Connection form and Christoffel Symbols

To show:

\[ \Gamma^i_k = \omega^i_k \]

where the \( \Gamma^i_k \) are the Christoffel symbols defined by

\[ \nabla_X e_i = \Gamma^i_k(X)e_k \]

and where \( \omega^i_j = -\omega^i_j \) are the connection one-forms defined by

\[ d\theta^i = \omega^i_k \land \theta^k \]

with \( e_i \) an orthonormal frame and \( \theta^i \) the corresponding dual coframe, so that

\[ \theta^i(X) = \langle e_i, X \rangle \]

where \( \langle \cdot, \cdot \rangle \) is the Riemannian metric, and \( \nabla \) its Levi-Civita connection.

The proof is based on the “S\( _3 \)-lemma”

**lemma.** Any 3-tensor \( \gamma^i_{jk} \) which is antisymmetric in two indices and symmetric in two other indices is the zero-tensor: \( \gamma^i_{jk} = 0 \).

Metric compatibility implies that \( \Gamma^i_j = -\Gamma^i_j \). Indeed, \( \Gamma^i_j(X) = \langle e_i, \nabla_X e_j \rangle = -\langle \nabla_X e_j, e_i \rangle = -\Gamma^i_j(X) \).

Torsion freeness asserts that

\[ \nabla_{e_i} e_j - \nabla_{e_j} e_i - [e_i, e_j] = 0 \]

Applying \( \theta^k \) to this vector equation yields

\[ \Gamma^k_j(e_i) - \Gamma^k_i(e_j) - \theta^k([e_i, e_j]). \]

Now

\[ d\theta^k(e_i, e_j) = e_i[\theta^k(e_j)] - e_j[\theta^k(e_i)] - \theta^k([e_i, e_j]) = -\theta^k([e_i, e_j]) \]

and the structure equations \( d\theta^k = \omega^k_s \land \theta^s \), upon application to the pair \( (e_i, e_j) \) yields

\[ d\theta^k(e_i, e_j) = \omega^k_j(e_i) - \omega^k_i(e_j). \]

Substituting, we find that

\[ \Gamma^k_j(e_i) - \Gamma^k_i(e_j) + \omega^k_j(e_i) - \omega^k_i(e_j) = 0 \]

View this equation as one in which the \( \Gamma^k_j(e_i) \) are given and the \( \omega^k_j(e_i) \) are unknowns. A particular solution is

\[ \omega^k_j(e_i) = \Gamma^k_j(e_i). \]

Any other solution differs from this one by a 3-tensor \( \gamma^k_{ji} \) which must satisfy \( \gamma^k_{ji} - \gamma^k_{ij} = 0 \), i.e. which is symmetric in \( ij \). But the \( \gamma^k_{ji} \) must be skew symmetric in \( k \) and \( j \), since both \( \omega^k_j \) and \( \Gamma^k_j \) are skew symmetric in these indices. By the \( S_3 \)-lemma, \( \gamma^k_{ji} = 0 \) and so our particular solution is the only solution.

Curvature. To show \( \Omega^l_j = d\omega^l_j - \Sigma_k \omega^l_k \land \omega^k_j \) encodes the curvature as a skew-symmetric matrix valued two-form, \( \Omega^l_j \), related in a simple way to the usual curvature tensor of the Levi-Civita connection viewed as an affine connection.

We have

\[ d\nabla e = \omega e \]

meaning the following: For a given orthonormal frame \( e = \{ e_i \}_{i=1, \ldots, n} \), the quantity \( d\nabla e \) is the collection of tangent-bundle valued one-forms which maps \( V \in T_x M \) to \( \{ \nabla_X e_i \} \). The right hand side it the collection \( V \mapsto \{ \Sigma_i \omega^i_l(V)e_j \}_{i=1, \ldots, n} \). Then the curvature is given by

\[ d\nabla d\nabla e = \Omega e \]

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where the left hand side takes $X, Y$ to \{${R(X, Y)(e_i)}\}$ while the right hand side yields \{${\sum_j \Omega^j_i(X, Y)e_j}$\} the one-form with values in the tangent bundle.

But

$$d\nabla^2 d\nabla e = d\nabla(\o e)$$

$$= (d\o)e - \o e$$

$$= (d\o - \o)e).$$

from whence we get that $\Omega = d\o - \o$, the desired result, when $\o$ is appropriately interpreted using a matrix wedge product of matrix valued one-forms.