Isometries are distance preserving maps. We will study them through the Poincare metric, which is the infinitesimal distance ds/y.

Our initial goal in studying isometries is to use them to verify that the geodesics are what we said they are: semicircles orthogonal to the ideal. Here is the strategy: we know already that the positive y axis is a geodesic. We will show that isometries take geodesics to geodesics, and, that given any semicircle orthogonal to the ideal, there is a hyerbolic isometry taking the positive y axis to this semi-cricle

We begin generally with Lagrangians. A transformation $F : x, y \mapsto (u, v) = (u(x, y), v(x, y)) := F(x, y)$ from the plane to the plane induces a (pointwise) linear transformation of the derivative variables $(\dot{x}, \dot{y}) \mapsto (\dot{u}, \dot{v})$ by

$$\dot{u} = \frac{\partial u}{\partial x}\dot{x} + \frac{\partial y}{\partial y}\dot{y}.$$

$$\dot{v} = \frac{\partial v}{\partial x}\dot{x} + \frac{\partial v}{\partial y}\dot{y}$$

In other words: $(\dot{u}, \dot{v}) = DF_{(x,y)}(\dot{x}, \dot{y})$ where $DF_{(x,y)}$ is the Jacobian matrix of the transformation F at the point (x, y). We say that this transformation leaves the Lagrangian L invariant if L's values remain unchanged under the transformation:

$$L(u(x, y), v(x, y), \dot{u}, \dot{v}) = L(x, y, \dot{x}, \dot{y})$$

Proposition 1. *If a transformation F of the plane (or a region of the plane) leaves the Lagrangian invariant and is invertible then it maps action mininimizers to action minimizers.*

More specifically, the curve c minimizes the action from P to Q if and only if the curve $F(c)$ minimizes the action from $F(P)$ to $F(Q)$.

Proof: If c is any curve from P to Q then $F(c)$ is a curve from $F(P)$ to $F(Q)$ and $A(c) = A(F(c))$.

Definition 1. *If L has the optical form $L(x, y, \dot{x}, \dot{y}) = n(x, y)\sqrt{\dot{x}^2 + \dot{y}^2}$ then we will call any such L-preserving map F an "isometry".*

Notation Set

$$ds_E = \sqrt{dx^2 + dy^2}.$$

Then we can also write our optical Lagrangian as

$$Ldt = n(x, y)ds_E$$

The logic of the notation is that as soon as x, y are parameterized by t then $dx = \dot{x}dt, dy = \dot{y}dt$ so that $ds_E = \sqrt{\dot{x}^2dt^2 + \dot{y}^2dt^2} = \sqrt{\dot{x}^2 + \dot{y}^2}dt$

By a slight abuse of notation we may also write simply $L = nds_E$

Example 1 (Isometries of the Euclidean plane)**.** Let $L = \sqrt{\dot{x}^2 + \dot{y}^2}$. Then any translation: $(x, y) \mapsto (x + x_0, y + y_0) = F(x, y)$ preserves L, as does any rotation $(x, y) \mapsto (cx - sy, sx + cy) = F(x, y)$, $c = \cos(\theta_0)$, $s = \sin(\theta_0)$

Complex notation.

Exercise: verify that translation and rotation are isometries of the Cartesian plane by using complex notation.

Write $z = x + iy$ thus identifying the complex number line $\mathbb{C}$ with the Cartesian xy plane $\mathbb{R}^2$ which we identify in turn with Euclid's plane. Then $ds_E = |dz|$ so

that the Lagrangian for Euclid is $L = |dz|$. And any translation can be written $z \mapsto z + z_0$ while the rotation can be written $z \mapsto e^{i\theta_0} z$.

IN CLASS EXERCISES. Verify, algegraically that the transformations of translation and rotation just defined algebraically leave the Euclidean arclength, viewed as a Lagrangian, invariant.

We have seen that the general optical Lagrangian can be written $n(x, y)|dz|$. Poincare's upper half plane arclength has this form with $n = 1/y$ or:

$$L = |dz|/Im(z)$$

since $y = Im(z)$. We call this Lagrangian the "hyperbolic element of arclength." or the Poincare arclength.

Exercises. [IN CLASS] Show that L is invariant under the following transformations:

- 1)translations in the x direction : $(x, y) \mapsto (x + x_0, y)$ or $z \mapsto z + x_0$ (Note $x_0 = x_0 * 1 = (x_0, 0)$.)
- 2)positive scalings $(x, y) \mapsto (\lambda x, \lambda y)$; or $z \mapsto \lambda z$, $\lambda > 0$ real
- 3)reflections about y-axis: $(x, y) \mapsto (-x, y) = -\bar{z}$
  and the kicker:
- 4)Reflection about the unit circle: $(x, y) \mapsto (x, y)/(x^2 + y^2)$ or $z \mapsto 1/\bar{z}$. HINT: Rewrite both the transformation and the hyperbolic arclength element in terms of polar coordinates.

Show that by combining the two reflections (3) and (4) we get the map $z \mapsto -1/z$

Show that by composing transformations of type (1), (2) and (4) we can realize any transformation of the form

$$z \mapsto (az + b)/(cz + d) \tag{1}$$

as long as a, b, c, d are real and $ad - bc > 0$.

Show that by combining transformations (1), (2) and (3) we can realize the reflection about any circle whose center lies on the ideal.

And the reason we are doing all this:

Show that given any semicircle orthogonal to the ideal, we can find a transformation (1 which sends the positive y-axis onto this semicircle.

Conclude that the geodesics- the solution to (1) are precisely the arcs of these semicircles.

****

A second proof that these semicircles are geodesics.

Sketch proof. Reflections have fixed curves. These curves are geodesics. The reflection about any semi-circle perpindicular to the ideal is a reflection in the hyperbolic geometry.

Details.

Definition 2. *A reflection in a two-dimensional geometry is an isometry such that $F \circ F = Id$ and the fixed point set of F is a curve. We will call this curve the "mirror" of F.*

Examples.

Reflections across a line in Euclidean geometry.
Reflection across the y-axis in hyperbolic geometry.
Reflection about the unit circle in hyperbolic geometry. This is item (4) above.

Theorem 1. *The mirror of a reflection is a geodesic.*

We assume the local uniqueness of geodesics: given any two sufficiently close points P, Q, there is a *unique* shortest geodesic joining them. So let P, Q be two sufficiently close points on the mirror. Consider the unique geodesic c joining them. Apply F to c. We must have $F \circ c = c$ otherwise c is not unique. So c lies in the fixed point set of F. But this fixed point set is a curve! So the arc of c makes up the part of the mirrror between P and Q. Now continue c.

Exercise 1. *If F is a reflection and g is an isometry with no fixed points then gFg^{-1} is also a reflection in that geometry.*

Exercise 2. *Using the examples (1) and (2) show that you can find a g as in the exercise which takes the unit circle to any circle whose center is on the x axis.*

Conclude that reflection about any such circle is a hyperbolic isometry and hence, by the theorem, that such circles are geodesics.