HW 1. (Classical Geometries.) Due Monday, Oct 1, 2012, i.e.: Next class.

1. A gradient flow. Let $\mathbb{T}$ be the flat torus with standard coordinates θ_1, θ_2 mod 2π. Let $V = -\cos(\theta_1) cos(\theta_2)$. Locate the equilibria. Describe each type (source, saddle, sink). Sketch the flow lines .

2. Hamiltonian flows. For $M = T^*\mathbb{R} = \mathbb{R} \times \mathbb{R}$ let $H = (1/2)p^2 + P(x)$ where P is a polynomial. Sketch the phase portraits in case
a) P is linear
b) P is homogeneous quadratic.
c) P is cubic. Do a few cases.
d) $P = (x-1)^2(x+1)^2$.

3. Again as in 2. Again P is polynomial. Is the flow complete? Find a proof or a counterexample.

4. N-dimensional oscillator. This has for its Hamiltonian $H(q,p) = (1/2)\langle p, p\rangle^2 + \langle q, Aq\rangle$ where $p, q \in \mathbb{R}^n$, where we use the standard inner product $\langle \cdot, \cdot \rangle$ to identify $\mathbb{R}^n$ with its dual, and where A is a positive definite symmetric matrix. Prove that the closure of the typical orbit is a k-torus, for some $k \leq N$. Describe the maximal k in terms of the eigenvalues of A.

5. Guckenheimer-Holmes. Exer. 5.1.2 and 5.1.3 of p. 234.
These exercises are on the Smale Horseshoe and are best solved using symbolic dynamics. Let $\Gamma \subset I^2$ be the subset of the square $S = I^2$ which never leaves the square in forward or backward time.
5.1.2. Show that all the periodic orbits are of saddle type. Locate the periodic orbits with period 4 or less and write out their symbol sequence. Show that Λ contains a countable infinity of heteroclinic and homoclinic orbits. Show that Λ contains an uncountable number of orbits which are not periodic.
5.1.3 Show that Λ contains a dense orbit.

6. A gradient system in $\mathbb{R}^n$ is given by $\dot{x} = -\nabla V(x), x \in \mathbb{R}^n$ where V is a smooth function. What is special about the linearization of a gradient system at an equilibrium, in comparison to a general linear system $\dot{x} = Ax$ with A a general n by n matrix.

7. Newton's equations on $\mathbb{R}^n$ are equations of the form $\ddot{x} = -\nabla V(x), x \in \mathbb{R}^n$ where V is a smooth function, called the potential.
a) First orderize the system by introducing $v = \dot{x}$ so as to make it an ODE on $\mathbb{R}^n \times \mathbb{R}^n = T\mathbb{R}^n$.
b. What is special about the linearization of Newton's equations at an equilibrium in in comparison to a general linear system $\dot{x} = Ax$ with A a general $2nby2n$ matrix.

8. Take the Cantor set to be the product space $\mathbb{Z}_2^{\mathbb{N}_+}$, ($\mathbb{N}_+$ is the set of all positive integers) endowed with the product topology. An element of the Cantor set is then an infinite sequence $(\sigma_i)_{i \in \mathbb{N}}$ of 1's and 0's; $\sigma_i \in \{0, 1\}$. the sequence labelled by the positive integers. Consider the map F whcih sends to

$$;_2 : C_2 \to [0,1]; F_2(\sigma) = \Sigma_{i \in \mathbb{N}_+} \sigma_i 2^{-i}$$

a) Show that F_2 is continous and onto.

b) If we give $\mathbb{Z}_2$ the "coin-flip" measure (each element has probability 1/2) then the Cantor set inherits a probability measure. (The product of probability spaces is a probability space, so that the Cantor set has a probability measure on it.) Show that F_2 is an isomorphism in the sense of measure theory: it is onto, and the map is measure preserving: $\mu(F_2^{-1}(I) = |I|$ for any interval I.

hint: consider dyadic intervals.

d)Show that F_2 is a measure preserving semi-conjugacy between the Bernoulli shift on the Cantor set and the doubling map (mod 1) on the interval.

e) Repeat (a)-(c) for $F_N : \mathbb{Z}_N^{\mathbb{N}_+} \to I$.

f) Use the fact that there is a bijective map from two disjoint copies of $\mathbb{N}_+$ to $\mathbb{N}_+$ to define an onto map $C \to I \times I$. Repeat, to establish the existence of an onto map from the Cantor set ONTO the n-cube. Onto any compact n-manifold.

9. Show that the doubling map $S^1 \to S^1$ is measure preserving.

10. Prove that rotation of the circle is NOT mixing.

11. Prove that the suspension of a map is NOT a mixing flow.

12. Construct a homeomorphism of the plane $\mathbb{R}^2 = \mathbb{C}$ which maps the spiral $exp(1+i)t), t \in \mathbb{R}$ to the ray $y = 0, x > 0$.