MATH 209 SUZANA MILEA

FINAL EXAM

Background. Let ω be a symplectic form and H a smooth function on a smooth even dimensional manifold P.
Hamilton's equations are the ODE's for X_H where X_H is called the "Hamiltonian vector field" for H and is defined by

$$\boxed{i_{X_H}\omega = dH}$$

Canonical coordinates, also known as Darboux coordinates, are coordinates $(q^1,\dots,q^n, p_1,\dots,p_n)$ such that $\boxed{\omega = \sum dq^i \wedge dp_i}$.

(1) Work out expressions for Hamilton's equations in canonical coordinates: $\begin{cases} \dot{q}^i = ? \\ \dot{p}_i = ? \end{cases}$

Fill in the question marks in terms of H.

Sol.

$$i_{X_H}\omega = i_{X_H}\Big(\sum_{i=1}^{n} dq^i \wedge dp_i\Big)$$

$$i_{X_H}\omega = \sum_{i=1}^{n} i_{X_H}(dq^i\wedge dp_i)$$

$$i_{X_H}\omega = \sum_{i=1}^{n} i_{X_H}(dq^i)\wedge dp_i - \sum_{i=1}^{n} dq^i \wedge i_{X_H}(dp_i)$$

$$\boxed{i_{X_H}\omega = \sum_{i=1}^{n}(a_i\, dp_i - b_i\, dq^i)}$$

since
- $i_{X_H}(dq^i) = dq^i(X_H)$

$$= X_H(q^i). \quad \Big(\text{Write } X_H = \sum\Big(a_i\frac{\partial}{\partial q^i} + b^i\frac{\partial}{\partial p_i}\Big).\Big)$$

$$= \sum\Big(a_i\frac{\partial q^i}{\partial q^i} + b^i\underbrace{\frac{\partial q^i}{\partial p_i}}_{=0}\Big) = a_i$$

- $i_{X_H}(dp_i) = dp_i(X_H)$

$$= X_H(p_i)$$

$$= \sum\Big(a_i\underbrace{\frac{\partial p_i}{\partial q^i}}_{=0} + b^i\frac{\partial p_i}{\partial p_i}\Big) = b_i$$

Since $i_{X_H}\omega = \sum_{i=1}^{n}(a_i\,dp_i - b_i\,dq^i)$

and $dH = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q^i}dq^i + \frac{\partial H}{\partial p_i}dp_i\right)$

we have that $\begin{cases} a_i = \frac{\partial H}{\partial p_i} \\ b_i = -\frac{\partial H}{\partial q^i} \end{cases} \Rightarrow \begin{cases} \dot{q}^i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q^i} \end{cases}$.

are Hamilton's equations.

② Write Φ_t for the flow of X_H. Prove (a) $\Phi_t^*\omega = \omega$.

(b) $\Phi_t^* H = H$.

Sol. (a) Using $\boxed{\frac{d}{dt}\Phi_t^*\omega = \Phi_t^* L_X\omega}$ and $\boxed{L_X\omega = i_X d\omega + d i_X\omega}$

we have:

$$\frac{d}{dt}\Phi_t^*\omega = \Phi_t^* L_{X_H}\omega = \Phi_t^*(\underbrace{i_{X_H}d\omega}_{=0} + d\,\underbrace{i_{X_H}\omega}_{=dH})$$

$$= \Phi_t^*(0 + \underbrace{d(dH)}_{=0 \text{ since } d^2=0}) = 0.$$

$\Rightarrow$ $\left.\begin{array}{l}\Phi_t^*\omega \text{ is constant with respect to } t \\ \Phi_0 = Id.\end{array}\right\} \Rightarrow \Phi_t^*\omega = \omega.$

(b) $$\frac{d}{dt}\Phi_t^* H = \Phi_t^* L_{X_H}H$$

$$= \Phi_t^*(\underbrace{i_{X_H}dH}_{=i_{X_H}\omega} + \underbrace{d\,i_{X_H}H}_{=0 \text{ since } i_X f = 0})$$

$$= \Phi_t^*\, i_{X_H}(i_{X_H}\omega)$$

$$= 0 \quad \text{since } i_{X_H}\circ i_{X_H} = 0.$$

$\Rightarrow \Phi_t^* H = H.$

7 [Hamilton-Jacobi Theory] Continuing in the framework of Riemannian geometry from problem 6, suppose that S is a smooth function defined on some open subset U of Q such that

$$\boxed{H(q, dS(q)) = \frac{1}{2}}$$

(A) Show that $\|\nabla S(q)\| = 1$ on U. Recall that $\langle \nabla S(q), v\rangle = dS(q)(v)$.

Sol. Write $\nabla S = \sum_{i=1}^{n} a^i \frac{\partial}{\partial q^i}$. Note: $dS(q) = \sum_{i=1}^{n} \frac{\partial S}{\partial q^i} dq^i$

- $\langle \nabla S(q), \frac{\partial}{\partial q^j}\rangle = dS(q)\left(\frac{\partial}{\partial q^j}\right)$

$$= \left(\sum_{i=1}^{n} \frac{\partial S}{\partial q^j} dq^i\right)\left(\frac{\partial}{\partial q^j}\right) = \frac{\partial S}{\partial q^j}$$

- $\langle \nabla S(q), \frac{\partial}{\partial q^j}\rangle = \langle \sum_{i=1}^{n} a^i \frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\rangle = \sum a_i \langle \frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\rangle$

$$= \sum a^i g_{ij}$$

$$\Rightarrow \sum a^i g_{ij} = \frac{\partial S}{\partial q^j} \Rightarrow \boxed{a^i = \sum_{j=1}^{n} g^{ij} \frac{\partial S}{\partial q^j}}$$

$$\Rightarrow \boxed{\nabla S(q) = \sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij} \frac{\partial S}{\partial q^j}\frac{\partial}{\partial q^i}.} = \sum_{i,j} g^{ij}\frac{\partial S}{\partial q^j}\frac{\partial}{\partial q^i}$$

- Since $H(q,p) = \frac{1}{2}\sum g^{ij}(q) p_i p_j$ we have that

$$\frac{1}{2} = H(q, dS(q)) = \frac{1}{2}\sum g^{ij}(q) dS_i dS_j \Rightarrow \boxed{\sum g^{ij} dS_i dS_j = 1}$$

- $\|\nabla S(q)\|^2 = \langle \nabla S(q), \nabla S(q)\rangle$

We want to show that $\langle \nabla S(q), \nabla S(q)\rangle = \sum g^{ij} dS_i dS_j$.

$\langle \nabla S(q), \nabla S(q)\rangle = dS(q)(\nabla S(q))$ since $\langle \nabla S(q), v\rangle = dS(q)(v)$.

$$= dS(q)\left(\sum g^{ij}\frac{\partial S}{\partial q^j}\frac{\partial}{\partial q^i}\right)$$

$$= \sum g^{ij}\frac{\partial S}{\partial q^j}\, \underbrace{dS(q)\frac{\partial}{\partial q^i}}$$

$$= \sum g^{ij}\frac{\partial S}{\partial q^j}\frac{\partial S}{\partial q^i} = \sum g^{ij} dS_i dS_j = 1$$

$$\Uparrow$$

$\|\nabla S(q)\| = 1.$

(B) Use Cauchy Schwartz to show that

$$\boxed{dS(\sigma)(\dot{\sigma}) \leq \|\dot{\sigma}\|}$$

holds along any curve σ in U.

Sol. We know that $dS(q)(v) = \langle \nabla S(q), v \rangle$.

$$\Rightarrow dS(\sigma)(\dot{\sigma}) = \langle \nabla S(\sigma), \dot{\sigma} \rangle$$

$$\leq \underbrace{\|\nabla S(\sigma)\|}_{\substack{\| \\ 1 \text{ by part (A)}}} \cdot \|\dot{\sigma}\| = \|\dot{\sigma}\|.$$

(c) Prove that $\boxed{S(\sigma(b)) - S(\sigma(a)) \leq \ell(\sigma)}$ for any curve σ in U.

Sol. Integrating $dS(\sigma)(\dot{\sigma}) \leq \|\dot{\sigma}\|$

we have $\int_a^b dS(\sigma(t))(\dot{\sigma}(t))dt \leq \underbrace{\int_a^b \|\dot{\sigma}(t)\| dt}_{\ell(\sigma)}$

$$\begin{cases} u = \sigma(t) \\ du = \dot{\sigma}(t)\,dt \end{cases}$$

$$\int_{\sigma(a)}^{\sigma(b)} dS(u)\,du \leq \ell(\sigma)$$

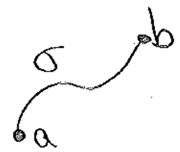
$$S(\sigma(b)) - S(\sigma(a)) \leq \ell(\sigma).$$

(D) Put together these steps to show that the integral curves of ∇S in U are geodesics.

Sol. σ is an integral curve $\Rightarrow \dot{\sigma} = \nabla S(\sigma)$

$$dS(\sigma)(\dot{\sigma}) = \langle \nabla S(\sigma), \dot{\sigma} \rangle$$
$$= \langle \dot{\sigma}, \dot{\sigma} \rangle$$
$$= \|\dot{\sigma}\|^2$$
$$= \|\dot{\sigma}\| \quad \text{since } \|\dot{\sigma}\| = \|\nabla S(\sigma)\| = 1.$$

Integrate $\|\dot{\sigma}\| = dS(\sigma)(\dot{\sigma})$



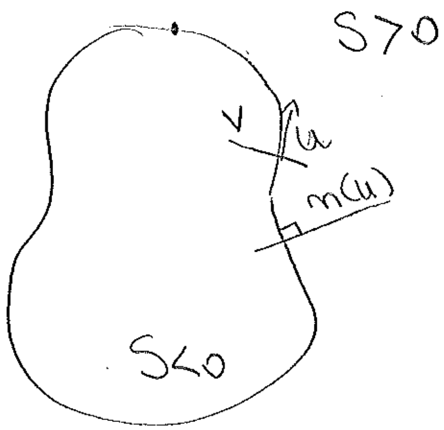
$$\ell(\sigma) = \int_a^b \|\dot{\sigma}\| dt = \int_a^b dS(\sigma(t))(\dot{\sigma}(t))dt$$
$$= S(\sigma(b)) - S(\sigma(a)).$$
$$= S(\alpha(d)) - S(\alpha(c)) \leq \ell(\alpha) \text{ by (C).}$$

where α is any other curve passing through the points $\alpha(c) = \sigma(a)$ and $\alpha(d) = \sigma(b)$

$\Rightarrow \sigma$ is a unit speed geodesic.

(E)

Sol.



Let $F(u,v) = \sigma(u) + v\,m(u)$

Define $\boxed{S = v}$.

You need to know the metric
in the u, v coordinates. It is not `standard'!
See my solutions.

• S satisfies the Hamilton-Jacobi equations $\frac{1}{2}|ds| = \frac{1}{2}$ is equivalent to showing that $\|\nabla S\| = 1$ (by part (A)).

③

(B) $P = T^*Q \xrightarrow{\pi} Q$

$(q,\alpha) \longmapsto q$ where $\alpha \in T_q^*Q$

$$T_{(q,\alpha)}(T^*Q) \xrightarrow{d\pi(q,\alpha)} T_qQ$$

$$T^*_{(q,\alpha)}(T^*Q) \xleftarrow{(d\pi_{(q,\alpha)})^*} T_q^*Q$$

$$\theta(q,\alpha) = (d\pi_{(q,\alpha)})^*\alpha = \alpha \circ d\pi_{(q,\alpha)}$$

$$\Leftrightarrow \theta(q,\alpha)(V) = \alpha(d\pi_{(q,\alpha)}V) \quad \forall V \in T_{(q,\alpha)}(T^*Q).$$

⑤

(A) Work $W = \int_\gamma F$.

It is more natural to think of "Force" as a one-form instead of vector fields because we are integrating 1-forms.