The 1 dimensional version of inverse stereo projection

$$\mathbb{R} \to S^1 \subset \mathbb{R}^2$$

is

$$u \mapsto (\frac{2u}{u^2+1}, \frac{u^2-1}{u^2+1})$$

We can make this map homogeneous by making the substitution:

$$u = m/\ell.$$

Multiply the top and bottom of the right hand side by ℓ^2 to get the map in the form

$$[m : \ell] \mapsto (\frac{2m\ell}{m^2+\ell^2}, \frac{m^2-\ell^2}{m^2+\ell^2}) := (x, y) \tag{1}$$

Now these components satisfy $x^2 + y^2 = 1$: they lie on the unit circle. Multiply both sides of this equation by the denominator $c = m^2 + \ell^2$ to get

$$a^2 + b^2 = c^2$$

where we have set:

$$a = 2m\ell, b = m^2 - \ell^2, c = m^2 + \ell^2.$$

We have shown that whenever we express a, b, c this way in terms of m, ℓ we have, automatically: that $a^2 + b^2 = c^2$ and so we have made a Pythagorean triple. This logic can be inverted. By letting m, ℓ run over pairs of relatively prime positive integers with $\ell < m$ we get all Pythagorean triples.

Relation to projective line.

In the notes 'Four incarnations of the sphere' we talk about the complex projective line $\mathbb{CP}^1$ and give a rational algebraic formula for its stereographic representation, which is an explicit isomorphism $\mathbb{CP}^1 \to S^2$. If we take the homogeneous coordinates $[z_0 : z_1]$ in that formula we get the real projective line as a subset of the complex projective line: $\mathbb{RP}^1 \subset \mathbb{CP}^1$. Restricting the stereographic representation to this $\mathbb{RP}^1$ yields as image a great circle in S^2. (Referring to the formulae there, $Im(z_0\bar{z}_1) = 0$ and so the image makes up the great circle $x_2 = 0$.) Standard Cartesian coordinates $(x_1, x_3) = (x, y)$ on that great circle yield the above map (1). Finally we have $\mathbb{ZP}^1 = \mathbb{QP}^1 \subset \mathbb{RP}^1$ and this restriction is implemented by further restricting homogeneous coordinates to be integers: $[z_0 : z_1] = [m : \ell]$.