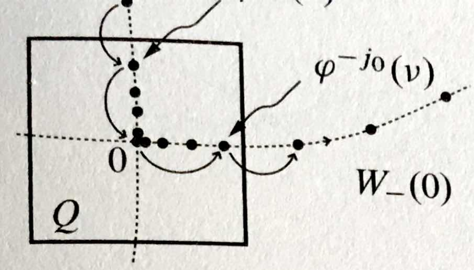

Figure III.6. Jump of the ε-pseudo orbit from $W_+(0)$ onto 0 and then into $W_-(0)$.

periodic ε-pseudo orbit on Λ, namely,

$$q_{j+N} = q_j, \quad j \in \mathbb{Z},$$

where $N = 2j_0 + k + 1$.

In view of the shadowing lemma the pseudo orbit q is shadowed by the unique δ-shadowing orbit $p = (p_j)_{j\in\mathbb{Z}}$ given by $p_j = \varphi^j(p_0)$. By construction, $p_0 \in V$ and $\mathcal{O}(p_0) \subset U(\Lambda)$. Since also the shifted sequence $\hat{p} = (\hat{p}_j)_{j\in\mathbb{Z}}$, defined by $\hat{p}_j = p_{j+N}$, is a δ-shadowing orbit of the pseudo orbit q, it follows from the *uniqueness* that $\hat{p} = p$, hence

$$p_{j+N} = p_j, \quad j \in \mathbb{Z}.$$

Therefore, the shadowing orbit p is a *periodic* orbit of φ having the minimal period N. The theorem follows if we set $N_0 = 2j_0 + 2$.

□

The next result goes back to H. Poincaré. It explains why the transversal homoclinic point forces the invariant manifolds $W_+(0)$ and $W_-(0)$ issuing from the hyperbolic fixed point to double back and pile up on themselves as illustrated in Figure III.7.

Theorem III.12 (H. Poincaré). *We assume that ν is a transversal homoclinic point associated with the hyperbolic fixed point 0 of the diffeomorphism φ. Let V be an open neighborhood of ν and let $U = U(\Lambda)$ be an open neighborhood of $\Lambda = \overline{\mathcal{O}(\nu)}$. Then there exist infinitely many homoclinic points associated with 0 in V, whose orbits run in U, and which are distinguished by two* **rotation numbers** *$r^\pm$.*

Proof [Shadowing lemma]. We again construct a suitable ε-pseudo orbit on Λ using the same notation as in the previous proof and let $r^\pm \in \mathbb{N}_0$ be two integers. Set

$$\begin{array}{ccccccccc} \nu & \varphi(\nu) & \dots & \varphi^{j_0}(\nu) & 0 & \varphi^{-j_0}(\nu) & \dots & \varphi^{-1}(\nu) \\ \| & \| & & \| & \| & \| & & \| \\ q_0 & q_1 & \dots & q_{j_0} & q_{j_0+1} & q_{j_0+2} & \dots & q_{2j_0+1}. \end{array}$$

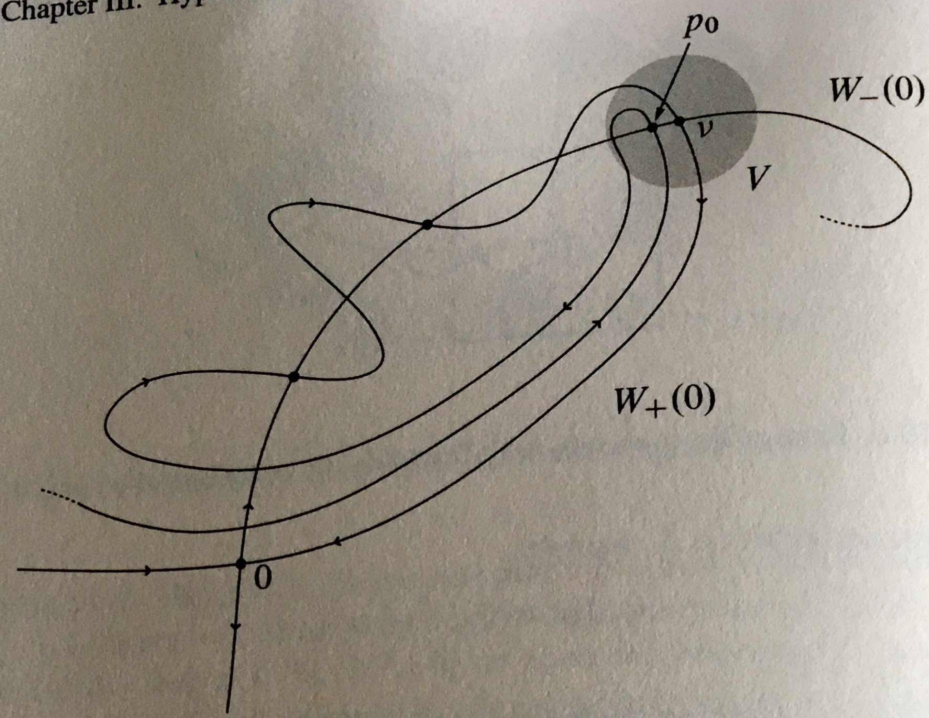

Figure III.7. One of the infinitely many homoclinic points p_0 near ν.

We repeat this finite sequence on the right r^+-times and on the left r^--times, then we add on the left $\varphi^{-j_0}(\nu), \dots, \varphi^{-1}(\nu)$ and on the right $\nu, \varphi(\nu), \dots, \varphi^{j_0}(\nu)$. Finally, we add on the left respectively on the right the infinite sequences $\dots, 0, 0, 0$ resp. $0, 0, 0, \dots$, which belong to the orbit of the fixed point 0. Hence, after choosing the integer j_0 and the open neighborhood Q as in the proof of the previous theorem, we have constructed an ε-pseudo orbit q on the hyperbolic set Λ, for which it holds true that

$$q_j = 0, \quad |j| \geq M,$$

for a suitable constant M.

This pseudo orbit is shadowed by the unique δ-shadowing orbit $p = (p_j)_{j \in \mathbb{Z}}$ which satisfies, by construction, $\mathcal{O}(p_0) \subset U$ and $p_0 \in V$ and, *in addition*,

$$|p_j| \leq \delta, \quad |j| \geq M.$$

Therefore, for $|j|$ large, all the orbit points lie in the δ neighborhood Q_δ of the hyperbolic fixed point 0. Hence, they lie on the *local manifolds* $W^+_{\text{loc}}(Q_\delta)$ respectively $W^-_{\text{loc}}(Q_\delta)$, introduced in II.2. Due to Theorem II.7, $\varphi^j(p_0) \to 0$ as $|j| \to \infty$, if we choose δ sufficiently small. Therefore, p_0 is a *homoclinic* point in V, and, by construction, $p_0 \neq \nu$. Shadowing orbits belonging to different rotation numbers are different from each other. The proof of the theorem is complete. □

More generally, we can consider two hyperbolic fixed points $x^* \neq y^*$ of the diffeomorphism φ. If the invariant manifolds intersect *transversally* in the heteroclinic