There are four principal incarnations of the sphere in mathematics, each contributing its own perspective and simplification of our overall understanding of the geometry of "a sphere". We review these incarnations, the interrelations between them and their principal uses.

## 1. The Incarnations.

I. **The standard sphere** $S^2$ is the solution set to the equation

$$x^2 + y^2 + x^2 = 1$$

in Euclidean 3-space with Cartesian coordinates $(x, y, z)$.

2. **The Riemann sphere** $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is formed by adding one point at infinity to the usual Euclidean plane. We identify that plane with $\mathbb{C}$ by sending $(x, y)$ to $x + iy$.

3. **The complex projective line** $\mathbb{CP}^1$ is the set whose points are complex lines through the origin in $\mathbb{C}^2$.

4. **The celestial sphere** is the set of light rays through the origin in space-time.

## 2. The Standard Sphere $S^2$.

The standard sphere has

*points:* vectors $\vec{P} = (x, y, z) \in \mathbb{R}^3$ of unit length: $\|\vec{P}\| = 1$.

*lines:* intersections of planes through the origin with the sphere. These 'lines' are also called "great circles" or "geodesics".

*distances:* the spherical distance between two points $\vec{P}, \vec{Q} \in S^2$ is the angle between the rays they define. Algebraically we can use the dot product and inverse cosine to compute this distance $d(\mathbf{P}, \mathbf{Q})$ as follows:

$$d(\vec{P}, \vec{Q}) = \theta \text{ where } \vec{P} \cdot \vec{Q} = \cos(\theta); 0 \leq \theta \leq \pi.$$

*angles:* the angle between two lines is the Euclidean angle between their two tangents at the point of intersection. That angle is also equal to the dihedral angle between the planes whose intersections with the sphere defines these lines.

*circles:* intersections of any plane, not necessarily thorugh the origin, with the sphere. If $\vec{N}$ is a unit normal to such a plane, then the (spherical) centers of the corresponding circle are either point $\pm\vec{N}$.

**Exercise.** If the plane is $Ax + By + Cz = D$ what is its center and radius in terms of the coefficients $A, B, C, D$? What conditions on the coefficients guarantees that the circle (intersection) is nonempty?

*element of arclength:* The integrand $ds = \sqrt{dx^2 + dy^2 + dz^2}$ restricted to the sphere is the "element of arclength" for the sphere. If $c$ is a curve lying on the sphere then $\int_c ds$ is the curve's length, which is the same length we get by thinking of the curve as a curve in $\mathbb{R}^3$. In spherical coordinates $ds^2 = d\phi^2 + \sin^2(\phi) d\theta^2$.

*element of area:* Integrate $\sin(\phi) d\phi d\theta$ over a spherical region to obtain that region's area.

*symmetries:* linear maps of $\mathbb{R}^3$ which map the sphere to itself.

The collection of all such symmetries form a *group* known as the orthogonal group and denoted by $O(3)$. When one speaks of "spherical symmetry" one is automatically speaking of this group. In matrix terms, it consists of all 3 by 3 matrices $A$ such that $AA^T = Id$. The set of such symmetries forms a group known as $O(3)$. The group $O(3)$ falls into two componets, those $A$ with $det(A) = 1$ and those with $det(A) = -1$. The first component forms a subgroup of $O(3)$ known as the *rotation group* and denoted by $SO(n)$. Any element $R$ of $SO(3)$ has an eigenvector $v$ with eigenvalue 1, and if $R \neq Id$ this eigenspace is one-dimensional. In physical terms $R$ is rotation about the axis spanned by $v$. In the plane orthogonal to this line $R$ acts by rotation by $\theta$ radians. The full list of eigenvalues of $A$ is $1, e^{i\theta}, e^{-i\theta}$.

Finite subgroups. We have a full list of finite subgroups of $SO(3)$. They correspond to the regular N-gons in the plane and the 5 platonic solids. The subgroups for the regular N-gons are the cyclic groups and dihedral groups and these form 2 infinite families (called $A$ and $D$). There are 3 for the platonic solids (associated to $E_6, E_7, E_8$) since a platonic solid and its dual have the same group. This list is ubiquitous in mathematics. In Lie group theory and singularity theory it is th "A-D-E simply laced Dynkin diagrams"

Exercise: Find the matrices which form the symmetry group of the standard cube, whose vertices are the eight points $(\pm 1, \pm 1, \pm 1)$.

Generalizations. The definiton of $O(n)$ and $SO(n)$ works in any dimension and these groups are basic to mathematics and physics.

Biology: Our brains and bodies are hardwired to do computations in the group $SE(3)$ of Euclidean motions. This group is comprised of two well-known subgroups: translations, denoted $\mathbb{R}^3$ and rotations $SO(3)$. The group $SE(3)$ is the group of isometries of space. So: you know $SO(3)$ way better than you might think you do.

## 3. Riemann Sphere

In the Reader, and in a lecture, we derive the stereographic projection map from the sphere minus a single point onto the Euclidean plane. In coordinates this map is

$$Stereo : (S^2 \setminus \{N\}) \mapsto \mathbb{C}; (x, y, z) \mapsto \frac{1}{1-z}(x + iy); N = (0, 0, 1).$$

If we add back in the missing point $N$ there is only one place for $N$ to go: a new point "$\infty$" infinitely far from the origin in any direction. The image under stereo projection of a tiny disc surrounding N is the *exterior* to a huge disc ($|z| > \frac{1}{\epsilon}$, $\epsilon$ tiny). In the language of point set topology, we have formed the one point compactification of the plane.

Going through our list, as before, the Riemann sphere has :

*points:* usual points in $\mathbb{C}$ or the point $\infty$. Notation: $z = x + iy$.

*circles:* usual circles in the planes, or usual lines. Lines pass through the point $\infty$ and are the only 'circles' which pass through infinity.

Exercise: show that passing through any three distinct points there is a unique circle.

*lines, or geodesics:* are the images of great circles under stereo projection. They are special kinds of circles. A Euclidean line is a geodesic on the Riemann sphere if and only if it passes through the origin.

*angles:* are measured exactly as in the Euclidean plane.

*element of arclength:* is $nds_E$ where $ds_E = |dz| = \sqrt{dx^2 + dy^2}$ is the standard Euclidean element of arclength and where $n = n(z) = \frac{2}{1+|z|^2}$ is a conformal factor (= 'index of refraction). One computes that with this measure, the length of a curve in the Riemann sphere equals the length of the curve on the standard sphere $S^2$ which it is the image of.

*element of area:* $n(x,y)^2 dxdy$ with $n$ as per "element of arclength".

*symmetries:* What are the symmetries of the Riemann sphere? Functions of the form:

$$z \mapsto \frac{az+b}{cz+d} := F(z)$$

with $a, b, c, d$ complex numbers such that $ad - bc \neq 0$. These are called *linear fractional* or *Möbius transformations*:

Exercise. With $F$ as above, what is $F(\infty)$? What is $F^{-1}(\infty)$?

In what sense are these maps symmetries? First, what do we mean by a symmetry? A symmetry is a map of a space to itself which preserves structure. Okay. Then the question becomes : "what structure on the Riemann sphere do the Möbius transformations preserve?"

**Theorem 1.** *If $F$ is a Möbius transformation then it is an invertible, orientation-preserving map of the Riemann sphere to itself which also*

- *(i) is holomorphic (see complex variables)*
- *(ii) preserves circles (maps circles to circles)*
- *(iii) is conformal (preserves angles between curves)*

*Conversely any invertible map of the Riemann sphere onto itself which preserves orientations and satisfies any one of (i) , (ii) or (iii) is a Möbius transformation.*

*reflections; anti-holomorphic maps.* Reflections about circles also map circles to circles. We might want to throw these in to our collection of "symmetries". They reverse orientation, and are anti-holomorphic. For example, reflection about the unit circle is the map $z \mapsto \frac{1}{\bar{z}}$.

**Theorem 2.** *Any invertible map from the Riemann sphere to itself which maps circles to circles is either a Möbius transformation (in which case it is orientation preserving) or the composition of a Möbius transformation with a reflection about a circle.*

We can use stereo projection to push forward But we might want our maps to be distance preserving. What distance do we use? Well the one described above, coming from the standard sphere. In this way we get

*rotations:* The rotations form a subgroup of the group of Möbius transformations as follows. Consider the maps of the form $z \mapsto Stereo \circ R \circ Stereo^{-1})(z)$ as $R$ varies over the rotation group $SO(3)$. These can all be written as Möbius transformations whose coefficients $a, b, c, d$ satisfy the constraints

$$: c = -\bar{b}, d = \bar{a}, |a|^2 + |b|^2 = 1$$

These preserve distances on the Riemann sphere, provided we use stereographic projection to carry over measurements on the standard sphere to measurements on the Riemann sphere. (Use the element of arclength for lengths.)

## 4. Complex Projective Line.

The complex projective line is probably the least familiar of our four incarnations of the sphere. Using this incarnation Möbius transformations become exceedingly simply.

Let us begin intuitively. A fraction is simply a ratio $[p:q] = p/q$ subject to the standard cancellation rule

$$[ap:aq] = [p:q], a \neq 0$$

In teaching kids fractions we begin with $p$ and $q$ being any positive integers and we let them write down $[0:q] = 0/q$ and tell them this equals $0/q = 0$, "because if you divide 0 into a bunch of parts you still have 0". But we do not let them write down $[p:0]$ since this looks like $p/0$. "Don't divide by 0" we command!. In making the projective line the only difference with usual fractions is that we allow and even encourage denominators $q = 0$. Our only rule is that $0/0$ is not allowed. We cannot write down $[0:0]$! $[1:0]$ is fine. According to our cancellation rule it equals $[a:0]$ for any $a \neq 0$. Formally we say that

$$1/0 = a/0 = -1/0 = \infty.$$

**A point of the complex projective line is a ratio of complex numbers, written $[z_0 : z_1]$ or more simply $[z_0, z_1]$. We allow $[1:0]$ as well as $[0:1]$ and there is real difference betweeen the two.**

*Formal Linear algebraic apparatus.* Let $\mathbb{C}^2$ be the complex vector space of dimension 2. A point of $\mathbb{C}^2$ is thus an ordered pair $(z_1, z_2)$ of complex numbers.

**Definition 1.** *A (complex) line in $\mathbb{C}^2$ is a one-dimensional complex subspace of $\mathbb{C}^2$. The set of all such lines is denoted $\mathbb{CP}^1$ and called the complex projective line. (The superscript "1" is for "one complex dimensional".)*

If $\ell \subset \mathbb{C}^2$ and $(z_0, z_1) \in \ell$ is a *nonzero* point on the line, then $\ell$ is the span of $(z_0, z_1)$: $\ell = span_{\mathbb{C}}(z_0, z_1) := \{(\lambda z_0, \lambda z_1) : \lambda \in \mathbb{C}\}$. Now $\ell$ is also a *point* in $\mathbb{CP}^1$ and when we want to think of it this way, we write

$$\ell = [z_0, z_1] := [z_0 : z_1] \in \mathbb{CP}^1$$

The expression $[z_0, z_1]$ are called "homogenous coordinates" on $\mathbb{CP}^1$. Thus

$$[z_0, z_1] = [\lambda z_0, \lambda z_1], \text{ for } \lambda \in \mathbb{C}, \lambda \neq 0$$

and conversely if $(w_0, w_1) \neq (0,0)$ we have:

$$[z_0, z_1] = [w_0, w_1] \text{ iff there is } \lambda \neq 0, \lambda \in \mathbb{C} : (w_0, w_1) = (\lambda z_0, \lambda z_1).$$

The expression $[0,0]$ is forbidden, being essentially like $0/0$.

Examples.

A. $[z_0, z_1] = [2z_0, 2z_1]$

B. $[1,0] = [i,0] = [\lambda, 0], \lambda \neq 0$

C. If $z_0, z_1 \neq 0$ then $[z_1, z_2] = [z_1/z_2, 1] = [1, z_2/z_1]$.

Now, suppose that $z_1 \neq 0$. Then $[z_1, z_2] = [z, 1]$ with $z = z_1/z_2$. There is only one point of $\mathbb{CP}^1$ with $z_1 = 0$, namely the point $[1,0]$. Thus, by throwing out the single point $P_\infty = [1,0]$ we map all of $\mathbb{CP}^1$ bijectively onto $\mathbb{C}$, by sending $[z_1, z_2] \mapsto z_1/z_2 := z$.

We have proved

**Lemma 1.** *$\mathbb{CP}^1 \setminus P_\infty \cong \mathbb{C}$ by sending $[z_1, z_2] \mapsto z_1/z_2$. Its inverse is the map $z \mapsto [z, 1]$*

**Definition 2.** *The complex coordinate $z$ is called the "affine coordinate" on $\mathbb{CP}^1$. And $[z_1, z_2]$ are homogeneous coordinates*

*Remark. What we have just described works with any field $\mathbb{K}$ in place of the complex field $\mathbb{C}$. The result is the projective line over that field, denoted $\mathbb{KP}^1$. As a set,$\mathbb{KP}^1 = \mathbb{K} \cup \{\infty\}$.*

Comparing the affine coordinate map of the lemma with stereo projection $S^2 \setminus \{N\} \to \mathbb{C}$ suggests that $S^2 \cong \mathbb{CP}^1$. Both $S^2$ and $\mathbb{CP}^1$ are abstractly isomorphic to the Riemann sphere $\mathbb{C} \cup \{\infty\}$. How can we build an isomorphism $\mathbb{CP}^2 \to S^2 \subset \mathbb{R}^3$? We will need three real-valued functions on $\mathbb{CP}^1$ $x_1, x_2, x_3$ with $x_1^2 + x_2^2 + x_3^2$. How do we write down even a single real function $f : \mathbb{CP}^1$? Such a function $f$ is a function on $\mathbb{C}^2 \setminus \{(0,0)\}$, i.e a function of $z_1, z_2$ (with $(z_1, z_2) \neq (0,0)$) which is invariant under complex scalar multiplication: $f(z_1, z_2) = f(\lambda z_1, \lambda z_1)$

The simplest non-constant functions on $\mathbb{CP}^1$ which we can write down are

$$\frac{|z_1|^2}{|z_1|^2 + |z_2|^2}, \frac{|z_2|^2}{|z_1|^2 + |z_2|^2}, \frac{z_1 \bar{z}_2}{|z_1|^2 + |z_2|^2}$$

Each is defined as long as $(z_1, z_2) \neq (0,0)$. The last one is complex-valued so is really two real-valued functions, its real and imaginary parts. Combine these four functions into a single vector function :

$$[z_1, z_2] \mapsto \frac{1}{|z_1|^2 + |z_2|^2}(2Re(z_1 \bar{z}_2), 2Im(z_1 \bar{z}_2), |z_1|^2 - |z_2|^2) := (x_1, x_2, x_3)$$

thus defining a map $\mathbb{CP}^1 \to \mathbb{R}^3$. Algebraically, it is more convenient to think of $\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}$ so that the map can be written as

$$(1) \qquad [z_1, z_2] \mapsto \frac{1}{|z_1|^2 + |z_2|^2}(2z_1 \bar{z}_2, |z_1|^2 - |z_2|^2) := (x_1 + ix_2, x_3) \in \mathbb{C} \oplus \mathbb{R}$$

Exercises. Verify that the components of this map satisfy $x_1^2 + x_2^2 + x_3^2 = 1$. Hint: check that $(|z_1|^2 - |z_2|^2)^2 + 4|z_1|^2|z_2|^2 = (|z_1|^2 + |z_2|^2)^2$.

Verify that this map takes $P_\infty$ to the north pole $(0,0,1)$.

Verify that when we express this map in terms of the affine coordinate $z$ that it becomes precisely the inverse of stereographic projection as worked out in the reader.

Verify that this map composed with stereo projection takes $[z_1, z_2]$ to $z_1/z_2$. Hint: WLOG set $z_2 = 1$.

These exercises establish the identification of $\mathbb{CP}^1$ with $S^2$.

Linear transformations take lines to lines so the (invertible) linear transformations of $\mathbb{C}^2$ act on $\mathbb{CP}^1$. Such a linear transformations is given by a two-by-two matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which acts on the vector $(z_1, z_2)$ by sending it to $(az_1 + bz_2, cz_2 + dz_2)$. Thus this matrix takes the point $[z_1, z_2] \in \mathbb{CP}^1$ to the point $[az_1 + bz_2, cz_2 + dz_2]$

Exercise. Verify that when we use the affine coordinate $z$ of the lemma then the transformation just written down has the effect

$$z \mapsto (az + b)/(cz + d).$$

Going through our list of geometric quantities:

*Points:* lines in $\mathbb{C}^2$. Notation: $[z_1, z_2]$, or $[z_1 : z_2]$.

*Symmetries:* invertible linear transformations of $\mathbb{C}^2$. Notation: as 2 by 2 matrices.

*Circles:* Define $\mathbb{RP}^1 \subset \mathbb{CP}^1$ to be the locus obtained by taking both homogeneous coordinates $z_1, z_2$ to be real. Declare this to be a “circle”. Verify that in terms of the map (1) this is the ‘xz’ equator: i.e the locus $x_2 = 0$. Declare a ‘circle’ in $\mathbb{CP}^1$ to be the sets which are the image of $\mathbb{RP}^1$ under any symmetry.

*distance, element of arclength, angles:* To describe these in $\mathbb{CP}^1$ -terms (rather than pushing them onto the standard sphere or Reimann sphere by one of our maps) requires Hermitian geometry on $\mathbb{C}^2$, which is to say the form $\langle (z_1, z_2), (w_1, w_2)\rangle = z_1\bar{w}_1 + z_2\bar{w}_2$ and some understanding of the geometry of the *three-sphere*: $S^3 \subset \mathbb{C}^2$ which is the locus of points $|z_1|^2 + |z_2|^2 = 1$.

Parameterizing Pythagorean triples.

The 1 dimensional version of inverse stereo projection

$$\mathbb{R} \to S^1 \subset \mathbb{R}^2$$

is

$$u \mapsto (\frac{2u}{u^2+1}, \frac{u^2-1}{u^2+1})$$

We can make this map homogeneous by making the substitution:

$$u = m/\ell.$$

Multiply the top and bottom of the right hand side by $\ell^2$ to get the map in the form

$$[m : \ell] \mapsto (\frac{2m\ell}{m^2+\ell^2}, \frac{m^2-\ell^2}{m^2+\ell^2}) := (x, y) \tag{2}$$

Now these components satisfy $x^2 + y^2 = 1$: they lie on the unit circle. Set

$$a = 2m\ell, b = m^2 - \ell^2, c = m^2 + \ell^2$$

We have shown that whenever we express $a, b, c$ this way in terms of $m, \ell$ we have, automatically: that

$$a^2 + b^2 = c^2$$

By letting $m, \ell$ run over positive integers with $\ell < m$ we get in this way all Pythagorean triples.

**Relation to projective line.**

In the notes ‘Four incarnations of the sphere’ we talk about the complex projective line $\mathbb{CP}^1$ and its “stereographic representation. See the explicity map $\mathbb{CP}^1 \to S^2$ there. If we take the homogeneous coordinates $[z_0 : z_1]$ to both be real we we define $\mathbb{RP}^1 \subset \mathbb{CP}^1$. If we restrict the ‘stereographic representation” to this $\mathbb{RP}^1$, then its image is a great circle since now $Im(z_0\bar{z}_1) = 0$. (In that coordinate representation that great circle is the great circle $x_2 = 0$.) And, in standard coordinates on that great circle this restricted stereographic representation becomes the above map (2). Finally we have $\mathbb{ZP}^1 = \mathbb{QP}^1$ and this is implemented by further restricting the homogeneous coordinates to be integers: $[z_0 : z_1] = [m : \ell]$.

## 5. Light Cone

Let $(x, y, z, t)$ be a point in space-time $\mathbb{R}^4$. Then a light ray is a line $x = v_1 t + x_0, y = v_2 t + y_0, z = v_3 t + z_0, t = t$ where the speed $v_1^2 + v_2^2 + v_3^2 = c^2$ is the speed of light. We set $c = 1$, or, if you prefer, introduce $x_4 = ct$. Then, the union of all light rays passing through the origin at time $t = 0$ forms the light cone [1] :

$$x^2 + y^2 + z^2 - t^2 = 0 \tag{3}$$

[1] Its future half $t > 0$ consists of all points, i.e events illuminated by light rays bursting forth from the origin. Its past half comprises the celestial sphere: the set of all stars whose light reaches our eye at the origin at time 0