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A Stable Manifold Theorem for Degenerate Fixed Points with Applications to Celestial Mechanics

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1. INTRODUCTION

Consider the classical three-body problem, i.e. the motion of three point masses under the laws of classical mechanics. We shall say an orbit is parabolic if two of the particles remain bounded for all positive time while the third approaches infinity with zero velocity. One can conjecture that the set of all parabolic orbits forms a smooth submanifold of the phase space.

In this paper we prove this conjecture for three special cases of the three-body problem. The first is the well-known "restricted three-body problem" [2], where one of the masses is zero and the other two move in circular orbits. The second is a problem discussed by Sitnikov [5], where two equal masses move in a plane and the third moves on a line perpendicular to the plane through the center of mass of the first two. The third is the collinear three-body problem, where the particles are confined to a line. In each case the problem has only two degrees of freedom. For these examples we prove that the parabolic orbits form an analytically immersed submanifold of the energy surface.

Our method is to introduce at infinity a periodic orbit. This orbit has as its asymptotic set the set of parabolic orbits. We study the Poincaré map of this periodic orbit. This map is a diffeomorphism of the plane to itself leaving the origin fixed. The points asymptotic to the origin correspond to parabolic orbits.

For a diffeomorphism $\mathbf{f}: R^2 \to R^2$ leaving the origin fixed, define the local asymptotic set of $\mathbf{f}$ as:

$$\mathcal{O}\!\mathcal{l}^+(\mathbf{f}, U) = \{\mathbf{x} \in U: \mathbf{f}^k(\mathbf{x}) \in U \; \forall \, k > 0, \mathbf{f}^k(\mathbf{x}) \to 0 \text{ as } k \to \infty\}.$$

Here U is an open set containing the origin. Let $D\mathbf{f}(\mathbf{x})$ denote the derivative (Jacobian) of $\mathbf{f}$ at $\mathbf{x}$. If $D\mathbf{f}(0)$ has no eigenvalues of modulus one, then the

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standard stable manifold theorem tells us that $\mathcal{A}^+$ is a submanifold of U. For small U, $\mathcal{A}^+$ is an embedded submanifold, with the embedding as smooth as $\mathbf{f}$.

However, for the Poincaré map of the periodic orbit introduced at infinity, $D\mathbf{f}(0)$ is the identity. In this paper we give sufficient conditions for $\mathcal{A}^+$ to be a submanifold in this degenerate case. This part of the work is closely related to a paper of Slotnik [6], although the proofs are fundamentally different. Also, Slotnik considers only symplectic $\mathbf{f}$, while we do not make this restriction.

The first half of this paper concerns a stable manifold theorem for degenerate fixed points. The second half contains the applications of this theorem to parabolic orbits in the special cases of the three-body problem. The relation between parabolic orbits, homoclinic points, and wildly oscillating orbits has been previously discussed [1, 4]. In this paper we are content to prove that the parabolic orbits form a smooth submanifold of the energy surface.

2. A Stable Manifold Theorem for Degenerate Fixed Points

For degenerate $\mathbf{f}$ the asymptotic set $\mathcal{A}^+(\mathbf{f}, U)$ is generally not a manifold, as one can easily see in the following example:

$$\mathbf{f}(x, y) = (x - x^2 + y^2, y + 2xy).$$

Here $\mathcal{A}^+(\mathbf{f}, R^2)$ is the union of three rays. Although each ray is a submanifold, $\mathcal{A}^-$ is not. We must therefore fix our attention on one branch of the asymptotic set at a time.

Consider the following sector centered on the positive x-axis:

$$B(\beta, \delta) = \{(x, y) \in R^2 : 0 \leqslant x \leqslant \delta, |y| \leqslant \beta x\}.$$

We can define the asymptotic set restricted to such a sector:

$$A^+(\mathbf{f}, B) = \{\mathbf{x} \in B : \mathbf{f}^k(\mathbf{x}) \in B \; \forall k > 0, \mathbf{f}^k(\mathbf{x}) \to 0 \text{ as } k \to \infty\}.$$

The following theorem gives sufficient conditions for A^+ to be a real analytic arc. To analyse all branches of the asymptotic set, one must rotate the coordinates (x, y) until the branch of interest is near the positive x-axis. The negatively asymptotic set is analysed by considering $\mathbf{f}^{-1}$.

Theorem 1. *Let* $\mathbf{f}: R^2 \to R^2$ *be real analytic and have the form*

$$\mathbf{f} = \mathbf{id} + \mathbf{p} + \mathbf{r},$$

where $\mathbf{id}$ *is the identity,* $\mathbf{p} = (p_1, p_2)$ *is a homogeneous polynomial of degree*

$n \geq 2$, and $\mathbf{r}$ consists of terms of degree at least $n - 1$. Suppose further that for $x > 0$,

$$p_1(x, 0) < 0$$

$$p_2(x, 0) = 0$$

$$\frac{\partial p_2}{\partial y}(x, 0) > 0.$$

Then there exist positive constants β and δ such that $A^-(\mathbf{f}, B(\beta, \delta))$ is the graph of a differentiable function φ: $[0, \delta] \to R^1$. Furthermore, $f|_{(0,\delta]}$ is real analytic.

Slotnik [6] gives an example for which φ is not analytic at the origin. He further proves for symplectic $\mathbf{f}$ that φ is C^∞ at the origin. The author conjectures that this is also true for arbitrary $\mathbf{f}$, but the proof does not seem to fit naturally with our methods. Furthermore, smoothness at the origin is of little interest in our examples.

The proof of Theorem 1 proceeds in two parts. In the next section we shall prove the following:

PROPOSITION 2. *Let $\mathbf{f}$ satisfy the hypotheses of Theorem 1. Then there is a positive constant β_0 such that, for any $\beta \in (0, \beta_0]$, we can find a $\delta > 0$ such that $A^-(\mathbf{f}, B(\beta, \delta))$ is the graph of a Lipschitz function φ: $[0, \delta] \to R^1$.*

In Section 4 we use the above result and some techniques from the theory of holomorphic functions to prove Theorem 1. Sections 5 and 6 contain the estimates needed in the proof of the theorem. The remaining sections are concerned with the applications to celestial mechanics.

3. THE GEOMETRIC ARGUMENT

In this section we give a geometric proof of Proposition 2. This is the part of the proof of Theorem 1 differing most drastically from the methods used by Slotnik.

Let α, β, and δ be positive. Define the following subsets of R^2:

$$\begin{aligned}
B &= B(\beta, \delta) = \{(x, y) \in R^2 : 0 \leqslant x \leqslant \delta, |y| \leqslant \beta x\} \\
b^+ &= \{(x, y) \in B : y = \beta x\} \\
b^- &= \{(x, y) \in B : y = -\beta x\} \\
B^+ &= \{(x, y) \in R^2 : 0 \leqslant x \leqslant \delta, y \geqslant \beta x\} \\
B^- &= \{(x, y) \in R^2 : 0 \leqslant x \leqslant \delta, y \leqslant -\beta x\} \\
S(\alpha) &= \{(x, y) \in R^2 : |y| \geqslant \alpha |x|\}
\end{aligned}$$

Let $\pi_2 : R^2 \to R^1$ be projection onto the y-axis, i.e., $\pi_2(x, y) = y$. We shall need the following proposition, the proof of which is given in Section 5.

PROPOSITION 3. *Let* $\mathbf{f} = (f_1, f_2)$ *satisfy the hypotheses of Theorem* 1. *Then there exist positive constants* α *and* β_1 *with the following properties. For any* $\beta \in (0, \beta_1]$ *there exists a* δ *such that*

$$0 < f_1(x, y) < x, \qquad \forall (x, y) \in B(\beta, \delta), \qquad x \neq 0 \tag{3.1}$$

$$\mathbf{f}(b^{\pm}) \subset B^{\pm} \tag{3.2}$$

$$D\mathbf{f}(\mathbf{x}): S(\alpha) \to S(\alpha), \qquad \forall \mathbf{x} \in B(\beta, \delta) \tag{3.3}$$

$$|\pi_2 D\mathbf{f}^{-1}(\mathbf{x})\xi| \leqslant |\pi_2 \xi|, \qquad \forall \mathbf{x} \in \mathbf{f}(B(\beta, \delta)), \qquad \xi \in S(\alpha). \tag{3.4}$$

We now proceed with the proof of Proposition 2. Let Γ be a C^1 arc, and let $\mathbf{x} \in \Gamma$. Let $T_x\Gamma$ be the tangent space of Γ at $\mathbf{x}$. Define the set of vertical arcs in $A \subset R^2$:

$$V(\alpha, A) = \{\Gamma \subset A : \Gamma \text{ is a } C^1 \text{ arc}, \ T_x\Gamma \subset S(\alpha) \ \forall \mathbf{x} \in \Gamma\}$$

By condition (3.3), $\mathbf{f}$ maps vertical arcs in B to vertical arcs in R^2, i.e.

$$\mathbf{f}: V(\alpha, B) \to V(\alpha, R^2). \tag{3.5}$$

Furthermore, $\mathbf{f}$ does not contract vertical arcs in the following sense:

PROPOSITION 4. *Let* $\Gamma \in V(\alpha, B)$, *with* $\mathbf{x}_1, \mathbf{x}_2 \in \Gamma$. *Then*

$$|\pi_2(\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2))| \geqslant |\pi_2(\mathbf{x}_1 - \mathbf{x}_2)|.$$

Proof. Since $\mathbf{f}$ maps vertical arcs to vertical arcs,

$$\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2) \in S(\alpha).$$

Let $\Psi: [0, 1] \to \mathbf{f}(B)$ be defined by

$$\Psi(t) = t\mathbf{f}(\mathbf{x}_1) + (1 - t)\,\mathbf{f}(\mathbf{x}_2).$$

Then

$$\begin{aligned}
|\pi_2(\mathbf{x}_1 - \mathbf{x}_2)| &= |(\pi_2 \circ \mathbf{f}^{-1} \circ \Psi)(1) - (\pi_2 \circ \mathbf{f}^{-1} \circ \Psi)(0)| \\
&= \left| \int_0^1 (\pi_2 \circ \mathbf{f}^{-1} \circ \Psi)'(t)\, dt \right| \\
&\leqslant \int_0^1 |\pi_2 \circ D\mathbf{f}^{-1}(\Psi(t))\, \Psi'(t)|\, dt \\
&\leqslant \int_0^1 |\pi_2 \Psi'(t)|\, dt \quad \text{(by (3–4))} \\
&= |\pi_2(\mathbf{f}(\mathbf{x}_1) - \mathbf{f}(\mathbf{x}_2))|,
\end{aligned}$$

which completes the proof of Proposition 4.

We now wish to consider the set V' of those vertical arcs extending all the way across B:

$$V' = \{\Gamma \in V(\alpha, B)\text{: one endpoint of } \Gamma \text{ is in } b^+\text{, the other in } b^-\}.$$

The following proposition concludes that a vertical arc extending across B intersects the asymptotic set A^+ exactly once. It then follows immediately that A^+ is the graph of a Lipschitz function $\varphi\colon [0, \delta] \to R^1$. The Lipschitz constant must be less than α.

PROPOSITION 5. *Choose $\beta_0 \in (0, \beta_1]$ such that $\alpha > \beta_0$. Let $\Gamma \in V'$. Then, for $\beta \in (0, \beta_0]$, there exists a $\delta > 0$ such that $\Gamma \cap A^+(\mathbf{f}, B(\beta, \delta))$ contains exactly one point.*

Proof. We first note that the condition $\alpha > \beta$ insures that if $\Gamma \in V(\alpha, R^2)$ and if $\mathbf{x}_1, \mathbf{x}_2 \in \Gamma \cap B$, then the subarc of Γ between $\mathbf{x}_1$ and $\mathbf{x}_2$ is a subset of B. Thus if $\Gamma \in V'$, properties (3.2) and (3.5) imply that $\mathbf{f}(\Gamma) \cap B \in V'$. Letting $\Gamma = \Gamma_0$, define

$$\Gamma_k = \mathbf{f}(\Gamma_{k-1}) \cap B \in V', \qquad \text{for} \quad k \geqslant 1.$$

Let

$$I_k = \mathbf{f}^{-k}(\Gamma_k), \qquad k \geqslant 0.$$

Then $\{I_k\}$ is a nested sequence of non-empty compact arcs, so $\bigcap I_k \neq \varnothing$.

Now suppose that $\mathbf{x}_1$ and $\mathbf{x}_2 \in \bigcap I_k$. Then $\mathbf{f}^k(\mathbf{x}_1)$ and $\mathbf{f}^k(\mathbf{x}_2) \in B$ for $k \geqslant 0$. By condition (3.1), $\mathbf{f}^k(\mathbf{x}_1)$ and $\mathbf{f}^k(\mathbf{x}_2) \to 0$ as $k \to \infty$. Therefore $\pi_2(\mathbf{f}^k(\mathbf{x}_1) - \mathbf{f}^k(\mathbf{x}_2)) \to 0$. But Proposition 4 implies that $|\pi_2(\mathbf{f}^k(\mathbf{x}_1) - \mathbf{f}^k(\mathbf{x}_2))| \geqslant |\pi_2(\mathbf{x}_1 - \mathbf{x}_2)|$. Therefore $\mathbf{x}_1 = \mathbf{x}_2$ and $\bigcap I_k$ contains only one point. Since $\Gamma \cap A^- = \bigcap I_k$, the proof of Proposition 5 is complete.

We have now completed the proof of Proposition 2, given the proof of Proposition 3 appearing in Section 5. In fact, we have proved more. One can see in the proof of Proposition 3 that α can be chosen arbitrarily small at the expense of choosing β and δ small. Thus the Lipschitz constant of φ goes to zero at the origin and φ is differentiable there, with $\varphi'(0) = 0$. One can also see in the proof of Proposition 3 that we do not use the analyticity of $\mathbf{f}$. We only need that $\mathbf{r}$ is C^1, and

$$\mathbf{r}(\mathbf{x}) = o(\|\mathbf{x}\|^n), \qquad D\mathbf{r}(\mathbf{x}) = o(\|\mathbf{x}\|^{n-1}).$$

Thus we have proved the following:

PROPOSITION 6. *Let $\mathbf{f}\colon R^2 \to R^2$ be C^1, $\mathbf{f} = \mathbf{id} + \mathbf{p} + \mathbf{r}$. Suppose $\mathbf{p}$ satisfies the hypotheses of Theorem 1, and let $\mathbf{r}$ satisfy the above conditions.*

Then there exist $\beta, \delta > 0$ such that $A^+(\mathbf{f}, B(\beta, \delta))$ is the graph of a Lipschitz function φ: $[0, \delta] \to R^1$. Furthermore, $\varphi'(0)$ exists and equals zero.

The author conjectures that φ is actually C^1. Indeed, one expects that φ is C^r if f is C^r.

4. The Analytic Argument

Using the results of the previous section, we can now prove Theorem 1. Since the mapping $\mathbf{f} = (f_1, f_2)$ is real analytic, we can extend f_1 and f_2 to holomorphic functions of two complex variables in a neighborhood of the origin in $\mathbf{C}^2$. We need the following proposition, which will be proved in Section 6.

Proposition 7. *Let* $\mathbf{f}$ *satisfy the hypotheses of Theorem* 1. *Then there exist* $\beta, \delta > 0$ *and an open set* $\Omega \subset \mathbf{C}$ *such that* $(0, \delta) \subset \Omega$ *and*

$$f_1(x, y) \in \Omega \quad \text{for} \quad x \in \Omega, \quad |y| \leqslant \beta |x|, \tag{4.1}$$

$$|f_2(x, y) - y| < |y| \quad \text{for} \quad x \in \Omega, \quad |y| = \beta |x|, \quad x \neq 0, \tag{4.2}$$

$$\beta |f_1(x, y)| < |f_2(x, y)| \quad \text{for} \quad x \in \Omega, \quad |y| = \beta |x|, \quad x \neq 0. \tag{4.3}$$

Furthermore, $A^-(\mathbf{f}, B(\beta, \delta))$ *is the graph of a function* φ: $[0, \delta] \to R^1$.

We now proceed with the proof of Theorem 1. Let $\mathscr{H} = \{h : h$ is holomorphic in Ω, $|h(x)| \leqslant \beta |x|$, and $h(x)$ is real for real $x\}$. We shall define a map $\mathscr{F}: \mathscr{H} \to \mathscr{H}$. Fix $h \in \mathscr{H}$. Let

$$H(x, y) = f_2(x, y) - h(f_1(x, y)).$$

Let

$$\Lambda^0 = \{(x, y) : x \in \Omega, |y| < \beta |x|\},$$

$$\Lambda = \{(x, y) : x \in \Omega, |y| \leqslant \beta |x|\}.$$

By (4.1), H is holomorphic on Λ^0. Fix $x = x_0$. By Rouché's Theorem and (4.2), $f_2(x, y)$ and y have the same number of zeros in

$$D(x_0) = \{y : |y| < \beta |x_0|\}.$$

By (4.3), $|h(f_1(x_0, y))| \leqslant \beta |f_1(x_0, y)| < |f_2(x_0, y)|$ for $y \in \partial D(x_0)$. So, again by Rouché's Theorem, $H(x_0, y)$ and $f_2(x_0, y)$ have the same number of zeros in $D(x_0)$. Thus, for each fixed $x_0 \in \Omega$, $H(x_0, y)$ has a unique simple zero in $D(x_0)$. Define $\mathscr{F}h(x_0)$ to be that zero. By the Implicit Function Theorem, $\mathscr{F}h$ is holomorphic in Ω. Since $\mathscr{F}h(x) \in D(x)$, $|\mathscr{F}h(x)| < \beta |x|$. For fixed real x_0, $H(x_0, y)$ is real for real y. Since $H(x_0, y)$ has only one zero in $D(x_0)$, that zero must be real. Hence $\mathscr{F}h(x)$ is real for real x, and $\mathscr{F}: \mathscr{H} \to \mathscr{H}$.

Note that we have constructed $\mathscr{F}$ so that $\mathbf{f}(\text{graph}(\mathscr{F}h)) \subseteq \text{graph}(h)$. Therefore, for $n \geqslant m$, we have

$$\mathbf{f}^m(\text{graph}(\mathscr{F}^n h)) \subset \text{graph}(\mathscr{F}^{n-m} h) \subset \Lambda. \tag{4.4}$$

Now fix $h_0 \in \mathscr{H}$. Let $h_n = \mathscr{F}^n h_0$, $n = 1, 2,\ldots$. The sequence $\{h_n\}$ is uniformly bounded, hence a normal family on Ω. Therefore there exists a subsequence of $\{h_n\}$ converging to $h \in \mathscr{H}$. We shall show that graph(h) remains in Λ under iterations of $\mathbf{f}$.

Suppose there is a positive integer m and an $x \in \Omega$ such that $\mathbf{f}^m(x, h(x)) \notin \Lambda$. By (4.1) we have $\pi_1 \mathbf{f}^m(x, h(x)) \in \Omega$, for all $x \in \Omega$, $k \geqslant 0$. Therefore we must have $|\pi_2 \mathbf{f}^m(x, h(x))| > \beta |\pi_1 \mathbf{f}^m(x, h(x))|$. Here $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Since $\mathbf{f}^m$ is continuous, there is an $\epsilon > 0$ such that

$$|y - h(x)| < \epsilon \Rightarrow \mathbf{f}^m(x, y) \notin \Lambda.$$

Pick $n \geqslant m$ so that $|h_n(x) - h(x)| < \epsilon$. Then $\mathbf{f}^m(x, h_n(x)) \notin \Lambda$, contradicting (4.4). Therefore $\mathbf{f}^m(x, h(x)) \in \Lambda$, for all $x \in \Omega$, $m \geqslant 0$.

Now consider real x. We have shown that

$$\text{graph}(h|_{(0,\delta)}) \subset \Lambda^+(\mathbf{f}, B(\beta, \delta)) = \text{graph}(\varphi).$$

Therefore $h|_{(0,\delta)} = \varphi$. Thus φ is real analytic and the proof of Theorem 1 is complete.

5. Estimates

In this section we prove Proposition 3. We need the following lemma, the proof of which is an elementary exercise.

Lemma 8. *Let $p: R^2 \to R^1$ be a homogeneous polynomial of degree n with $p(x, 0) > 0$ for $x > 0$. Then there exist positive constants β and K such that $p(x, y) \geqslant Kx^n$ for $x \geqslant 0$, $|y| \leqslant \beta x$.*

We begin by establishing the following estimates for the mapping $\mathbf{f}$. Let $D_1 = \partial/\partial x$ and $D_2 = \partial/\partial y$.

Proposition 9. *Let $\mathbf{f} = (f_1, f_2)$ satisfy the hypotheses of Theorem 1. Then there exist positive constants α, β_1, and δ_1 such that for $(x, y) \in B(\beta_1, \delta_1)$ and $x \neq 0$,*

$$x > f_1 > 0 \tag{5.1}$$

$$\begin{gathered} D_2 f_2 - D_1 f_1 > |\alpha D_2 f_1 - \alpha^{-1} D_1 f_2| \\ D_2 f_2 + D_1 f_1 > |\alpha D_2 f_1 + \alpha^{-1} D_1 f_2| \end{gathered} \tag{5.2}$$

$$\begin{gathered} \det(D\mathbf{f}) - D_1 f_1 > \alpha^{-1} |D_1 f_2| \\ \det(D\mathbf{f}) + D_1 f_1 > \alpha^{-1} |D_1 f_2|. \end{gathered} \tag{5.3}$$

Furthermore, for any $\beta \in (0, \beta_1]$, there exists a positive $\delta < \delta_1$ such that

$$xy^{-1}f_2 > f_1, \quad \text{for} \quad x \in (0, \delta], \quad y = \pm\beta x. \tag{5.4}$$

Proof. Let

$$p_1(x, y) = a_0x^n + a_1x^{n-1}y + \cdots$$
$$p_2(x, y) = b_0x^n + b_1x^{n-1}y + \cdots .$$

The hypotheses of Theorem 1 imply that $a_0 < 0$, $b_0 = 0$, and $b_1 > 0$. Choose α so that

$$0 < \alpha < (b_1 - na_0)/|a_1|.$$

Each of the following homogeneous polynomials is positive along the positive x-axis. Lemma 8 implies the existence of positive constants β_1 and K_1 such that, for $x \geqslant 0$ and $|y| \leqslant \beta_1 x$,

$$\begin{aligned} -p_1 &\geqslant K_1x^n \\ D_2p_2 - D_1p_1 + \alpha D_2p_1 - \alpha^{-1}D_1p_2 &\geqslant K_1x^{n-1} \\ D_2p_2 - D_1p_1 - \alpha D_2p_1 + \alpha^{-1}D_1p_2 &\geqslant K_1x^{n-1} \\ D_2p_2 - \alpha^{-1}D_1p_2 &\geqslant K_1x^{n-1} \\ D_2p_2 + \alpha^{-1}D_1p_2 &\geqslant K_1x^{n-1} \\ xy^{-1}p_2 - p_1 &\geqslant K_1x^n. \end{aligned} \tag{5.5}$$

By hypothesis we have $\mathbf{r}(\mathbf{x}) = o(\|\mathbf{x}\|^n)$ and $D\mathbf{r}(\mathbf{x}) = o(\|\mathbf{x}\|^{n-1})$. Thus we can choose $\delta_2 > 0$ so that, for $x \in (0, \delta_2]$ and $|y| \leqslant \beta_1 x$,

$$\begin{aligned} |r_1| &< K_1x^n \\ |D_2r_2 - D_1r_1 + \alpha D_2r_1 - \alpha^{-1}D_1r_2| &< K_1x^{n-1} \\ |D_2r_2 - D_1r_1 - \alpha D_2r_1 + \alpha^{-1}D_1r_2| &< K_1x^{n-1} \\ |D_2r_2 - \alpha^{-1}D_1r_2| &< \tfrac{1}{2}K_1x^{n-1} \\ |D_2r_2 + \alpha^{-1}D_1r_2| &< \tfrac{1}{2}K_1x^{n-1}. \end{aligned}$$

Now let $q_i = p_i + r_i$, for $i = 1$ or 2. Combining the above corresponding inequalities we have, for $(x, y) \in B(\beta_1, \delta_2)$ and $x \neq 0$,

$$-q_1 > 0 \tag{5.6}$$

$$\begin{aligned} D_2q_2 - D_1q_1 + \alpha D_2q_1 - \alpha^{-1}D_1q_2 &> 0 \\ D_2q_2 - D_1q_1 - \alpha D_2q_1 + \alpha^{-1}D_1q_2 &> 0 \end{aligned} \tag{5.7}$$

$$\begin{aligned} D_2q_2 - \alpha^{-1}D_1q_2 &> \tfrac{1}{2}K_1x^{n-1} \\ D_2q_2 + \alpha^{-1}D_1q_2 &> \tfrac{1}{2}K_1x^{n-1} \end{aligned} \tag{5.8}$$

Let $\mathbf{q} = (q_1, q_2)$. Note that $\mathbf{q}(\mathbf{x}) = O(\|\mathbf{x}\|^n)$, $D\mathbf{q}(\mathbf{x}) = O(\|\mathbf{x}\|^{n-1})$, and $\det(D\mathbf{q}(\mathbf{x})) = o(\|\mathbf{x}\|^{n-1})$. We can therefore choose constants K_2 and δ_3, with $K_2\delta_3^{n-1} < 1$, such that, for $(x, y) \in B(\beta_1, \delta_3)$ and $x \neq 0$,

$$|q_1| < K_2x^n \tag{5.9}$$

$$|\alpha D_2q_1 + \alpha^{-1}D_1q_2| + |D_1q_1 + D_2q_2| < 2K_2x^{n-1} \tag{5.10}$$

$$|2D_1q_1 + D_2q_2| + \alpha^{-1}|D_1q_2| < K_2x^{n-1} \tag{5.11}$$

$$|\det(D\mathbf{q})| < K_2x^{n-1} \tag{5.12}$$

$$|\det(D\mathbf{q})| < \tfrac{1}{2}K_1x^{n-1} \tag{5.13}$$

Now let $\delta_1 = \min(\delta_2, \delta_3)$ and recall that $f_1(x, y) = x + q_1(x, y)$ and $f_2(x, y) = y + q_2(x, y)$. For $(x, y) \in B(\beta_1, \delta_1)$, (5.6) implies $f_1(x, y) < x$, while (5.9) implies $f_1 \geqslant x - |q_1| > x(1 - K_2x^{n-1}) > 0$. Thus we have established (5.1). Inequalities (5.2) follow from (5.7) and (5.10), since

$$\begin{aligned} D_2f_2 - D_1f_1 = D_2q_2 - D_2q_1 &> |\alpha D_2q_1 - \alpha^{-1}D_1q_2| \\ &= |\alpha D_2f_1 - \alpha^{-1}D_1f_2| \\ D_2f_2 + D_1f_1 \geqslant 2 - |D_2q_2 + D_1q_1| &> |\alpha D_2q_1 + \alpha^{-1}D_1q_2| \\ &= |\alpha D_2f_1 + \alpha^{-1}D_1f_2|. \end{aligned}$$

For inequalities (5.3) we note that

$$\det(Df) = 1 + D_1q_1 + D_2q_2 + \det(D\mathbf{q}).$$

Therefore

$$\det(Df) - D_1f_1 \geqslant D_2q_2 - |\det(D\mathbf{q})| > \alpha^{-1}|D_1f_2|,$$

by (5.8) and (5.13). Furthermore, by (5.11) and (5.12),

$$\det(Df) + D_1f_1 \geqslant 2 - |2D_1q_1 + D_2q_2| - |\det(D\mathbf{q})| > \alpha^{-1}|D_1f_2|.$$

Now let $\beta \in (0, \beta_1]$. Choose δ so that

$$|xy^{-1}r_2 - r_1| < K_1x^n$$

for $x \in (0, \delta]$ and $y = \pm\beta x$. Then by (5.5) and the above inequality we have

$$xy^{-1}f_2 - f_1 \geqslant xy^{-1}p_2 - p_1 - |xy^{-1}r_2 - r_1| > 0$$

This establishes (5.4), and the proof of Proposition 9 is complete.

We can now proceed with the proof of Proposition 3. Properties (3.1) and (3.2) follow immediately from (5.1) and (5.4). Inequalities (5.2) imply

$$\begin{aligned} D_2f_2 - D_1f_1 &> \alpha D_2f_1 - \alpha^{-1}D_1f_2 \\ D_2f_2 + D_1f_1 &> -\alpha D_2f_1 - \alpha^{-1}D_1f_2. \end{aligned}$$

Multiplying by α and rearranging terms we get

$$| D_1 f_2 + \alpha D_2 f_2 | \geqslant \alpha | D_1 f_1 + \alpha D_2 f_1 |,$$

for $\mathbf{x} \in B(\beta, \delta)$. Now let $\mathbf{a}^+ = (1, \alpha) \in R^2$ and $S^+(\alpha) = S(\alpha) \cap \{(x, y) : y \geqslant 0\}$. The above inequality implies that $D\mathbf{f}(\mathbf{x})\, \mathbf{a}^+ \in S^+(\alpha)$. Let $\mathbf{a}^- = (-1, \alpha)$. A similar argument shows that $D\mathbf{f}(\mathbf{x})\, \mathbf{a}^- \in S^+(\alpha)$. It follows that $D\mathbf{f}(\mathbf{x})\xi \in S^+(\alpha)$ for $\xi \in S^+(\alpha)$. Therefore $D\mathbf{f}(\mathbf{x})$: $S(\alpha) \to S(\alpha)$ for all $\mathbf{x} \in B(\beta, \delta)$, and we have established (3.3).

Now consider inequalities (5.3). These imply

$$| D_1 f_1 - \alpha^{-1} D_1 f_2 | \leqslant \det(D\mathbf{f}), \tag{5.14}$$

for $\mathbf{x} \in B(\beta, \delta)$. Now suppose $\mathbf{x} \in \mathbf{f}(B(\beta, \delta))$. Then $D\mathbf{f}^{-1}(\mathbf{x}) = (D\mathbf{f}(\mathbf{f}^{-1}(\mathbf{x})))^{-1}$, so

$$| \pi_2 D\mathbf{f}^{-1}(\mathbf{x})\, \mathbf{a}^+ | = | \det(D\mathbf{f})|^{-1} | -D_1 f_2 + \alpha D_1 f_1 | \leqslant \alpha,$$

by (5.14). Similarly we can show $| \pi_2 D\mathbf{f}^{-1}(\mathbf{x})\, \mathbf{a}^- | \leqslant \alpha$. Thus $| \pi_2 D\mathbf{f}^{-1}(\mathbf{x})\xi | \leqslant | \pi_2 \xi |$, for all $\mathbf{x} \in \mathbf{f}(B(\beta, \delta))$, $\xi \in S(\alpha)$, and the proof of Proposition 3 is complete.

6. Estimates in the Complex Plane

In this section we finish the estimates by proving Proposition 7. Recall that

$$\begin{aligned} p_1(x, y) &= a_0 x^n + a_1 x^{n-1} y + \cdots \\ p_2(x, y) &= b_0 x^n + b_1 x^{n-1} y + \cdots \end{aligned}$$

and that the hypotheses of Theorem 1 imply $a_0 < 0$, $b_0 = 0$, and $b_1 > 0$. Let

$$\begin{aligned} g_1(x) &= x + a_0 x^n \\ g_2(x, y) &= y + b_1 x^{n-1} y \\ s_1(x, y) &= p_1(x, y) - a_0 x^n \\ s_2(x, y) &= p_2(x, y) - b_1 x^{n-1} y \\ \Omega(\gamma, \delta) &= \{x \in \mathbf{C} : 0 < | x | < \delta, | \arg x | < \gamma\} \\ \Omega' &= \text{complement } (\Omega). \end{aligned}$$

Proposition 10. *There exist positive constants γ, δ_4, and K_3 so that, for $\delta \in (0, \delta_4]$ and $x \in \Omega(\gamma, \delta)$,*

$$\operatorname{dist}(g_1(x), \Omega'(\gamma, \delta)) \geqslant 2K_3 | x |^n \tag{6.1}$$

$$| 1 + b_1 x^{n-1} | - | 1 + a_0 x^{n-1} | \geqslant 2K_3 | x |^{n-1}. \tag{6.2}$$

Proof. It is an elementary exercise to show that one can take $\gamma = \pi/4(n-1)$, $8^{1/2}K_3 = \min(b_1, -a_0)$, and $-a_0 n\delta_4^{n-1} < 1$. The details are omitted.

We now proceed with the proof of Proposition 7. Let γ, δ_4, and K_3 be given by Proposition 10. Choose β_2 so that

$$|s_1(x, y)| \leqslant \tfrac{1}{2}K_3 |x|^n$$
$$|s_2(x, y)| \leqslant \tfrac{1}{2}K_3 |x|^{n-1} |y|$$

for $|y| \leqslant \beta_2 |x|$. By (6.1),

$$\operatorname{dist}(g_1 + s_1, \Omega') \geqslant \operatorname{dist}(g_1, \Omega') - |s_1| \geqslant K_3 |x|^n.$$

By (6.2),

$$\begin{aligned}|g_2 + s_2| &- |y|\,|x|^{-1}|g_1 + s_1| \\ &\geqslant |y|\,(|1 + b_1x^{n-1}| - |1 + a_0x^{n-1}|) - |s_2| - |y|\,|x|^{-1}|s_1| \\ &\geqslant K_3 |x|^{n-1} |y|.\end{aligned}$$

Restating the above two estimates we have, for $\delta \in (0, \delta_4]$ and $\beta \in (0, \beta_2]$,

$$\operatorname{dist}(x + p_1(x, y), \Omega'(\gamma, \delta)) \geqslant K_3 |x|^n \tag{6.3}$$
$$|y + p_2(x, y)| - |y|\,|x|^{-1}|x + p_1(x, y)| \geqslant K_3 |x|^{n-1} |y|, \tag{6.4}$$

if $x \in \Omega(\gamma, \delta)$ and $|y| \leqslant \beta |x|$.

Now let β_1 be given by Proposition 9 and fix $\beta \leqslant \min(\beta_1, \beta_2)$. We know from our estimates in Sections 3 and 5 that there exists a δ_5 so that $A^+(\mathbf{f}, B(\beta, \delta_5))$ is the graph of a Lipschitz function φ: $[0, \delta_5] \to R^1$. Choose $\delta \leqslant \min(\delta_4, \delta_5)$ so that

$$|r_1(x, y)| \leqslant \tfrac{1}{2}K_3 |x|^n \tag{6.5}$$
$$\beta |r_1(x, y)| + |r_2(x, y)| \leqslant \tfrac{1}{2}K_3\beta |x|^n \tag{6.6}$$
$$|p_2(x, y) + r_2(x, y)| \leqslant \tfrac{1}{2}\beta |x|, \tag{6.7}$$

for $|x| \leqslant \delta$ and $|y| \leqslant \beta |x|$. Then (6.3) and (6.5) imply $\operatorname{dist}(f_1(x, y), \Omega') \geqslant \frac{1}{2}K_3 |x|^n$, which proves (4.1). Inequality (4.2) follows immediately from (6.7). Finally, (6.4) and (6.6) imply, for $|y| = \beta |x|$,

$$|f_2| - \beta |f_1| \geqslant \tfrac{1}{2}K_3\beta |x|^n,$$

which establishes (4.3) and completes the proof of Proposition 7.

7. Parabolic Orbits in the Restricted Three-Body Problem

We now turn to an application of Theorem 1 arising in celestial mechanics. The first example we wish to consider is the well-known restricted three body problem. [2] In this problem we consider the motion of three particles in a plane. Two of the particles, of mass ν and μ with $\mu + \nu = 1$, move in circular orbits of angular velocity 1 about their common center of mass. The third particle has zero mass and moves in the gravitational field of the first two bodies. Choose a rotating complex coordinate system so that mass ν is fixed on the real axis at $-\mu$ and mass μ is fixed at $+\nu$. If $z \in \mathbf{C}$ is the position of the third particle, the equation of motion is

$$\ddot{z} = z - 2i\dot{z} - z \mid z \mid^{-3} - g(z), \tag{7.1}$$

where $g(z) = \nu(z + \mu) \mid z + \mu \mid^{-3} + \mu(z - \nu) \mid z - \nu \mid^{-3} - z \mid z \mid^{-3} = O(\mid z \mid^{-4})$. This equation admits the so-called Jacobi integral

$$\tfrac{1}{2} \mid \dot{z} \mid^2 - \tfrac{1}{2} \mid z \mid^2 - \mid z \mid^{-1} - v(z) = h$$

where $v(z) = \nu \mid z + \mu \mid^{-1} + \mu \mid z - \nu \mid^{-1} - \mid z \mid^{-1} = O(\mid z \mid^{-3})$.

We shall say an orbit is parabolic if $z \to \infty$ and the radial component of $\dot{z}$ goes to zero as $t \to \infty$. We wish to use Theorem 1 to show that the set of parabolic orbits is a smooth submanifold of the energy surface given by the Jacobi integral. Let

$$\begin{aligned} z &= 2x^{-2}e^{-i\theta} \\ \dot{z} &= e^{-i\theta}[y + i(\tfrac{1}{2}x^2\omega - 2x^{-2})]. \end{aligned} \tag{7.2}$$

Equation (7.1) can then be written

$$\begin{aligned} \dot{x} &= -\tfrac{1}{4}x^3y \\ \dot{\theta} &= 1 - \tfrac{1}{4}x^3\omega \\ \dot{y} &= -\tfrac{1}{4}x^4 + \tfrac{1}{8}x^6\omega^2 - \operatorname{Re}\{e^{i\theta}g(2x^{-2}e^{-i\theta})\} \\ \dot{\omega} &= -2x^{-2}\operatorname{Im}\{e^{i\theta}g(2x^{-2}e^{-i\theta})\}, \end{aligned}$$

and the Jacobi integral becomes

$$\tfrac{1}{2}(y^2 - x^2) - \omega + \tfrac{1}{8}x^4\omega^2 - v(2x^{-2}e^{-i\theta}) = h.$$

We note that these equations have a 2π-periodic solution at $(x, y, \omega) = (0, 0, -h)$. Near this periodic solution we can use (x, y, θ) as coordinates and solve the Jacobi integral for ω:

$$\omega = -h + v_1(x, y, \theta).$$

Here $v_1(x, y, \theta)$ is second order in x and y and 2π-periodic in θ. The differential equations become

$$\begin{aligned}\dot{x} &= -\tfrac{1}{4}x^3y\\ \dot{y} &= -\tfrac{1}{4}x^3(x + g_1(x, y, \theta))\\ \dot{\theta} &= 1 - g_2(x, y, \theta).\end{aligned}$$

Here g_1 and g_2 are 2π-periodic in θ, g_1 is third order in (x, y) and g_2 is fourth order. The Poincaré map of the periodic orbit $(x, y) = (0, 0)$ has the form:

$$\mathbf{f}: \begin{aligned} x &\to x - Kx^3(y + r_1(x, y))\\ y &\to y - Kx^3(x + r_2(x, y)),\end{aligned} \tag{7.3}$$

where K is a positive constant and r_1 and r_2 are real analytic and contain terms of at least second order. We shall also encounter this map in the other two examples; here $K = \frac{1}{2}\pi$ and r_1 and r_2 are actually third order.

Note that the parabolic orbits are exactly those orbits such that $(x, y) \to (0, 0)$ as $t \to \infty$. Recall that for open $U \subset R^2$ containing the origin,

$$\mathcal{O}^+(\mathbf{f}, U) = \{\mathbf{x} \in U : \mathbf{f}^k(\mathbf{x}) \in U \ \forall k > 0, \mathbf{f}^k(\mathbf{x}) \to 0 \text{ as } k \to \infty\}.$$

We shall consider only positive x to avoid ambiguity in transformation (7.2). The parabolic orbits are then exactly $\mathcal{O}^+(\mathbf{f}, U) \cap \{x > 0\}$. The following proposition concludes that this set is a real analytic arc and hence that the parabolic orbits form a real analytic submanifold of the integral surface.

PROPOSITION 11. *Let* $\mathbf{f}$ *have the form* (7.3). *Then there exists an open* $U \subset R^2$ *containing the origin such that* $\mathcal{O}^+(\mathbf{f}, U) \cap \{x > 0\}$ *is a real analytic arc.*

Proof. If we write $\mathbf{f}$ in polar coordinates we have

$$\mathbf{f}: \begin{aligned} r &\to r - Kr^4 \cos^3\theta(2 \sin\theta \cos\theta + O(r))\\ \theta &\to \theta + Kr^3 \cos^3\theta(\sin^2\theta - \cos^2\theta + O(r)).\end{aligned}$$

Candidates for stable and unstable manifolds can occur only when θ is approximately constant, i.e., where $\cos^3\theta(\sin^2\theta - \cos^2\theta) = 0$, or $\theta = \pm\frac{1}{2}\pi$, $\pm\frac{1}{4}\pi$.

Consider first $\theta = \frac{1}{4}\pi$. If we make the transformation

$$\begin{aligned} x &= u - v\\ y &= u + v,\end{aligned}$$

The map $\mathbf{f}$ becomes

$$\mathbf{f}^*: \begin{array}{l} u \to u - p_1(u, v) + \cdots \\ v \to v - p_2(u, v) + \cdots, \end{array}$$

where $p_1(u, v) = -K(u - v)^3 u$ and $p_2(u, v) = K(u - v)^3 v$. For $u > 0$ we have $p_1(u, 0) = -Ku^4 < 0$, $p_2(u, 0) = 0$, and $\partial p_2/\partial v(u, 0) = Ku^3 > 0$, so $\mathbf{f}^*$ satisfies the hypotheses of Theorem 1. Hence we can find $\beta_1, \delta > 0$ so that the set

$$\mathscr{B}^+ = \{(r, \theta) : r \leqslant \delta, | \theta - \tfrac{1}{4}\pi | \leqslant \beta_1\}$$

has the following properties:

$$A^+ = \{x \in \mathscr{B}^+ : \mathbf{f}^k(\mathbf{x}) \in \mathscr{B}^+ \, \forall k > 0\} - \{0\} \quad \text{is an analytic arc.} \tag{7.4}$$

$$\text{If} \qquad \mathbf{x} \in A^+, \quad \text{then} \quad \mathbf{f}^k(\mathbf{x}) \to 0 \quad \text{as} \quad k \to \infty. \tag{7.5}$$

$$\text{If} \qquad \| \mathbf{x} \| < \delta, \quad \| \mathbf{f}(\mathbf{x}) \| < \delta, \quad \text{and} \quad \mathbf{x} \notin \mathscr{B}^+, \quad \text{then} \quad \mathbf{f}(\mathbf{x}) \notin \mathscr{B}^+ \tag{7.6}$$

Properties (7.4) and (7.5) follow immediately from Theorem 1, while property (7.6) follows from (3.1) and (3.2).

We shall prove that $A^+ = \mathscr{A}^+ \cap \{x > 0\}$. To do so we must show that there are no points outside $\mathscr{B}^+$ which tend to the origin under positive iterates of $\mathbf{f}$. Our plan is to divide $U \cap \{x > 0\}$ into sectors and eliminate the other sectors one by one.

Rotate the coordinates so the line $y = -x$ becomes the u-axis. By applying Theorem 1 to $\mathbf{f}^{-1}$ we can find a set

$$\mathscr{B}^- = \left\{(r, \theta): r \leqslant \delta, \left| \theta + \frac{\pi}{4} \right| \leqslant \beta_2 \right\}$$

with properties analogous to those of $\mathscr{B}^+$. Note that the analog of (7.6) can be stated:

$$\text{If} \qquad \| \mathbf{x} \| < \delta, \quad \| \mathbf{f}(\mathbf{x}) \| < \delta, \quad \text{and} \quad \mathbf{x} \in \mathscr{B}^-, \quad \text{then} \quad \mathbf{f}(\mathbf{x}) \in \mathscr{B}^-. \tag{7.7}$$

Now let

$$\mathscr{C}^+ = \{(r, \theta) : r \leqslant \delta, \beta_3 \leqslant \theta \leqslant \tfrac{1}{2}\pi\}$$

If we choose $\beta_3 > \frac{1}{4}\pi$ and δ small enough, then $\mathscr{C}^+$ will have the following properties:

$$\text{If} \qquad \mathbf{x} \in \mathscr{C}^+ \quad \text{and} \quad \|\mathbf{f}(\mathbf{x})\| \leqslant \delta, \quad \text{then} \quad \mathbf{f}(\mathbf{x}) \in \mathscr{C}^+. \tag{7.8}$$

$$\text{If} \qquad \mathbf{x} \neq 0 \quad \text{and} \quad \mathbf{f}^k(\mathbf{x}) \in \mathscr{C}^+ \quad \text{for all} \quad k \geqslant 0,$$

$$\text{then} \qquad \mathbf{f}^k(\mathbf{x}) \nrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{7.9}$$

Property (7.8) follows from the polar form for $\mathbf{f}$, since θ is increasing in $\mathscr{C}^+$. Property (7.9) follows if we choose δ so small that

$$|r_1(x, y)| + |r_2(x, y)| \leqslant (1 - \cot \beta_3) y.$$

Since $x \leqslant y \cot \beta_3$ in $\mathscr{C}^+$, it follows that

$$x + r_2(x, y) \leqslant y + r_1(x, y),$$

and hence

$$f_1(x, y) - x \leqslant f_2(x, y) - y.$$

Let $\mathscr{D}(x_0, y_0) = \{(x, y) : y - y_0 \geqslant x - x_0\}$. We have shown that if $\mathbf{f}^k(x_0, y_0) \in \mathscr{C}^+$ for $k \geqslant 0$, then $\mathbf{f}^k(x_0, y_0) \in \mathscr{D}(x_0, y_0)$. Since $0 \notin \mathscr{D}(x_0, y_0)$ for $(x_0, y_0) \in C^+ - \{0\}$, we have proved (7.9).

Similarly we can define

$$\mathscr{C}^- = \left\{(r, \theta): r \leqslant \delta, -\frac{\pi}{2} \leqslant \theta \leqslant \beta_4\right\}$$

with analogous properties for $\mathbf{f}^{-1}$. Note that the analog of (7.8) becomes:

$$\text{If} \quad \|\mathbf{x}\| \leqslant \delta, \quad \|\mathbf{f}(\mathbf{x})\| \leqslant \delta, \quad \text{and} \quad \mathbf{x} \notin \mathscr{C}^-, \quad \text{then} \quad \mathbf{f}(\mathbf{x}) \notin \mathscr{C}^-. \tag{7.10}$$

Now let $U = \{(r, \theta) : r < \delta\}$. We shall show that

$$A^+ = \mathscr{A}^+(\mathbf{f}, U) \cap \{x > 0\}.$$

Let $\mathbf{x} \in \mathscr{A}^+(\mathbf{f}, U) \cap \{x > 0\}$, i.e., $\|\mathbf{f}^k(\mathbf{x})\| < \delta$ for all $k \geqslant 0$ and $\mathbf{f}^k(\mathbf{x}) \to 0$ as $k \to \infty$. Since r is increasing in $\mathscr{B}^-$, (7.7) implies that $\mathbf{f}^k(\mathbf{x}) \notin \mathscr{B}^-$ for all k. If $\mathbf{f}^k(\mathbf{x}) \in \mathscr{C}^+$ for some $k = k_1$, then $\mathbf{f}^k(\mathbf{x}) \in \mathscr{C}^+$ for all $k \geqslant k_1$ by property (7.8). But property (7.9) implies $\mathbf{f}^k(\mathbf{x}) \nrightarrow 0$. Therefore $\mathbf{f}^k(\mathbf{x}) \notin \mathscr{C}^+$ for all k. Since θ is increasing in $\mathscr{C}^- \cap \{x > 0\}$, (7.10) implies there exists k_2 such that $\mathbf{f}^k(\mathbf{x}) \notin \mathscr{C}^-$ for all $k \geqslant k_2$. Suppose there is a $k_3 \geqslant k_2$ such that $\mathbf{f}^k(\mathbf{x}) \notin \mathscr{B}^+$ for $k = k_3$. Then by (7.6), $\mathbf{f}^k(\mathbf{x}) \notin \mathscr{B}^+$ for all $k \geqslant k_3$. Therefore

$$\mathbf{f}^k(\mathbf{x}) \notin \mathscr{B}^+ \cup \mathscr{B}^- \cup \mathscr{C}^+ \cup \mathscr{C}^- = \mathscr{B} \qquad \text{for all} \quad k \geqslant k_3.$$

But θ is strictly monotone on the complement of $\mathscr{B}$, which is a contradiction.

Therefore $\mathbf{f}^k(\mathbf{x}) \in \mathscr{B}^-$ for all $k \geqslant k_2$. Property (7.6) implies $\mathbf{f}^k(\mathbf{x}) \in \mathscr{B}^+$ for all $k \geqslant 0$. Therefore $\mathbf{x} \in A^-$ by (7.4).

Now suppose $\mathbf{x} \in A^+ - \{0\}$. Property (7.5) implies that

$$x \in \mathcal{Cl}^-(\mathbf{f}, U) \cap \{x > 0\}.$$

Therefore

$$\mathcal{Cl}^-(\mathbf{f}, U) \cap \{x > 0\} = A^-,$$

and is an analytic arc by (7.4). The proof of Proposition 11 is complete.

8. The Collinear Three Body Problem

The next example is the three body problem in R^1, i.e., when all three particles move along a line. Let particle k have position $z_k \in R^1$ and mass m_k. The motion is described by the differential equations:

$$\ddot{z}_k = -\sum_{j\neq k} \frac{m_j(z_k - z_j)}{|z_k - z_j|^3}$$

We take the total energy to be negative:

$$\frac{1}{2}\sum m_j \dot{z}_j^2 - \frac{1}{2}\sum\sum_{j\neq k} \frac{m_j m_k}{|z_j - z_k|} = -h < 0.$$

Now assume $z_1 < z_2 < z_3$. We shall regularize double collisions by a Levi-Civita transformation, therefore such a collision results in a "bounce" and the ordering remains the same. We shall consider parabolic orbits such that $z_1 \to -\infty$ and $\dot{z}_1 \to 0$ as $t \to \infty$. Since the total energy is negative, the distance $z_3 - z_2$ will remain bounded.

We take the center of mass at the origin: $m_1 z_1 + m_2 z_2 + m_3 z_3 = 0$. Let $m = (m_2 + m_3)^{1/3}$, $M = (m_1 + m_2 + m_3)^{1/3}$, and make the following change of variables:

$$z = -M^2 m^{-3} z_1$$
$$\zeta = m^2 m_2^{-1}(z_3 + m_1 m^{-3} z_1).$$

The equations become

$$\ddot{z} = -m^{-3} z^{-2}[m_2(1 - m_3 u)^{-2} + m_3(1 + m_2 u)^{-2}]$$
$$\ddot{\zeta} = -\zeta^{-2} + m_1 m^{-1} M^{-2} z^{-2}[(1 - m_3 u)^{-2} - (1 + m_3 u)^{-2}],$$

where $u = m^{-2} M^{-1} \zeta z^{-1}$. As $z_1 \to -\infty$, $z \to +\infty$ and $\zeta = m^{-1}(z_3 - z_2)$ remains bounded. Therefore $u \to 0$. Expanding in powers of u, we get

$$\ddot{z} = -z^{-2}(1 + g_1(u))$$
$$\ddot{\zeta} = -\zeta^{-2}(1 + g_2(u)),$$

where $g_1(u) = O(u^2)$ and $g_2(u) = O(u^3)$. The energy integral becomes:

$$2a^2(\tfrac{1}{2}\dot{z}^2 - z^{-1}) + 2b^2(\tfrac{1}{2}\dot{\zeta}^2 - \zeta^{-1}) + z^{-1}h^{-1}v(u) = -1,$$

where $2a^2 = h^{-1}m_1m^3M^{-1}$, $2b^2 = h^{-1}m_2m_3m^{-1}$, and $v(u) = O(u^2)$.

As in the previous example, we wish to bring $z = \infty$ to the origin. We must also regularize the singularity at $\zeta = 0$ with a Levi-Civita transformation. Let

$$\begin{aligned} z &= 2x^{-2} \\ \dot{z} &= y \\ \zeta &= 2b^2\xi \\ \dot{\zeta} &= b^{-1}\eta\xi^{-1}. \end{aligned}$$

The energy relation can be written

$$\eta^2 + \xi^2 + a\xi^2(y^2 - x^2) + v_1(u) = 1.$$

The differential equations become

$$\begin{aligned} x' &= -b^3\xi^2x^3y \\ y' &= -b^3\xi^2x^4(1 + g_1(u)) \\ \xi' &= \eta \\ \eta' &= \xi(1 + g_3(x, y, \xi)). \end{aligned}$$

Note that $u = (\text{const})\,\xi^2x^2$, $v_1(u) = O(u^3)$, $g_1(u) = O(u^2)$, and $g_3(x, y, \xi)$ is second order in x and y and fourth order in ξ. Also we have made the time transformation $dt = 4b^3\xi^2\,d\tau$, so that the prime denotes differentiation with respect to τ.

As in the previous example $(x, y) = 0$ is a periodic orbit. Taking a Poincaré section, we get a map of the form (7.3), where $K = \pi b^3$. Again the parabolic orbits are exactly those for which $(x, y) \to 0$ as $t \to \infty$. Applying Proposition 11 we have that the parabolic orbits form a real analytic submanifold of the energy surface.

9. Sitnikov's Problem

The final problem we wish to consider is one that has been discussed by Sitnikov [5] and Alekseev [1]. In this problem we have three non-zero masses moving in R^3, but we impose enough symmetry conditions so the problem is reduced to one of two degrees of freedom.

Let particle k have position $(w_k, z_k) \in \mathbf{C}^1 \times R^1$ and mass m_k. Assume that $w_1 = 0$, $z_2 = z_3$, $w_2 = -w_3$, $m_2 = m_3$, and the center of mass is

(0, 0). The three particles will retain this symmetry, and their positions are determined by z_1 and w_2. The equations of motion are:

$$\ddot{z}_1 = -2m_2(z_1 - z_2)[(z_1 - z_2)^2 + |w_2|^2]^{-3/2}$$
$$\ddot{w}_2 = -2m_2 w_2 |2w_2|^{-3} - m_1 w_2[(z_1 - z_2)^2 + |w_2|^2]^{-3/2}.$$

Let the total energy be negative:

$$\tfrac{1}{2}m_1\dot{z}_1^2 + m_2(\dot{z}_2^2 + |\dot{w}_2|^2) - 2m_1m_2[(z_1 - z_2)^2 + |w_2|^2]^{-1/2} - m_2^2 |2w_2|^{-1} = -h < 0.$$

Let $m = (2m_2)^{1/3}$, $M = (m_1 + 2m_2)^{1/3}$, and make the change of variables

$$z = M^2 m^{-3} z_1$$
$$\zeta = 2m^{-1} w_2 .$$

The differential equations become

$$\ddot{z} = -(1 + u^2)^{-3/2} z |z|^{-3}$$
$$\ddot{\zeta} = -\zeta |\zeta|^{-3} - m_1 M^{-3}(1 + u^2)^{-3/2} \zeta |z|^{-3},$$

where $u = \tfrac{1}{2}mM^{-1} |\zeta| z^{-1}$.

The parabolic orbits we wish to consider are those for which $z_1 \to \infty$, $\dot{z}_1 \to 0$, and hence $z \to \infty$, $\dot{z} \to 0$, as $t \to \infty$. Since the total energy is negative, $|w_2|$ will remain bounded. Therefore $u \to 0$ and we can write the equations:

$$\ddot{z} = -z |z|^{-3}(1 + g_1(u))$$
$$\ddot{\zeta} = -\zeta |\zeta|^{-3}(1 + g_2(u))$$

where $g_1(u) = O(u^2)$, $g_2(u) = O(u^3)$. The energy integral can be written

$$2a^2(\tfrac{1}{2}\dot{z}^2 - |z|^{-1}) + 2b^2(\tfrac{1}{2} |\dot{\zeta}|^2 - |\zeta|^{-1}) + |z|^{-1} v(u) = -1,$$

where $2a^2 = m_1 m^3 M^{-1} h^{-1}$, $2b^2 = \tfrac{1}{4} m^5 h^{-1}$, and $v(u) = O(u^2)$. Again we wish to bring $z = \infty$ to $x = 0$, so we make the further transformation

$$z = 2x^{-2}$$
$$\dot{z} = y$$
$$\sigma = b^{-2}[|\zeta| - b^2 - ib\,\mathrm{Re}(\bar{\zeta}\dot{\zeta})]$$

Note that angular momentum $\omega = \mathrm{Im}(\bar{\zeta}\dot{\zeta})$ is another integral for this problem. Note also that

$$|\zeta| = b^2(1 + \mathrm{Re}(\sigma)).$$

The energy integral becomes

$$a^2 |\zeta|^2(y^2 - x^2) + b^4 |\sigma|^2 + b^2\omega^2 - v_1(x, \sigma) = b^4,$$

where $v_1(x, \sigma)$ is sixth order in x. The differential equations become

$$\begin{aligned} x' &= -\tfrac{1}{4}b |\zeta| x^3y \\ y' &= -\tfrac{1}{4}b |\zeta| x^4(1 + g_3(x, \sigma)) \\ \sigma' &= i(\sigma + g_4(x, y, \sigma)), \end{aligned}$$

where g_3 is fourth order in x and g_4 is second order in x and y. Again we have made a time transformation $dt = b |\zeta| d\tau$ and the differentiation is with respect to τ.

Once more we have a periodic orbit at $(x, y) = 0$ and its Poincaré map has the form (7.3), with $K = \frac{1}{2}\pi b^3$. The set of parabolic orbits is therefore a real analytic submanifold of the energy surface.

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