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SPECTRAL INVARIANTS IN ERGODIC THEORY.

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Koopmanism and Spectral Theory.

A map $T : X \to X$ induces a linear operator on the space of functions $X \to \mathbb{C}$ by pull-back: $f \mapsto T^* f$ If T is invertible and preserves a measure μ on X, then this pull-back operator T^*, upon restriction to the space of L^2 functions defines a unitary operator $U = U_T$ on $L^2 = L^2(X, \mu)$. Unitary operators U have a spectrum $\sigma(U) \subset S^1 \subset \mathbb{C}$. Koopman's idea, sometimes called "Koopmanism" was to build a dictionary between dynamical properties of T and spectral properties of U.

Definition 0.1. *By the "spectrum" of an invertible measure preserving map $T : (X, \mu) \to (X, \mu)$ we mean the spectrum of its associated unitary operator U_T on $L^2(X, \mu)$.*

Exercise. Prove that $U = U_T$ is indeed unitary.

Note: Ruelle calls U the "transfer operator".

The spectrum of a bounded operator on a separable Hilbert space breaks up into various pieces with multiplicities attached to each. These pieces are called the "point spectrum", "absolutely continuous spectrum" and residual spectrum, also known as singular continuous. The elements of each piece also receive a multiplicity, this being either a positive integer or ∞, for a countable infinity. We will recall these different spectra momentarily.

Example 0.1. *A. If $T : S^1 \to S^1$ is rotation by an angle $\alpha = 2\pi p/q$ which is a rational multiple p/q or 2π then its spectrum consists of the qth roots of unity, each eigenvalue occurring with countable multiplicity.*

B. If $T : S^1 \to S^1$ is rotation by an irrational angle $e^{2\pi i \alpha}$, α irrational, then the spectrum of T consists of the entire unit circle, with the countable dense set $exp^{2\pi i k \alpha}$ consisting of point spectrum, each with multiplicity one.

C. If $T : C \to C$ is the shift operator of symbolic dynamics, where $C = (\mathbb{Z}_2)^{\mathbb{Z}}$ then its spectrum consists of the entire unit circle. 1 is the only eigenvalue and occurs with multiplicity one. The rest of the unit circle occurs with infinity multiplicity.

For A and B, use the basis $e_k = e^{i2\pi kx}, k \in \mathbb{Z}$ for $L^2(S^1)$. Note that if T is rotation by $2\pi\alpha$ then $T^* e_k = e^{2\pi i k \alpha} e_k$.

For C,

Theorem 0.1. *The spectrum of a Bernoulli shift is the entire unit circle, with 1 an eigenvalue of multiplicity one and all other elements lying in the continuous spectrum and having multiplicity ∞.*

In particular, all Bernoulli shifts are spectrally equivalent.

Entropy. There is an invariant of a measure preserving dynamical system called 'entropy' which can distinguish different Bernoulli shifts.

We recall some of the definitions.

and that the spectrum breaks up into pieces – continuous, discrete and then the mysterious 'singular continuous". – eigenvalues and continuous spectrum, and, possibly discontinuous spectrum. We recall the definitions. Let A be a bounded operator on a Hilbert space.

Eigenvalues. $\lambda \in \mathbb{C}$ is an eigenvalue of A if there is a nonzero vector v in the Hilbert space such that if $Av = \lambda v$. .

Spectrum. The spectrum of A, denoted $\sigma(A)$ consists of the set of all complex numbers λ for which $A - \lambda I$ is not invertible.

The spectrum is a closed subset of the complex plane sitting inside the disc of radius $\|A\|$ where the norm is the operator norm.

Continuous spectrum vs eigenvalues. When H is finite-dimensional, the only way for $A - \lambda I$ to be non-invertible is for λ to be an eigenvalue. In infinite dimensions there are other ways.

The model example of "another way" is given by a multipication operator by a nonconstant continuous function.

Theorem 0.2. *Let X be a Hausdorff space endowed with a Borel measure μ, $f : X \to \mathbb{C}$ be a bounded continuous nonconstant function and let $A = M_f$ be the bounded operator of multiplication by f acting on $L^2(X) = L^2(X, \mu)$. Then $\sigma(M_f) = f(X)$. Moreover, any number $\lambda \in f(X)$ for which the closed set $f^{-1}(\lambda)$ has measure zero is not an eigenvalue.*

$\sigma(M_f) \subset f(X)$. Suppose that $\lambda \notin f(X)$. Then $M_f - \lambda I = M_{f-\lambda}$. But the function $f - \lambda$ misses zero, so the function $g = 1/(f - \lambda)$ is everywhere defined on X and satisfies $g(f - \lambda) = 1$. Thus $M_f - \lambda I)^{-1} = M_g$, showing that $\mathbb{C} \setminus f(X) \subset \mathbb{C} \setminus \sigma(M_f)$ which is to say $\sigma(M_f) \subset f(X)$.

Next, suppose that $\lambda \in f(X)$ while the set $C = f^{-1}(\lambda)$ has positive measure. Write ϕ for the characteristic function of this set. Then $\int \phi^2 d\mu = \mu(C) \neq 0$, so that $\phi \neq 0$ (a.e.) while $M\phi = 0$. This proves that λ is an eigenvector with eigenvector ϕ.

Finally, suppose that $\lambda \in f(X)$ and that $f^{-1}(\lambda)$ has measure zero. I will show that $\lambda \in \sigma(M)$ but that λ is not an eigenvalue of M. Observe that $\lambda \in f(X)$ if and only if 0 is in the range of the continuous function $g = f - \lambda$, and that the set $f^{-1}(\lambda) = g^{-1}(\lambda)$. Thus it suffices to show that if g is a continuous function on X for which $g^{-1}(0)$ has measure zero, then $0 \in \sigma(M_g)$ but that 0 has no eigenfunction. The method is to construct a sequence of functions ψ_n of unit length, ie $\|\psi_n\| = 1$, such that $M_g\psi_n \to 0$. This would be impossible if M_g had a bounded inverse, for if L were this inverse and if its operator norm were $\|L\| = C$ then we would have on the one hand $LM_g\psi_n = \psi_n$ but on the other hand $\|\psi_n\| = \|LM\psi_n\| \leq C\|M\psi_n| \to 0$, which is impossible. . Consider the nested sequence of open sets $U_n = g^{-1}(-1/n, 1/n)$. For each n, apply Urysohn's lemma to the complementary closed sets $\bar{U}_{n+1}$ and $X \setminus U_n$. We find there exist "bump functions" $\phi_n : X \to \mathbb{R}$ which are 1 on U_{n+1} and 0 off of U_n. Normalize the ϕ_n by dividing them by $\sqrt{\int \psi_n^2 d\mu}$. Then these functions ψ_n are unit vectors: $\|\psi_n\| = 1$, having support on U_n. In particular $|g\psi_n| \leq (1/n)\psi_n$ from which it follows that $M\psi_n\| \leq (1/n) \to 0$.

To see that the $M = M_g$ of the previous paragraph has no eigenvector for the eigenvalue 0, suppose it did. Thus, there is a $\phi \in L^2$ wih $M_g\phi = 0\phi$ or, $g\phi = 0$. But $g(x)\phi(x) = 0$ for $g(x) \neq 0$ if and only if $\phi(x) = 0$. Thus $\phi = 0$ a.e. off of the set $g^{-1}(0)$. But $g^{-1}(0)$ has measure zero so that ϕ must be zero on a set of full measure, and consequently is zero as an element of L^2.

QED

Remark. The functions ψ_n constructed above are "trying" to converge to the delta function supported on the closed measure zero set $C = f^{-1}(\lambda)$. We say that M_f has $\lambda = f(x)$ as a "continuous eigenvalue" with corresponding "generalized eigenvector" δ_C, the distribution D on X for which $\langle D, g \rangle = \int_C g(y) du_C(y)$. The measure $d\mu_C$ is unclear, and could be made sense of with more structure. Perhaps.

Bi-infinite shift. A separable Hilbert space has a countable basis $e_k, k \in \mathbb{Z}$. The bi-infinite shift is the bounded linear map induced by $Ue_k = e_{k+1}$. It is unitary.

Theorem 0.3. *The spectrum of the shift map is the entire circle and is all absolutely continuous, each point occuring with multiplicity one.*

Proof. Think of $e_k = exp(i2\pi k\theta) \in L^2(S^1)$. The operator of multiplication M_z by the embedding function $z = exp(2\pi i\theta)$, $z : S^1 \to \mathbb{C}$, sends e_k to e_{k+1}. But the function z has image the unit circle and is nowhere constant. QED

If μ is a probability measure then the constant function $1 \in L^2(X, \mu)$ is an eigenvector of U_T. Consequently, $1^{\perp} := \{f \in L^2 : \int f = 0\}$ is a closed invariant subspace of the Hilbert space. We will write this space as L^2_0.

Theorem 0.4. *Let μ be a probability measure on X. Then T is ergodic if and only if the dimension of the space of $U = U_T$ -invariant functions is 1, if and only if the multiplicity of the eigenvalue 1 is equal to one.*

Following Arnold-Avez, we say a map $T : (X, \mu) \to (X, \mu)$ is a "K-system" if $L^2(X, \mu)_0 = 1^{\perp}$ admits an orthonormal basis $e_{a,i}$, doubly indexed by the integers : $a, i \in \mathbb{Z}$ such that $U_T e_{a,i} = e_{a,i+1}$. Fixing a and letting i vary and taking the closure, we get a separable Hilbert space $H_a \subset L^2$. On each H_a the operator U is a bi-infinite shift, so its spectrum is as given above: it consists of the entire circle as absolutely continuous part. Thus L^2, U is the countable direct sum of shift operators $\bigoplus_a H_a$, shift and U has the entire circle as absolutely continous spectrum, each element occuring with countable multiplicity.

Example. The Bernouli shift $B(1/2, 1/2)$ is a K-system. Recall that the elements of $B(1/2, 1/2)$ are bi-infinite sequences of 0's and 1's. For $n \in \mathbb{Z}$ write $f_n(\sigma) = -(-1)^{\sigma_n}$. In other words $f_n = -1$ if $\sigma_n = 0$ and $f_n = 1$ if $\sigma_n = 1$. Then $Uf_n = f_{n+1}$. Now for any finite subset $S \subset \mathbb{Z}$ we set $f_S = \Pi_{i \in S} f_i$. One checks without difficulty that the f_S form an orthonormal basis for L^2_0 and that $Uf = f_{S'}$ where S' is obtained by adding 1 to each element of S, or , for simplicity $S' = S + 1$. More generally $U^k f_S = f_{S'}$ where $S' = S + k$ and k is any integer. It follows that our countable basis f_S can be relabelled $f_{a,j}$ with $Uf_{a,j} = f_{a,j+1}$ and we have expressed the shift as a K-system.

Lemma 0.1. *T is ergodic if and only if $1 \in \sigma(U_T)$ is an eigenvalue with multiplicity 1.*

T is mixing if and only if for all $f, g \perp 1$ we have that $\lim_n \langle g, T^n f \rangle = \langle g, f \rangle$???

T is mixing if and only if the only eigenvalue of U is $1 \in S^1$.

Let us suppose, for simplicity, that μ is a probability measure: $\mu(X) = 1$.

REFERENCES

Arnold-Avez

Brin

Reed and Simon, section 8.4

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