where $ds = \sqrt{dx^2 + dy^2}$. For the integral to be well defined and positive we restrict attention to curves in the upper half–plane. The modified length $d\rho = ds/y$ is called the Poincaré metric. As an example, all the horizontal segments in Figure 11 have the same Poincaré lengths.

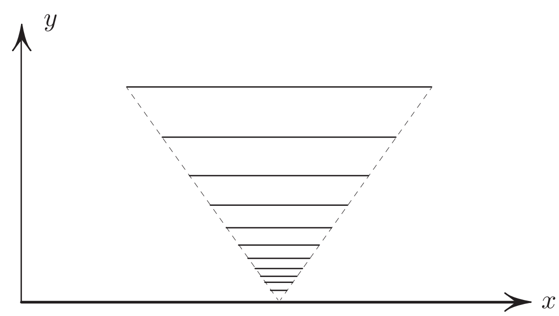

Figure 10. All these horizontal segments have the same length in Poincaré's metric.

A *geodesic* in the Poincaré metric is, by definition, a curve which makes the above integral stationary for variations of the curve with ends fixed. Since the above integral is of the conformal type $\int F(y)\, ds$ discussed in sections 3 and 4, geodesics satisfy $F(y)\sin\theta = c$ (see (5.4)), with $F(y) = 1/y$, or

$$\frac{\sin\theta}{y} = \text{const.} \tag{5.11}$$

Theorem 5.3. *Every geodesic of Poincaré's metric is a semicircle in $y > 0$ with the center on the x–axis or, as a limiting case, a vertical ray $x = c$, $y > 0$. Conversely, every such semicircle or ray is a geodesic in Poincaré's metric.*

Proof. Referring to Figure 11, consider a semicircle with the center O on the x–axis. Since

$$\theta \equiv \angle CAN = \angle BOA,$$

we conclude from ΔBOA that

$$\sin\theta = \frac{y}{r},$$

showing that (5.11) holds for such semicircles. Vertical rays satisfiy (5.11) with $\theta \equiv 0$ and the zero constant. This proves the first half of the theorem.

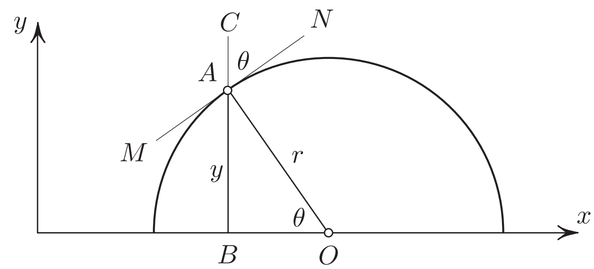

Figure 11. Geodesics in Poincaré's metric are semicircles or vertical rays.

To prove the converse, let us show that any solution of (5.11) is a semicircle with the center on the x–axis, or else a vertical ray. Indeed, (5.11) is a first order ODE . Given a point A and a slope y' at A, there is a unique solution with these initial data. But the semicircle in Figure 11 already satisfies this ODE (as we showed), and hence there is no other solution by the uniqueness theorem. It remains to consider the case when the initial slope is infinite. But in that case the geodesic must be a vertical ray. Indeed, otherwise there is a point on the geodesic where the slope is finite, but we already showed that such a geodesic is a semicircle whose slope is finite everywhere. The contradiction shows that the geodesic whose slope is vertical at one point is indeed a vertical ray. $\diamondsuit$

Poincaré metric is the simplest imaginable non–Euclidean metric in the sense explained by following exercise.

Exercise 5.2. Assume that the speed of light in an optical medium in the upper half–plane varies linearly with y, namely, $c(y) = y$. Using Fermat's principle, show that the rays are the Poincaré geodesics, i.e. semicircles perpendicular to the x–axis.

Solution. According to Fermat's principle, rays minimize (or make critical) the travel time

$$\int_A^B \frac{ds}{c} = \int_A^B ds/y.$$

$\diamondsuit$