system at an equilibrium, in comparison to a general linear system $\dot{x} = Ax$ with A a general n by n matrix.

7. Newton's equations on $\mathbb{R}^n$ are equations of the form $\ddot{x} = -\nabla V(x), x \in \mathbb{R}^n$ where V is a smooth function, called the potential.
a) First orderize the system by introducing $v = \dot{x}$ so as to make it an ODE on $\mathbb{R}^n \times \mathbb{R}^n = T\mathbb{R}^n$.
b. What is special about the linearization of Newton's equations at an equilibrium in in comparison to a general linear system $\dot{x} = Ax$ with A a general $2nby2n$ matrix.

8. Take the Cantor set to be the product space $\mathbb{Z}_2^{\mathbb{N}_+}$, ($\mathbb{N}_+$ is the set of all positive integers) endowed with the product topology. An element of the Cantor set is then an infinite sequence $(\sigma_i)_{i \in \mathbb{N}}$ of 1's and 0's; $\sigma_i \in \{0, 1\}$. the sequence labelled by the positive integers. Consider the map F whcih sends to

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9. Show that the doubling map $S^1 \to S^1$ is measure preserving.

10. Prove that rotation of the circle is NOT mixing.

11. Prove that the suspension of a map is NOT a mixing flow.

12. Construct a homeomorphism of the plane $\mathbb{R}^2 = \mathbb{C}$ which maps the spiral $exp(1+i)t), t \in \mathbb{R}$ to the ray $y = 0, x > 0$.