We recall that an integral of the vector field X is a function $H: \mathbb{R}^2 \to \mathbb{R}$ satisfying $dH(X) = 0$. Equivalently, the flow of X leaves the function H invariant, so that $H(\varphi^t(z)) = H(z)$ for all t and z. Therefore, the orbits lie on the level lines

$$E_c := \{(x, y) \in \mathbb{R}^2 \mid H(x, y) = c\},$$

consisting of the two branches $y = \pm\sqrt{2(c + \cos x)}$. Figure III.11 shows that the mathematical pendulum possesses the following *orbit types*.

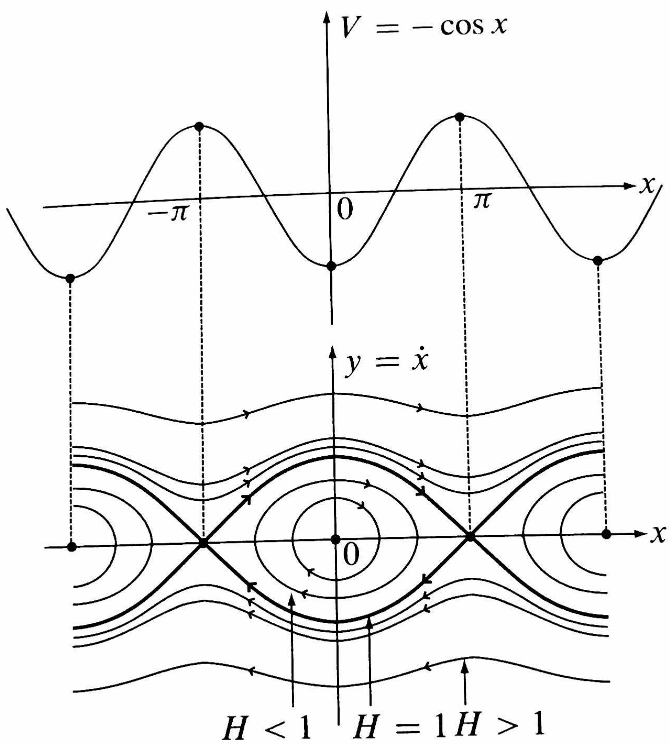
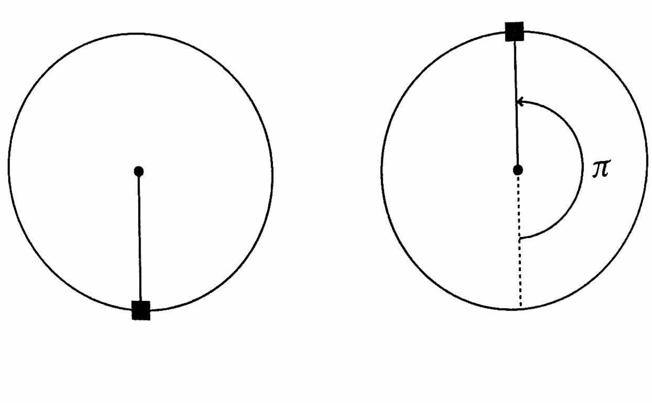
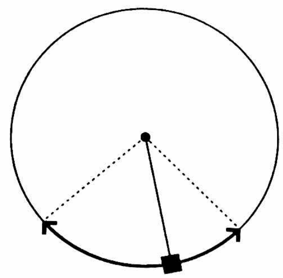


Figure III.11. Level sets of the integral H. The separatrix is marked.

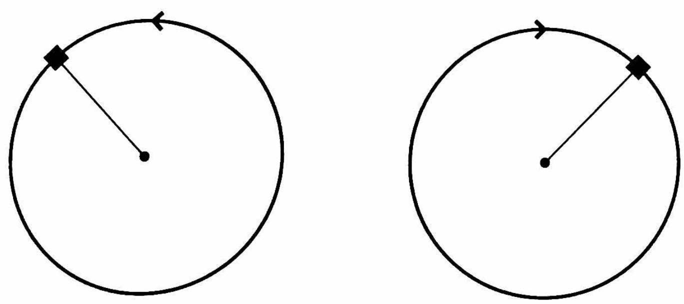
Equilibrium points. The equilibrium points are, on one hand, the constant orbits in the level set $\{H = -1\}$, these are the so-called elliptic equilibrium points located in $(x, y) = (2\pi n, 0)$ (on the left figure). On the other hand, the hyperbolic equilibrium points located in $(x, y) = ((2n + 1)\pi, 0)$ are on the level set $\{H = 1\}$.



Oscillation. The oscillations around the lowest point are on the level sets $\{-1 < H < 1\}$, in Figure III.12 described by the closed curves.



Rotations. The rotational solutions lie on the level sets $\{H > 1\}$. The angle is either strictly increasing (left) or else strictly decreasing (right).



Heteroclinic orbits. The level set $\{H = 1\}$ carries the homoclinic orbits (in $S^1 \times \mathbb{R}$) and the heteroclinic orbits (in $\mathbb{R}^2$) respectively, and the hyperbolic equilibrium points. This level set is called a *separatrix* because it separates the oscillations from the rotations.

Keeping the time $T > 0$ fixed, the flow in time T,

$$\varphi^T : \mathbb{R}^2 \to \mathbb{R}^2,$$

is a diffeomorphism possessing the hyperbolic fixed points $P_n = ((2n+1)\pi, 0)$ for $n \in \mathbb{Z}$. This is easily verified using Lemma II.12. They are 2π-periodically distributed (in the projection on $S^1 \times \mathbb{R}$ they all correspond to the same point). Their stable and unstable invariant manifolds coincide in the sense that $W_+(P_n) = W_-(P_{n+1})$ for all $n \in \mathbb{Z}$. We now *perturb* the pendulum by means of a *time T-periodic excitation* and consider the equation

$$\ddot{x} + \sin x = \mu \sin \omega t, \quad T = \frac{2\pi}{\omega}.$$

The energy function H is *no longer* an integral of the system and the orbit structure changes drastically. The new vector field

$$\dot{z} = X(t, \mu, z) \in \mathbb{R}^2$$

is now *time dependent* and *T-periodic* in time t, so that $X(t+T,\mu,z) = X(t,\mu,z)$ for all t,μ,z. The flow solves the initial value problem

$$\begin{cases} \frac{d}{dt}\varphi^t(z,\mu) = X(t,\mu,\varphi^t(z,\mu)), & t \in \mathbb{R}, \\ \varphi^0(z,\mu) = z. \end{cases}$$

Due to the *uniqueness* of the Cauchy initial value problem, it follows from the 2π periodicity of the vector field X in the variable x that

$$\varphi^t(z + 2\pi j e_1, \mu) = \varphi^t(z,\mu) + 2\pi j e_1,$$

for all $t,\mu \in \mathbb{R}$ and $j \in \mathbb{Z}$, where $e_1 = (1,0)$. Moreover, it follows from the T-periodicity of the vector field in time t that

$$\varphi^{t+T}(z,\mu) = \varphi^t(\varphi^T(z,\mu))$$

for every $t \in \mathbb{R}$ and $z \in \mathbb{R}^2$ (recall that the relation $\varphi^t \circ \varphi^s = \varphi^{t+s}$ is only valid for the flow of a time independent vector field). Keeping the parameter μ fixed, the mapping

$$\psi(z) := \varphi^T(z,\mu)\colon \mathbb{R}^2 \to \mathbb{R}^2$$

is a diffeomorphism satisfying $\psi^j(z) = \varphi^{jT}(z,\mu)$ for every $z \in \mathbb{R}^2$.

Let us assume that there exists a solution $x(t)$ of the equation $\ddot{x} + \sin x = \mu \sin \omega t$ possessing *infinitely many zeros* (mod 2π), at the times $(t_k)_{k\in\mathbb{Z}}$ which are all nondegenerate. The times are ordered according to $t_k < t_l$ if $k < l$, so that

$$x(t_k) = 0 \mod 2\pi, \quad \dot{x}(t_k) \neq 0, \quad k \in \mathbb{Z}.$$

In other words, the pendulum passes the lowest point infinitely often with a non-vanishing velocity. We associate with this solution a two-sided sequence $\sigma(x(t)) = (\sigma_k(x(t)))_{k\in\mathbb{Z}}$, defined by

$$\sigma_k(x(t)) = \operatorname{sign}(\dot{x}(t_k)) = \begin{cases} +1, & \dot{x}(t_k) > 0, \\ -1, & \dot{x}(t_k) < 0. \end{cases}$$

In the *unperturbed* case $\mu = 0$ there exist precisely three types of such sequences, namely

(a) constant $+1$, i.e., $\sigma_k(x(t)) = +1$ for all $k \in \mathbb{Z}$,
(b) constant -1, i.e., $\sigma_k(x(t)) = -1$ for all $k \in \mathbb{Z}$,
(c) alternating, i.e., $\sigma(x(t)) = (\dots, +1, -1, +1, -1, +1, \dots)$.

In sharp contrast to this unperturbed situation, the perturbed mathematical pendulum possesses a solution for every prescribed random sequence as the following theorem shows.

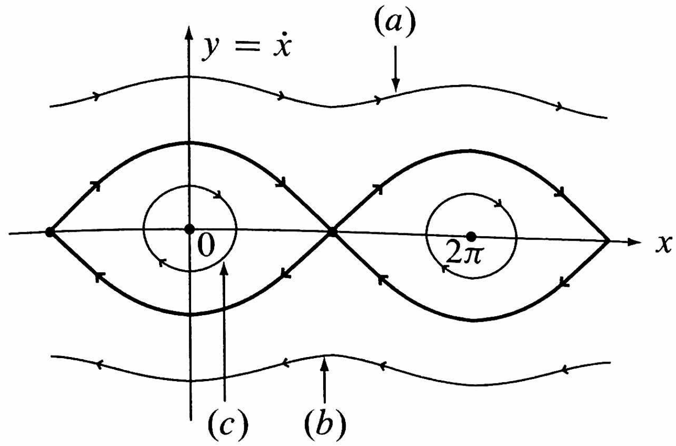


Figure III.12. Types in the unperturbed case.

Theorem III.19. *Let $U \subset \mathbb{R}^2$ be an open neighborhood of the separatrix (in the case $\mu = 0$). If $|\mu| > 0$ is sufficiently small, then there exists for every two-sided sequence $(s_k)_{k \in \mathbb{Z}}$ of integers $s_k \in \{-1, 1\}$ a solution $x(t)$ of the perturbed pendulum equation $\ddot{x} + \sin x = \mu \sin \omega t$ such that $(x(t), \dot{x}(t)) \in U$ which possesses infinitely many nondegenerate zeros (mod 2π) satisfying*

$$\sigma_k(x(t)) = s_k, \quad k \in \mathbb{Z}.$$

In addition, for every finite sequence $s_k \in \{-1, 1\}$ where $-N \le k \le M$ there exists a solution $x(t)$ possessing only finitely many nondegenerate zeros (mod 2π) and solving the equations $\sigma_k(x(t)) = s_k$ for $-N \le k \le M$. The same applies to half finite sequences s_k for $-\infty < k \le M$ or for $-N \le k < \infty$.

In short, one can prescribe any sequence of directions with which the pendulum should consecutively pass through the lowest point and there exists a solution doing precisely that.

Proof [Transversal heteroclinic point, shadowing lemma]. Assuming $\mu \neq 0$, we consider the diffeomorphism ψ of $\mathbb{R}^2$, defined by the time T flow map

$$\psi(z, \mu) := \psi_\mu(z) := \varphi^T(z, \mu),$$

at the time $T = 2\pi/\omega > 0$. In the case $\mu = 0$, the map ψ has the hyperbolic fixed point $P := P_{-1} = (-\pi, 0)$. We shall show that also the diffeomorphism ψ_μ has a *unique hyperbolic fixed point $P(\mu)$ near $P = P(0)$*, which depends differentiably on μ, if μ is small enough. For this, we define the mapping $F: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ by

$$F(z, \mu) = \varphi^T(z, \mu) - z.$$

If $\mu = 0$ then $F(P,0) = \phi^T(P,0) - P = 0$ and the partial derivative in the variable z is given by

$$D_1(P,0) = D_1\varphi^T(P,0) - \mathbb{1} \in \mathcal{L}(\mathbb{R}^2).$$

The linear mapping $D_1F(P,0)$ is an isomorphism, since the hyperbolic matrix $D_1\varphi^T(P,0)$ does not have an eigenvalue equal to 1. In a neighborhood of $\mu = 0$ there exists, by the implicit function theorem, a unique continuously differentiable function $\mu \mapsto P(\mu) \in \mathbb{R}^2$, solving $F(P(\mu),\mu) = 0$ and $P(0) = P$. In other words,

$$P(\mu) = \psi_\mu(P(\mu))$$

is a fixed point of the mapping ψ_μ. The eigenvalues of the derivative $d\psi_\mu(P(\mu))$ depend continuously on μ, hence the linear map $d\psi_\mu(P(\mu))$ possesses for small μ an eigenvalue whose absolute value is > 1 and an eigenvalue whose absolute value is < 1. Consequently, $P(\mu)$ is a hyperbolic fixed point of ψ_μ, if μ is small. Also the points $P(\mu) + 2n\pi$ are hyperbolic fixed points and $P(\mu) + 2n\pi = P_{-1}(\mu) + 2n\pi = P_{n-1}(\mu)$.

From the proof of Theorem II.8 (construction of h) we know that the *local invariant manifolds* issuing from the hyperbolic fixed point $P(\mu)$, denoted by

$$W^{\pm}_{\text{loc}}(P(\mu)),$$

depend differentiably on μ (by the implicit function theorem). For small μ they can, therefore, be represented locally as graphs over the invariant manifolds of the unperturbed system (the branches of the separatrix). If $t \mapsto \gamma(t)$ is a *heteroclinic solution* in the unperturbed case $\mu = 0$ having the x-coordinate at time $t = 0$ equal to $(\gamma(0))_1 = 0$, then $\gamma(t)$ lies on the separatrix. In formulas,

$$\frac{d}{dt}\gamma(t) = X(0,\gamma(t)), \quad t \in \mathbb{R},$$

and $\gamma(t) \to P = P_{-1}$ as $t \to -\infty$ and $\gamma(t) \to P + 2\pi e_1 = P_0$ as $t \to +\infty$.

Denoting by $n(\gamma(t))$ the unit normal vector of the homoclinic orbit γ in the point $\gamma(t)$ as depicted in Figure III.14, we can represent the relevant pieces of the invariant manifolds as follows:

$$\{\gamma(r) + u^-(r,\mu)\cdot n(\gamma(r)) \mid -\infty < r \le M\} \subset W_-(P_{-1}(\mu))$$

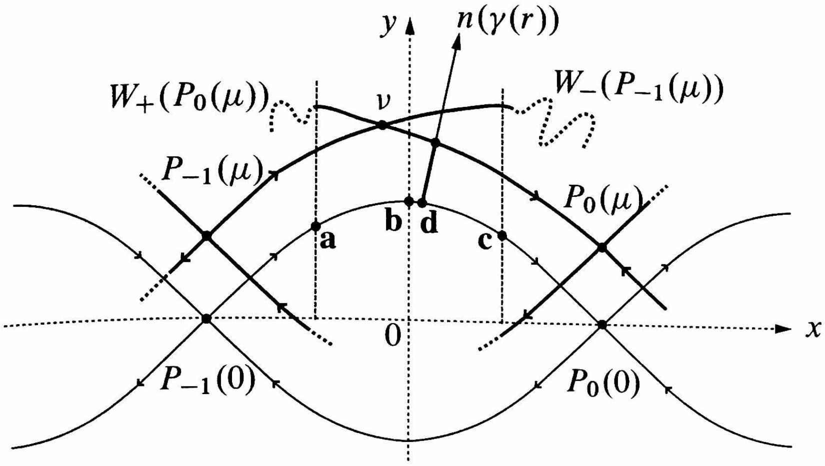
and

$$\{\gamma(r) + u^+(r,\mu)\cdot n(\gamma(r)) \mid -M \le r < \infty\} \subset W_+(P_0(\mu)),$$

with a sufficiently large constant $M > 0$ and where in the case $\mu = 0$ the functions $u^+(r,0) = u^-(r,0) = 0$ vanish.

If for a parameter value $r \in \mathbb{R}$,

$$u^-(r,\mu) = u^+(r,\mu) \quad \text{and} \quad \frac{\partial}{\partial r}u^-(r,\mu) \neq \frac{\partial}{\partial r}u^+(r,\mu),$$



Legend: **a**: $\gamma(-M)$ **b**: $\gamma(0)$ **c**: $\gamma(M)$ **d**: $\gamma(r)$

Figure III.13. The perturbed invariant manifolds possessing a transversal intersection.

then we have found the *transversal* intersection point

$$\nu := \gamma(r) + u^-(r,\mu)\cdot n(\gamma(r)) \in W_-(P_{-1}(\mu)) \cap W_+(P_0(\mu)).$$

In order to study the first-order term in μ of the function $(u^- - u^+)$ we introduce the so-called Melnikov function

$$d(r) := \frac{\partial}{\partial\mu}(u^- - u^+)|_{\mu=0}(r).$$

If $d(r_0) = 0$ and $\frac{d}{dr}d(r_0) \neq 0$, then there exists a transversal intersection point near $\gamma(r_0)$, for small $\mu \neq 0$. This follows from

$$(u^- - u^+)(r,\mu) = \mu\big(d(r) + O(\mu)\big)$$

in view of the implicit function theorem. The first approximation $d(r)$ can be explicitly calculated by means of the following *Melnikov formula.*

Theorem III.20 (Melnikov). *Let*

$$\dot{z} = f(z) + \mu g(t,z) \in \mathbb{R}^2,$$

$z \in \mathbb{R}^2$ *be a smooth vector field satisfying* div $f = 0$, *where* g *is a* T*-periodic vector field for some* $T > 0$, *so that* $g(t+T,z) = g(t,z)$. *We assume that for* $\mu = 0$ *there exists a homoclinic (resp. heteroclinic) orbit* γ *of the vector field* f, *hence satisfying* $\frac{d}{dt}\gamma(t) = f(\gamma(t))$ *for all* $t \in \mathbb{R}$ *and*

$$\gamma(t) \to P, \quad t \to -\infty,$$
$$\gamma(t) \to Q, \quad t \to +\infty$$

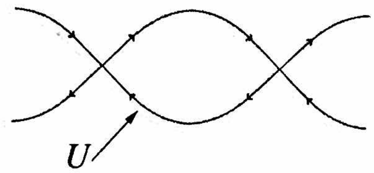


Figure III.14. A neighborhood U of the separatrix.

for two hyperbolic fixed points P and Q of f. Then, setting $f = (f_1, f_2)$ and $g = (g_1, g_2)$ the following formula holds true:

$$d(r) = \frac{1}{|f(\gamma(r))|} \int_{-\infty}^{\infty} (f_1 g_2 - f_2 g_1)(s, \gamma(r+s))\, ds.$$

For a *proof* we refer to C. Robinson in [91, S. 304]. In order to apply the formula to our pendulum, we consider the upper branch of the unperturbed separatrix,

$$\begin{cases} -\pi \le x \le \pi, \\ y = +\sqrt{2(1+\cos x)} = 2\cos(\frac{x}{2}) \end{cases}$$

where $y = \dot{x}$. The solution of the equation $\dot{x} = 2\cos(\frac{x}{2})$ is given by

$$x(t) = 2\arcsin(\tanh(t)), \quad t \in \mathbb{R},$$

and differentiating we obtain

$$\dot{x}(t) = y(t) = \frac{2}{\cosh(t)}.$$

Hence $\gamma(t) = (x(t), y(t))$ is the heteroclinic orbit. Inserting the curve γ into the Melnikov formula results in

$$d(r) = \frac{1}{|X(\gamma(r))|} \frac{2\pi \sin(\omega r)}{\cosh(\frac{\omega\pi}{2})}.$$

The function $d(r)$ has the *nondegenerate* zeros $r = \frac{\pi}{\omega} j$ for all $j \in \mathbb{Z}$. Therefore, there exists a transversal heteroclinic point ν. In the same way, there exists near the lower branch of the separatrix a transversal heteroclinic point η. The closure of the heteroclinic orbits is the *hyperbolic* set

$$\Lambda = \bigcup_{j,k \in \mathbb{Z}} \psi^j(\nu + 2\pi k e_1) \cup \{P_k(\mu)\} \cup \psi^j(\eta + 2\pi k e_1).$$

In order to finish the proof of Theorem III.19 we choose a neighborhood U of the separatrix and we choose the parameters ε, δ as in the shadowing lemma. For the given sequence $s = (s_k)_{k\in\mathbb{Z}}$ we construct the following ε-pseudo orbit q, described by Figure III.15. If $s_0 = 1$, we start in the heteroclinic point $q_0 = \nu$ and if $s_0 = -1$ we start in the heteroclinic point $q_0 = \eta$. Then we follow the heteroclinic orbit $\psi^j(\nu)$, resp. $\psi^j(\eta)$ into the $(\varepsilon/2)$-neighborhood of the next hyperbolic fixed point. There one has again two possibilities. If $s_1 = 1$ we jump onto the heteroclinic orbit of the upper branch to the *right* while if $s_1 = -1$ we jump onto the heteroclinic orbit of the lower branch to the *left*, and so on.

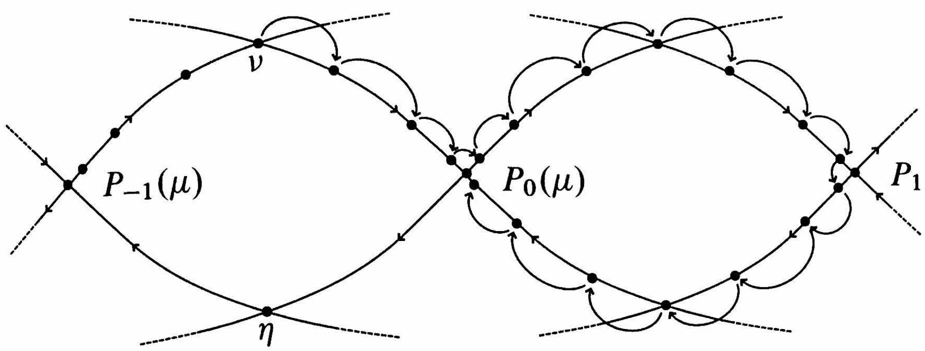


Figure III.15. The ε-pseudo orbit associated with the sequence $(s_k)=(\dots,s_0,s_1,s_2,\dots) = (\dots,1,1,-1,\dots)$.

The associated δ-shadowing orbit $p = (p_j)_{j\in\mathbb{Z}}$ guaranteed by the shadowing lemma,

$$p_j = \psi^j(p_0) = \varphi^{jT}(p_0,\mu), \quad j \in \mathbb{Z},$$

lies on the desired *solution* $t \mapsto \varphi^t(p_0,\mu)$ of the perturbed vector field $X(t,\mu,z)$ starting at the point $\varphi^0(p_0,\mu) = p_0$ at the time $t = 0$ and remaining in the open neighborhood U of the separatrix. Of course, this solution loses a lot of time near the hyperbolic equilibrium points, away from these neighborhoods it moves quite fast. We point out that all the solutions found this way start in a small neighborhood of the homoclinic points ν resp. η!

The passages near the transversal heteroclinic points ν resp. η correspond to the passages of the pendulum through the point $x = 0 \mod 2\pi$, which is the lowest position of the pendulum. In order to obtain a solution defined by a finite sequence (s_k), one constructs an ε-pseudo orbit q as before which, however, at the ends is equal to the orbits of hyperbolic fixed points. Then, the corresponding solution of the pendulum equation makes finitely many swings back and forth and then remains almost immobile near the highest position of the pendulum! This completes the proof of Theorem III.19. □

For a detailed study of the chaotic behavior of the periodically perturbed pendulum we refer to U. Kirchgraber and D. Stoffer in [59].