2×2 Linear Systems

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Introduction

Let us consider a first-order system of linear differential equations given by $X' = \mathbf{A}X$ where $\mathbf{A} \in \text{Mat}_{2\times 2}(\mathbb{R})$. The *phase portrait* associated to the system is a representative set of solutions plotted as parametric curves on the Cartesian plane and very much so like the slope field of a first-order differential equation, the phase portrait is a graphic tool used to visualize the long-term behavior of trajectories once an initial condition has been provided. To draw out the phase portrait take points $X \in \mathbb{R}^2$ and compute X' via $\mathbf{A}X$ in order to draw a vector at each point in the plane from which the general trajectories can be traced out for varying initial conditions.

Now in order to define a classification of such systems we aim to determine all of the possible phase portraits where the most natural place to start is the calculation of the *equilibrium points*, the points at which we have $X' = 0$. This corresponds to determining all solutions to the system $\mathbf{A}X = 0$, which can be separated into cases depending upon the structure of $\mathbf{A}$. More specifically, if $\det(\mathbf{A}) \neq 0$ then we know that the only solution is the zero vector, otherwise there are infinitely many solutions. Thus, for the purpose of this presentation we will from now on assume that $\det(\mathbf{A}) \neq 0$ so as to worry only about a single equilibrium point.

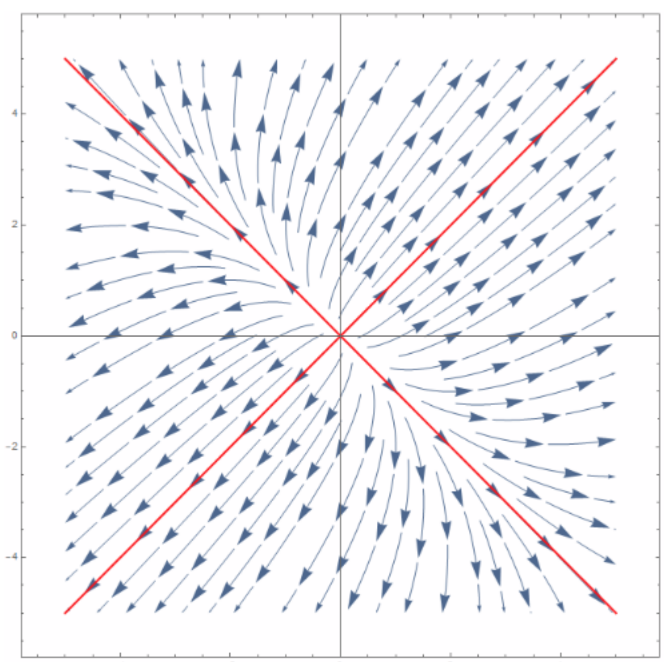
As we are going to be only concerned with $\det(\mathbf{A}) \neq 0$ we have to consider the different scenarios under which this can occur. By definition $\det(\mathbf{A}) = \lambda_1\lambda_2$ where λ_1 and λ_2 are the two eigenvalues of $\mathbf{A}$, allowing us to break down the possibilities based on the signs of the eigenvalues.

Classification

Let V_1 and V_2 represent the corresponding eigenvectors for λ_1 and λ_2 respectively. Now in general we have that for arbitrary $X = \xi_1 V_1 + \xi_2 V_2$ where $\xi_1, \xi_2 \in \mathbb{R}$ the following $\mathbf{A}X = \mathbf{A}(\xi_1 V_1 + \xi_2 V_2) = \xi_1 \mathbf{A} V_1 + \xi_2 \mathbf{A} V_2 = \xi_1 \lambda_1 V_1 + \xi_2 \lambda_2 V_2$.

Unstable Node

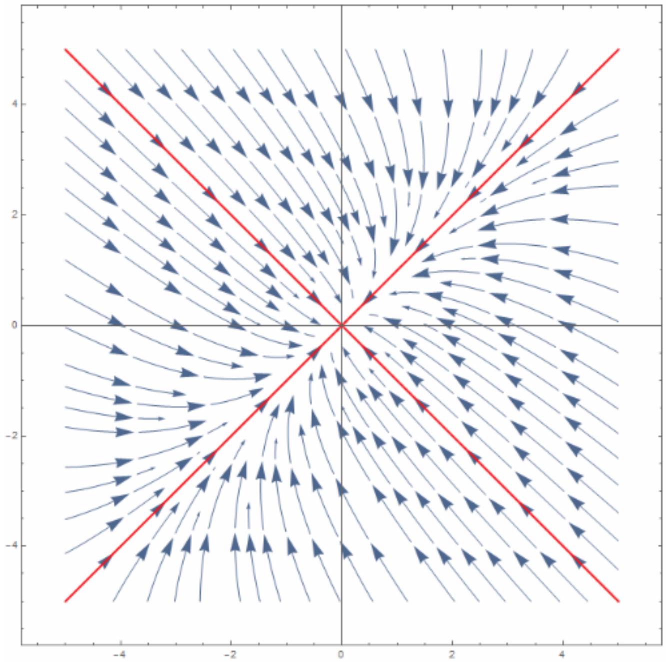
The above shows that if $\lambda_1, \lambda_2 > 0$, then the resulting vector given by $X' = \mathbf{A}X$ points in the same direction as the vector representing X. It is important to note that eigenvectors point away from the origin, thereby showing that each time the matrix is applied onto a point, the vector associated to the point is always pointing away from the origin, i.e.:



Such an equilibrium point is classified as an *unstable node*.

Asymptotically Stable Node

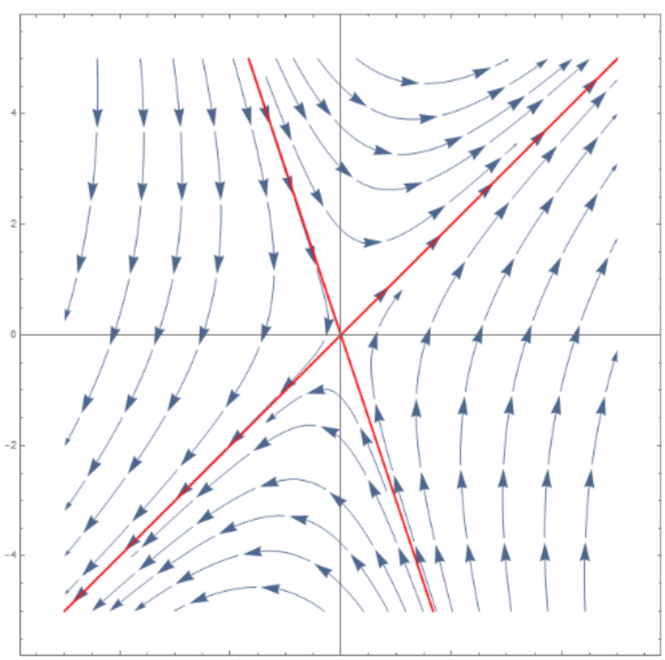
Similarly, for $\lambda_1, \lambda_2 < 0$ the resulting vector given by $X' = \mathbf{A}X$ points in the opposite direction as the vector representing X. Thus, opposite of the situation above, each time the matrix is applied onto a point, the resulting vector is always pointing towards the origin, i.e.:



Such an equilibrium point is classified as an *asymptotically stable node*.

Saddle Point

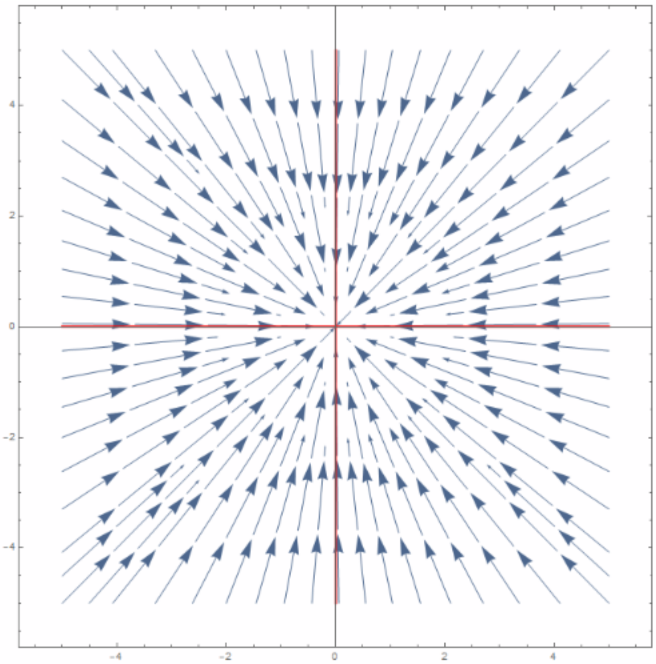
Now in the condition that $\det(\mathbf{A}) < 0$ we may assume without loss of generality that $\lambda_1 > 0$ and $\lambda_2 < 0$. Whenever this occurs we have that vectors lying along the direction of V_1 are pointing away from the origin, while any vector lying along the direction of V_2 points towards the origin, i.e.:



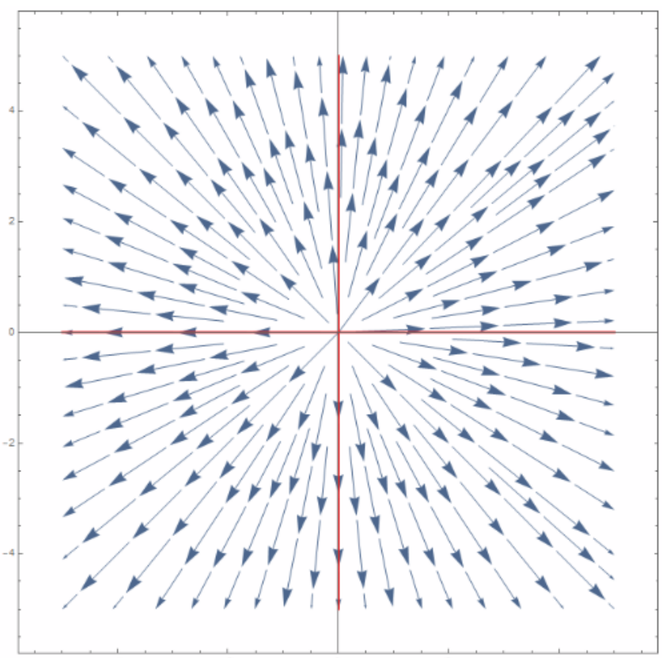
Such an equilibrium point is classified as a *saddle point* and is always *unstable*.

Proper Nodes

In the condition that $\lambda_1 = \lambda_2$ with geometric multiplicity two, we call the equilibrium point an *asymptotically stable proper node* if $\lambda_1 < 0$:

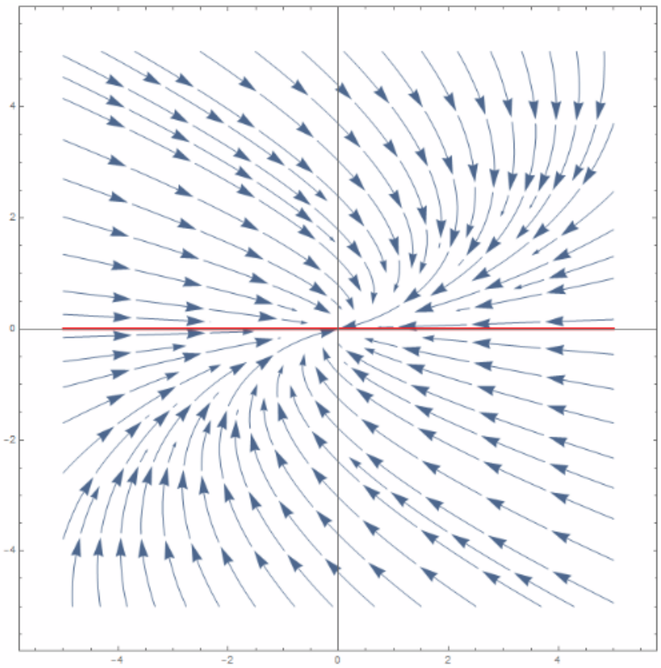


or an *unstable proper node* if $\lambda_1 > 0$:

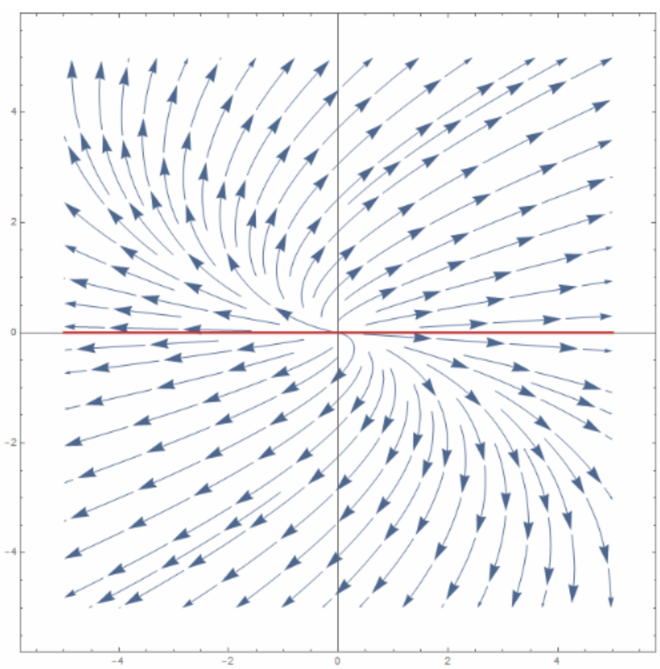


Improper Nodes

Unfortunately, the situation may not always be as nice as above, i.e. the matrix may not be diagonalizable. In the case that the geometric multiplicity is one we have an *asymptotically stable improper node* if $\lambda_1 < 0$:

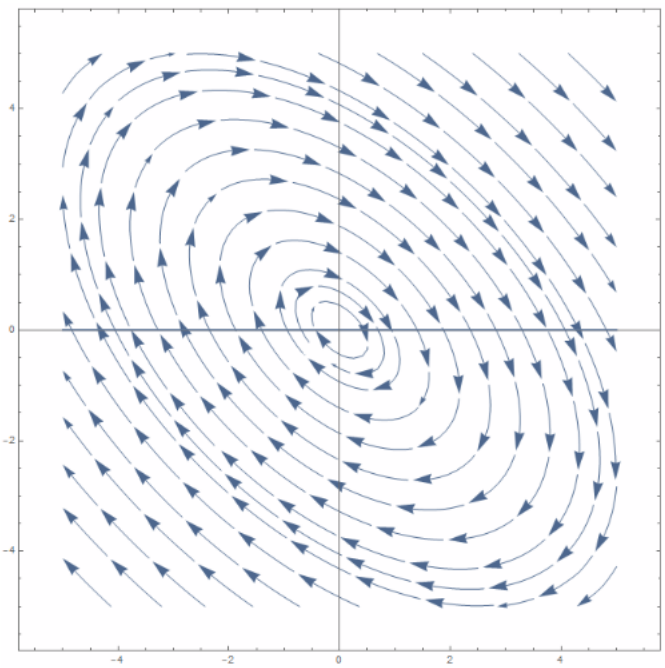


or an *unstable improper node* if $\lambda_1 > 0$:

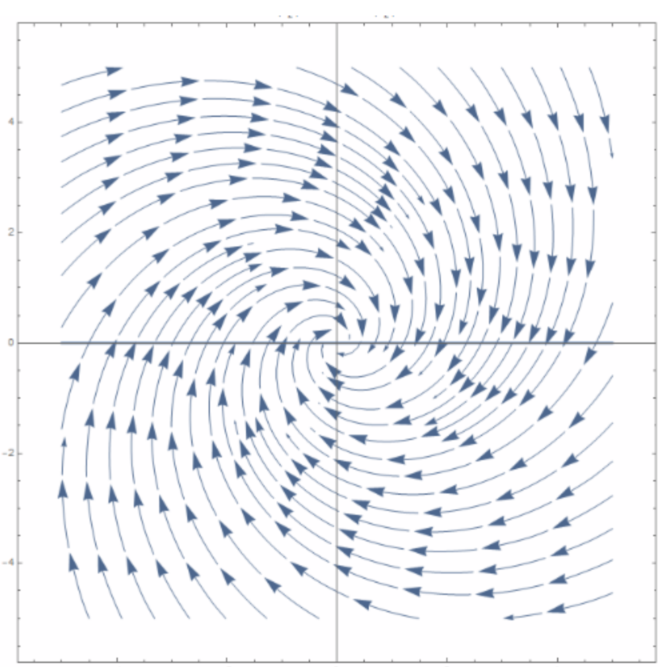


Spirals

Now the last three scenarios correspond to when $\lambda_{1,2} = \alpha \pm \beta i$ for $\alpha, \beta \in \mathbb{R}$. Knowing what the general solution will look like for such a system we can say with certainty that the magnitude of a vector is only going to be determined by the real part α. In the case that $\alpha = 0$ we obtain what is known as a *neutrally stable center*:



where the vectors rotate around the equilibrium point for all time. If this is not the case, then either $\alpha < 0$ giving an *asymptotically stable spiral point*:



or an *unstable spiral point* if $\alpha < 0$: